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Asymptotic equivalence of difference systems

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Abstract

Using comparison theorems and Schauder's fixed point theorem we obtain the asymptotic equivalence of solutions of quasilinear difference systems.

1 Introduction.

Consider the difference systems

$$\Delta x = A(n)x + f(n,x), \tag{1}$$

and

$$\Delta y = A(n)y, \tag{2}$$

where $n \in N(n_0) = \{n_0, n_0+1, \cdots\}$ (n_o is a fixed nonnegative integer), x and f are m dimensional vectors, $A(n)$ is a $m \times m$ matrix for $n \in N(n_0)$.

The function $f = f(n,x)$ is defined on the product space $N(n_0) \times \mathbf{R}^m$, and $\mathbf{R}^m$ denote the m-dimensional real euclidean m space. Δ is the forward difference operator, i.e.

$$\Delta u(n) = u(n+1) - u(n)$$

for any function $u : N(n_0) \to \mathbf{R}^m$.

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We are concerned with the asymptotic relationships between the solutions of Eq. (1) and those of the unperturbed linear equation (2).

The problem of asymptotic equivalence for systems of ordinary differential equations has been studied by many authors [1, 2, 3, 4]. However, the only result that we know concerning the asymptotic equivalence between equation (1) and (2) is due to Benzaid-Lutz [5]. They assume that the linear difference systems $y(n+1) = A(n)y$ has ordinary dichotomies.

In this paper, we attempt to investigate some asymptotic relationships between the solutions of these two equations for cases when the function $f(n,x)$ is not necessarily small for all x, but for sufficiently small initial values becomes small compared to solutions of Eq. (2).

The basic technique used for our investigation is based on the well-known Schauder's fixed point theorem. See Drozdowicz-Popenda [6], Medina-Pinto [7], and Szmanda [11].

For this we will use the following compactness criterium in $\ell_\infty = \ell_\infty(N(n_0), \mathbf{R}^m)$, the Banach space of all bounded functions $x : N(n_0) \to \mathbf{R}^m$ with the supremun norm:

Compactness criterium.

A set $E \subset \ell_\infty$ is relatively compact if E is bounded and equiconvergent to some $\eta \in \mathbf{R}^m$.

We recall that $E \subset \ell_\infty$ is said equiconvergent to $\eta \in \mathbf{R}^m$ if for every $\varepsilon > 0$ there exists $M \geq n_0$ such that

$$|x(n) - \eta| < \varepsilon$$

for any $x \in E$ and $n \geq M$.

2 Main results

Theorem 1 *Let $\Phi(n)$ be the fundamental solution matrix of Eq. (2) satisfying $\Phi(n_0) = I$ for some n_0, where I denote the identity matrix. Let $D(n)$ be a nonsingular matrix satisfying*

$$|D(n)\Phi(n)| \leq \alpha(n), \tag{3}$$

where $\alpha(n)$ is a positive function for $n \geq n_0$. Suppose also that $f(n,x)$ satisfies

$$|\Phi^{-1}(n+1)f(n,x)| \leq F(n, \frac{1}{\alpha(n)}|D(n)x|), \tag{4}$$

where $F(n,u)$ is monotone nondecreasing in u for each n on $n \geq n_0$, $0 \leq u < \infty$, and the scalar equation

$$\Delta u = F(n,u) \tag{5}$$

has a positive solution which is bounded on $n \geq n_0$.

Then corresponding to each solution $x(n)$ of Eq. (1), with $|x(n_0)|$ sufficiently small, there exists a constant vector ξ such that

$$x(n) = \Phi(n)\xi + D^{-1}(n) \cdot o(\alpha(n)), \quad n \to \infty. \tag{6}$$

Proof: Using the discrete variation of constants formula, we can represent any solution $x = x(n, n_0)$ such that $x(n_0) = x_0$ by

$$x(n) = \Phi(n)x_0 + \Phi(n) \sum_{j=n_0}^{n-1} \Phi^{-1}(j+1)f(j, x(j)). \tag{7}$$

Then,

$$D(n)x(n) = D(n)\Phi(n)x_0 + D(n)\Phi(n) \sum_{j=n_0}^{n-1} \Phi^{-1}(j+1)f(j, x(j)),$$

which implies

$$\begin{aligned} |D(n)x(n)| &\leq \alpha(n)|x_0| + \alpha(n)\textstyle\sum_{j=n_0}^{n-1} |\Phi^{-1}(j+1)f(j, x(j))| \\ &\leq \alpha(n)|x_0| + \alpha(n)\textstyle\sum_{j=n_0}^{n-1} F(j, \frac{1}{\alpha(j)}|D(j)x(j)|), \end{aligned}$$

or

$$\frac{1}{\alpha(n)}|D(n)x(n)| \leq |x_0| + \sum_{j=n_0}^{n-1} F(j, \frac{1}{\alpha(j)}|D(j)x(j)|). \tag{8}$$

Hence

$$v(n) = \frac{1}{\alpha(n)}|D(n)x(n)|$$

satisfies

$$v(n) \leq |x_0| + \sum_{j=n_0}^{n-1} F(j, v(j)).$$

Then by [12, Th. 1.9.1., pp. 51-52],

$$v(n) \leq u(n), \tag{9}$$

for any $n \geq n_0$ if

$$|x_0| \leq u(n_0), \tag{10}$$

where $u = u(n)$ is a positive and bounded solution of equation

$$u(n) = u(n_0) + \sum_{j=n_0}^{n-1} F(j, u(j)).$$

Next, consider the expresion

$$x(n_0) + \sum_{j=n_0}^{n-1} \Phi^{-1}(j+1)f(j, x(j)).$$

Using (4) and (9) and the monotonicity of F, we obtain

$$\begin{aligned} \textstyle\sum_{j=n_0}^{n-1} |\Phi^{-1}(j+1)f(j,x(j))| &\leq \textstyle\sum_{j=n_0}^{n-1} F(j, \frac{1}{\alpha(j)}|D(j)x(j)|) \\ &\leq \textstyle\sum_{j=n_0}^{n-1} F(j, u(j)) \qquad (11) \\ &= u(n) - u(n_0); \end{aligned}$$

which is bounded as $n \to \infty$. As a consequence of the Lebesgue dominated convergence Theorem for the counting measure, we have

$$\begin{aligned} \lim_{n\to\infty} \Phi^{-1}(n)x(n) &= \lim_{n\to\infty} \textstyle\sum_{j=n_0}^{n-1} \Phi^{-1}(j+1)f(j,x(j)) + x(n_0) \\ &= \xi \quad \text{exists.} \end{aligned} \qquad (12)$$

Using (12), we may rewrite (7) as:

$$\begin{aligned} D(n)x(n) &= D(n)\Phi(n)x_0 + D(n)\Phi(n)\textstyle\sum_{j=n_0}^{\infty} \Phi^{-1}(j+1)f(j,x(j)) \\ &- D(n)\Phi(n)\textstyle\sum_{j=n}^{\infty} \Phi^{-1}(j+1)f(j,x(j)) \qquad (13) \\ &= D(n)\Phi(n)\xi - D(n)\Phi(n)\textstyle\sum_{j=n}^{\infty} \Phi^{-1}(j+1)f(j,x(j)) \end{aligned}$$

So, it follows from (11) that

$$|D(n)(x(n) - \Phi(n)\xi)| \leq \alpha(n)|\sum_{n}^{\infty} \Phi^{-1}(j+1)f(j,x(j))| = o(\alpha(n)).$$

Theorem 2 *Under the hypothesis of Theorem 1 corresponding to each solution x of Eq. (1), satisfying $|x(n_0)| = u(n_0)$ there exists n_0 big enough such that (6) holds with $\xi \neq 0$.*

Proof: Proceed as in the proof of Theorem 1, except that $x(n_0)$ is choosen such that $|x(n_0)| = u(n_0)$, we obtain the limiting vector ξ given by (12). Choose n_0 so large that $2u(n_0) > u(\infty)$, then

$$|x(n_0) \;+\; \textstyle\sum_{j=n_0}^{n-1} \Phi^{-1}(j+1)f(j,x(j))|$$

$$\geq |x(n_0)| - \sum_{j=n_0}^{n-1} |\Phi^{-1}(j+1)f(j,x(j))|$$

$$\geq |x(n_0)| - \sum_{j=n_0}^{n-1} F(j,\frac{1}{\alpha(j)}|D(j)x(j)|)$$

$$\geq |x(n_0)| - \sum_{j=n_0}^{n-1} F(j,u(j))$$

$$= |x(n_0)| - u(n) + u(n_0)$$

$$\geq 2u(n_0) - u(\infty) > 0.$$

Since

$$|x(n_0) + \sum_{j=n_0}^{n-1} \Phi^{-1}(j+1)f(j,x(j))|$$

is bounded away from zero for all n, the limiting vector ξ is not the zero vector.

Conversely, we obtain:

Theorem 3 *Let the hypotheses of Theorem 1 be satisfied, if in addition we assume that $f(n,x)$ is continuous in x for any n fixed. Then corresponding to each $\xi \in \mathbf{R}^n$ with $|\xi|$ sufficiently small, there is a solution $x(n)$ of Eq. (1) such that (6) holds.*

Proof: We consider the eq. (7) rewritten in the form

$$\check{x}(n) = \check{\Phi}(n)[\xi - \sum_{j=n}^{\infty} \Phi^{-1}(j+1)f(j,\alpha(j)D^{-1}(j)\check{x}(j))], \tag{14}$$

where

$$\check{x}(n) = \frac{D(n)x(n)}{\alpha(n)},$$

and

$$\check{\Phi}(n) = \frac{D(n)\Phi(n)}{\alpha(n)}.$$

For a given vector ξ with sufficiently small $|\xi|$, we will show that there exists a solution $x(n)$ of Eq. (14). It then follows that $x(n)$ satisfies Eq.(1) and (6), for the argument used in the proof of Theorem 1 proves the convergence of the series in Eq. (14). We use the notations

$$\|f\|_\infty = \sup_{n\geq n_0} |f(n)|, \qquad n_0 \geq 0$$

$$\|x\|_0 := \|\check{x}\|_\infty,$$

and

$$C_0(n_0) = \{x : N(n_0) \to \mathbf{R}^m | \|x\|_0 < \infty\}.$$

We will understand $C_0(n_0)$ as a Banach vectorial space with $\|x\|_0 = \|\check{x}\|_\infty$. We define a bounded, convex and closed subset $B_0(0,\rho)$ of $C_0(n_0)$ as:

$$B_0 := B_0(0,\rho) = \{\check{x}/x \in C_0(n_0) \text{ and } \|x\|_0 \leq \rho\}$$

and the operator S defined by

$$Sx(n) = \check{\Phi}(n)(\xi - \sum_n^\infty \Phi^{-1}(j+1)f(j,\alpha(j)D^{-1}(j)\check{x}(j))). \tag{15}$$

We will prove that the mapping S satisfies the assumptions of the Schauder fixed point Theorem.

Previously, we note that if u is the given bounded solution of Eq.(5), then $u = u(n)$ is an increasing function, and therefore $\lim_{n\to\infty} u(n)$ exists. Hence

$$F(n,\gamma) \in \ell_1(N(n_0)), \tag{16}$$

for any γ fixed.

Consider the operator $\check{S}$ on B_0 defined by the relation

$$\check{S}\check{x}(n) = \xi - \sum_n^\infty \Phi^{-1}(i+1)f(i,\alpha(i)D^{-1}(i)\check{x}(i)).$$

It satisfies the following:

a) There exist ξ and t_0 such that $\check{S}$ maps B_0 into B_0. In fact, taking $|\xi| \leq \rho/2$ and n_0 so large that

$$\sum_{n_0}^\infty F(i,\rho) \leq \rho/2,$$

then $\check{x} \in B_0$ implies:

$$\begin{aligned} |\check{S}\check{x}(n)| &\leq |\xi| + \textstyle\sum_n^\infty |\Phi^{-1}(i+1)f(i,\alpha(i)D^{-1}(i)\check{x}(i))| \\ &\leq |\xi| + \textstyle\sum_n^\infty F(i,\rho) \\ &\leq \rho. \end{aligned}$$

Note that $\|\check{\Phi}\|_\infty \leq 1$ from condition (3).

b) $\check{S}$ is continuous. Let $\{\check{x}_j\}_{j=1}^\infty$ be a sequence of elements of B_0 such that

$$\lim_{j\to\infty} \|x_j - x\|_0 = 0.$$

Since B_0 is closed, $\check{x} \in B_0$. Let $\varepsilon > 0$ and choose $n_1 \in N(n_0)$ so large that

$$\sum_{n_1}^{\infty} F(i,\rho) < \varepsilon,$$

by (16). Then, using (15) we get

$$\begin{aligned}
|\check{S}\check{x}_j(n) \;&-\; \check{S}\check{x}(n)| \\
&\leq \textstyle\sum_n^{n_1-1} |\Phi^{-1}(i+1)(f(i,\alpha(i)D^{-1}(i)\check{x}_j(i)) - f(i,\alpha(i)D^{-1}(i)\check{x}(i)))| \\
&+ \textstyle\sum_{n_1}^{\infty} |\Phi^{-1}(i+1)(f(i,\alpha(i)D^{-1}(i)\check{x}_j(i)) - f(i,\alpha(i)D^{-1}(i)\check{x}(i))| \\
&\leq \textstyle\sum_n^{n_1-1} |\Phi^{-1}(i+1)(f(i,\alpha(i)D^{-1}(i)\check{x}_j(i)) - f(i,\alpha(i)D^{-1}(i)\check{x}(i)))| \\
&+ 2\textstyle\sum_{n_1}^{\infty} F(i,\rho).
\end{aligned}$$

From which, by the continuity of f, follows that

$$\lim_{j\to\infty} \sup_{n\geq n_1} |\check{S}\check{x}_j(n) - \check{S}\check{x}(n)| = 0.$$

c) $\check{S}B_0$ is relatively compact. It suffices to prove that $\check{S}B_0$ is bounded and equiconvergent to ξ. For any $\check{x} \in B_0$, we have

$$\begin{aligned}
|\check{S}\check{x}(n)| &\leq |\xi| + \textstyle\sum_n^{\infty} |\Phi^{-1}(i+1)f(i,\alpha(i)D^{-1}(i)\check{x}(i))| \\
&\leq |\xi| + \textstyle\sum_n^{\infty} F(i,\|\check{x}\|_{\infty}) \\
&\leq |\xi| + \textstyle\sum_n^{\infty} F(i,\rho) < \infty,
\end{aligned}$$

by (16). Therefore, the set $\check{S}B_0$ is a uniformly bounded subset of the space $C_0(n_0)$. Moreover, it is equiconvergent to ξ, since for every $\varepsilon > 0$ there exists $M = M(\xi)$ such that

$$|\check{S}\check{x}(n) - \xi| \leq \sum_n^{\infty} F(i,\rho) < \varepsilon,$$

for every $n \geq M$ and all $\check{x} \in B_0$. Then the compactness criterium of section 1 proves that $\check{S}B_0$ is relatively compact, hence $\check{\Phi}\check{S}B_0$ is relatively compact. Therefore the function $S = \check{\Phi}\check{S}$ is compact.

By Schauder's fixed point theorem we conclude that there exists $\check{x} \in B_0$ such that $\check{x} = S\check{x}$, that is a solution of Eq. (14). Hence the proof of Theorem is complete.

Theorem 4 *Let the hypotheses of Theorem 3 be satisfied , and the additional hypothesis that all the solution of scalar equation (5) are bounded. Then corresponding to each solution $x(n)$ of Eq. (1) there exists a solution $y(n)$ of Eq. (2) such that (6) holds.*

Proof: We note that the restriction of the smallness of $|x(n_0)|$ and $|y(n_0)|$ depends only on the given bounded solution $u(n)$ of Eq. (5).

If all solution of Eq.(5) are bounded, then the arguments presented in the proofs of Theorem 1 and 3 remains valid for arbitrary solutions of equation (1) or equation (2).

Theorem 4 above implies the following corollary.

Corollary 1 *If all solutions of Eq.(2) and of the scalar equation (5) are bounded, if f satisfies (4) and if in addition $|det\Phi(n)| \geq c > 0$ for $n \geq n_0$. Then equation (1) and (2) are asymptotically equivalent.*

Proof: It suffices to take $D(n) = I$ in this case.

Example We consider the non-autonomous equation

$$Ex(n) = A(n)x(n) + f(n, x(n)), \tag{17}$$

where $E = \Delta + I$, $x = (x_1, x_2)$,

$$f(n, x(n)) = \left(\frac{x_1(n)}{(n+1)^2}, \frac{x_2(n)}{n!}\right)$$

and

$$A(n) = \begin{pmatrix} 1 & 0 \\ 0 & 1+\frac{1}{n} \end{pmatrix};$$

as a perturbation of the linear equation

$$Ez(n) = A(n)z(n),$$

whose fundamental matrix is given by

$$\Phi(n; n_0) = \begin{pmatrix} 1 & 0 \\ & \\ 0 & n \end{pmatrix}, \quad n \geq n_0 = 1.$$

If we take

$$D(n) = \begin{pmatrix} 1/n & 0 \\ & \\ 0 & 1/n \end{pmatrix},$$

then (3) will be satisfied with $\alpha(n) = 2$ for $n \geq n_0 \geq 1$. Also note that

$$|\Phi^{-1}(n+1)f(n,x(n))| = \frac{|x_1(n)|}{(n+1)} + \frac{|x_2(n)|}{(1+n)!}$$
$$\leq \lambda(n)|x|,$$

where $\{\lambda(n)\}_{n=n_0}^{\infty}$ is a positive real sequence such that for all $n \in N(n_0)$

$$\prod_{i=n_0}^{n-1} i\lambda(i) < M, \quad M > 0 \text{ constant.} \tag{18}$$

(The norm $|\cdot|$ of a vector or matrix is the sum of the absolute values of its elements).

Taking $F(n,u) = 2n\lambda(n)u$, we get

$$F\left(n, \frac{|D(n)x|}{\alpha(n)}\right) = F(n, \frac{|x|}{2n}) = \lambda(n)|x|.$$

Hence (4) is satisfied, and thus Eq.(5) becomes

$$Eu(n) = 2n\lambda(n)u(n), \quad u(1) = u_1 > 0.$$

By (18) it is easily seen that this solution is bounded. The conclusion (6) of Theorem 1 follows with sufficiently small initial values.

References

[1] Brauer, F., On asymptotic behavior of perturbed linear systems, J. Diff. Eqs. Vol. 6, (1), 1969, pp. 142-153.

[2] Brauer, F., Wong, J. S., On the asymptotic relationships between solutions of two systems of ordinary differential equations. J. Diff. Eqs. 9 (1969), 527-543.

[3] Coppel, W. A., Stability and Asymptotic Behavior of Differential Equations, D. C. Heath and Company, Boston, 1965.

[4] Coppel, W. A., Dichotomies in Stability Theory. Lectures Notes in Mathematics, 629, Springer-Verlag, Berlin, 1978.

[5] Benzaid, Z. - Lutz, D. A., Asymptotic representation of solutions of perturbed systems of linear difference equations. Studies in Applied Mathematics 77 (1987), 195-221.

[6] Drozdowicz, A., Popenda, J., Asymptotic behavior of the solutions of the second order difference equation. Proc. Amer. Math. Soc. 99 (1987), 135-140.

[7] Medina, R., Pinto, M., Asymptotic equilibrium of ordinary difference systems. Submitted.

[8] Medina. R., Pinto. M., Asymptotic representations for solutions of linear difference equations. J. Math. Anal. Appl. 165 (1992). N^o 2. 505-516.

[9] Medina. R., Pinto. M., Asymptotic behavior of solutions of nonlinear second order difference equations. Nonlinear Analysis. T.M.A. Vol. 19. N^o 2 (1992) 187-195.

[10] Pinto. M., Discrete dichotomies. (To appear).

[11] Szmanda. B., Characterization of oscillation of second order nonlinear differences equations. Bull. Polish. Acad. Sc. Math. Vol. 34 (3-4). 1986.

[12] Lakshmikantham. V., Leela. S., Martynyuk. A. A., Stability Analysis of Nonlinear Systems. Pure and Applied Mathematics. Marcel Dekker. Inc., 1989.