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THE CENTENARY OF ALEXANDRU FRODA

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One hundred years have elapsed since the birth of Alexandru Froda, a famous Rumanian mathematician, Professor at the University of Bucharest, whose personality has left a strong imprint on the training of several generations of mathematicians who are nowadays at their full professional maturity.

About 45 years ago, in a lecture hall on the ground floor of the University of Bucharest, across the present headquarters of the Society of Mathematical Sciences, I was one of those who were attending his lectures on set theory, a course given outside the compulsory curriculum. I was attracted by his contemplative and meditative attitude, his discourse in a low voice, with modulations which were accompanying the rythm of his thoughts. It was an attitude both lyrical and philosophical, of ecstasy for the cantorian constructions of the cardinal and ordinal transfinite. An ecstasy kept under control by a strong critical spirit, a very special ability of laying special emphasis upon controversial aspects. The old, beginning of century, passionate debates on the foundations of mathematics, were rekindled under our very eyes. Indeed, in the early 20-th century, mathematics was considered from many angles, such as logicism, formalism or intuitionism. The polemics as to the more or less constructive nature of set theory were thus revived in front of us by a masterly staging of conflicting ideas, every direction being presented with total affective and intellectual participation. It was our turn to estimate the merits and weak points of each school of thought. Professor Froda "acted" consecutively Cantor, Kronecker, Dedekind, Zermelo, Borel, Hilbert, Russel, and Brouwer, and each with impartial devotion. Every "part" was perfectly acted. At first sight every one of them seemed to be right. Alexandru Froda was fully aware that one can not have a truthful image of mathematics without a good understanding of the merits and limitations of every school of thought, and by including it in a broader perspective. His "acting" (and we mean it in the best sense of the word) was a complete success with his audience, which comprised not only the best students, but also several faculty members. The Froda style, as described so far, was typical of the university

atmosphere in our country until the mid-fifties. The bureaucratic changes which eventually occurred made him feel ill at ease, and the "real" Froda could hardly be recognized after 1955. In order to grasp him, one needed to look into his scientific works, his doctoral thesis at the Sorbonne in 1929, or turn to the recollections of his fellow professors. We attempt to do this in the following paragraphs.

There is no doubt as to the fact that the best known result connected with his name is the one wich asserts that, for every real function of a real variable, the set of discontinuities of the first kind is at most countable. This result was somewhat "up in the air" early in this century, and yet, what had existed explicitly formulated until Froda, was the weaker result which asserts that, if a real function defined on an interval of the real line does not have points of discontinuity of the second kind, then those of the first kind form a set which is at most countable. Froda's theorem, published in the *Comptes rendus de l'Académie des Sciences de Paris*, 1928, and afterwards extended into a thesis, forced its way with difficulty in the Anglo-Saxon mathematical world. For a long time it was not well known even in our country. This was due to the fact that– until the middle of the fifth decade– university education in mathematical analysis was mainly oriented towards the calculatory aspects, while the qualitative ones were neglected. I clearly remember that in the early fifties, Professor G. Mihoc, who was Dean at the time, once attended my exercise class in which we were studying the discontinuities of a given function. After the class, the Dean blamed me, with his characteristic delicacy, for not having mentioned Professor's Froda name in our class discussion. For me it was a well deserved lesson. I subsequently did my best to convey this lesson to my own students, but, unfortunately, most of the time with no success. The respect for the paternity of results which do not belong to ourselves is hard to acquire, as can often be seen in the scientific literature.

Afterwards, while studying the theory of real functions, I ran into his doctoral thesis. Thus, I could gradually understand the deeper significance of several theorems of his thesis, or of some obtained subsequently, in the framework of what one calls *the structure of arbitrary real functions of a real variable*. This concept is an accomplishment of our century. The Froda theorem was received with astonishment (for instance by Traian Lalescu) when

he presented it at a session of the Rumanian Mathematical Society. Professor G. Moisil recounts the episode as follows: *Alexandru Froda was one of our first mathematicians to start working on the theory of functions of a real variable. I can recall what powerful an impression he made with his first talk, in which he proved that the most general functions can exhibit specific properties.*

To study in depth the idea of discontinuity was one of Froda's main interests, and it went well beyond the sixth decade. For this purpose, Froda uses the oscillation operator. At first sight this attempt seems past hope, because, for every function of a real variable, the third oscillation coincides with the second (Sierpinski). Consequently, one can not obtain – in this way – a variety of possible behaviours as far as discontinuity is concerned (i.e. when the oscillation is strictly positive). This deadlock was broken by H. Blumberg (1917) who replaced the usual oscillation by different concepts of general oscillation. These are obtained by eluding the values of the function on sets belonging to a certain class of negligible sets. Some of the oscillations thus obtained cease to be stationary after iteration and allow for a fine classification of the discontinuities. In an article going back to 1948, Froda succeeds to obtain a subtle classification of discontinuities without being acquainted with Blumberg's article, and actually he does it in a much simpler way than Blumberg. Instead of avoiding the sets of a whole class of negligible parts, Froda has the ingenious idea to avoid a single value of the function, which is not any value, but namely the value at the point where the discontinuity is studied. Thus one obtains "the oscillation in a neighbourhood" and this allows Froda to obtain a more refined and elegant classification of discontinuity points. Even today this work deserves to receive the attention of the researchers, whether their work is on real functions or if they look for possible extensions.

We do not want to deal in detail with the actual presentation of the above results, nor with presenting other Froda results. For this the reader is referred to a special article of our book *Din gândirea matematică românească*, 1975 (in English: *On the Rumanian Mathematical Thought*), and to a joint paper with Alexandru Solian published in *Gazeta matematică* (1977). In the book mentioned above we study other results, obtained subsequently by Froda, to which one should now add the ones given by Mihai Eşanu more

recently.

Besides what we have written in other places, we would like to point to the new significance of Froda's work on the axiomatics of mechanics, in the light of the fractal geometry of nature, which was initiated by Benoit Mandelbrot, some 20 years ago. In 1957-1960, in a series of works, Froda proposes (both for theoretical grounds and empirical motivation) the study of some movements which are more general than those considered in classical mechanics. These movements are obtained by replacing the axioms (implicit or explicit) of the existence of continuous speed and of acceleration by other axioms, which can be compared with experience. Movements with no speed correspond to curves with no tangent and to continuous, nondifferentiable functions. At the time, the interest in such functions seemed to many to be at least strange, because continuous functions with no derivative did not seem to be of any relevance in the study of nature. And yet, those who worked in the field were very much aware of some warnings which had started to point out that it could be otherwise. This is why Professor Froda was invited to attend two important meetings, one at Berkeley, the other one at Stanford, in order to make known his results on the axiomatics of mechanics (invitations he could not honour, because he was not granted the necessary passport). Denjoy, during the second decade of this century, had already noticed that the discontinuous structure of the matter imposes the replacement of the classical concepts of continuity and derivability by more general notions (approximate continuity and derivability). In the fourth decade, Banach and Mazurkiewicz proved that, in the space those continuous functions which have, at least in one point, a finite derivative, form a negligible set–from the topological point of view – and this conflicted with the common intuition. In the eighth decade, the Mandelbrot book *The fractal geometry of nature* brings into the fore just those strange creatures of yore, of curves without tangent, and it is proved convincingly that these "ugly sets" account for most shapes and movements in nature. The Froda work in the foundations of mechanics should be rethought in relation with this.

Froda thus belongs to a double tradition: on one hand he is the disciple of Dimitrie Pompeiu, on the other hand he is a successor of the mathematics of negligible sets, whose

founders are the three great French mathematicians of the beginning of the 20-th century: Borel, Lebesgue and Baire. However, the French master of Froda was Arnaud Denjoy, the president of his thesis committee, the same Denjoy who was among the first to acknowledge the value of Pompeiu's thesis. I can recall that at the Third Congress of the Rumanian Mathematicians (Bucharest, 1956) Denjoy and Froda could be seen together most of the time; the master and the disciple (who meanwhile had become himself the master of a new generation of mathematicians) who got reunited after a long separation, imposed by the regulations of the regime. The Pompeiu-Froda filiation can be followed on several levels. Froda dealt with some of the problems put forward by Pompeiu, and made use of notions and results of the former. However, the link between them goes deeper, into their common mentality, the similarity in style, the same cultural basis, predominantly francophone. Like Pompeiu, Froda kept moving from one problem to another, but also revisited some of them; like his master, he created gems, but left to others the task of studying them in more depth. Just to give an idea of how Pompeiu appreciated Froda we relate the following episode. Professor N. N. Mihăileanu, who was a student of Pompeiu's at the beginning of the fourth decade, recalls that Pompeiu wrote to him, in a letter dated March 1st 1932, how, for one whole month, as part of the course given by Pompeiu, and at his invitation, and in his presence, Froda gave lectures on some of the results he had obtained in his thesis.

While he emphasized set theory and real functions, Froda studied also problems in mathematical statistics, algebra, foundations of mechanics, and number theory. He proved himself to be a *total* mathematician. In everything he did he brought a spirit of rigour, combined with the understanding of the empirical aspects. The filiation Pompeiu-Froda can be noticed as well in their common interest for subtleties exposed to obstacles, dangereous turns in reasoning. In this respect, his book ***Error and Paradox in Mathematics***, published by Editura Enciclopedică in 1971, is very significant. It is, to a great extent, autobiographical, and very much related to his formative years. Froda deals with moments in the thought itinerary of such thinkers as Borel, Comberousse, Ky Fan, and spots logical lacunae that many a scholar failed to notice. The entire approach is like a therapeutics

of the reasoning. Therefore, we recommend the book to be republished and translated in a foreign language. Along the same lines goes his article *Problèmes singuliers appartenant à la mathématique de précision* (1966) where he deals with the distinction made by Felix Klein, about 70 years ago, between the mathematics of the approximation and the mathematics of the precision. In the same article, Froda discusses the problem of Ulam, regarding the existence of a plane set everywhere dense, whose points are at rational distances of one another. Froda looks at this problem in a totally different way. In the same article he makes subtle remarks on the behaviour of the Riemann sums of functions which are not Riemann integrable.

Attracted by singular problems, Alexandru Froda was himself singular among Rumanian mathematicians. He has created his own mathematical universe, in spite of fashions or facile aspects. Froda has always chosen to build his own problems, rather than taking them from others. A typical case is his series of works on what he calls *propriétés à distance*, in contrast with the neighbourhood properties he had studied in his youth. Thus he was guided by esthetic criteria, of symmetry, an attitude which puts him next to Denjoy, who also liked to build his own problems, and who even had his own terminology, different from the current one. The character of relay-racer of the research is somewhat shadowed, and the dominant tendency becomes the set up of a relatively autonomous universe. Like a wayfarer who is more in love with the road than its end, Froda was rather passionate about the mathematical reflection itself than by its more or less spectacular results.

Such an attitude is paid dearly for, both in science and in the social life. In mathematics he was quite seldom cited, and he does not deserve this fate. As scholar and professor, Alexandru Froda did not enjoy advantages from the political power, a power to which he showed total indifference. This uprightness of his backbone did not make him popular, but did not pass unnoticed either, among those who still believed in the moral conduct of the teacher and the scholar. Now, at the hundreth anniversary of Professor Alexandru Froda, his conduct appears in its full exemplariness, and we feel that we have the duty to bring forward his life and work, so that the younger generation gets acquainted with them.