LIBERTAS MATHEMATICA, VOL. XIV (1994)

On Solutions of Singularly Perturbed Stochastic Volterra Equations

Marica Lewin

This note is based on the results obtained by C. Corduneanu for the existence of a solution for abstract singularly perturbed Volterra equations [2].

In [2] C. Corduneanu proved the existence of a solution for the abstract Volterra equation

$$x(t) = (Vx)(t) \quad . \tag{1}$$

and for the singularly perturbed functional-differential equation

$$\varepsilon x_\varepsilon(t) = -x_\varepsilon(t) + (Vx_\varepsilon)(t) \quad , \qquad \varepsilon > 0 \quad , \tag{2}$$

using the Schauder fixed point theorem in $C([0,T], R^n)$ and $L^p([0,T], R^n)$
$1 \leq p \leq \infty$.

He also proved that the solutions of (2) are defined on intervals not depending on ε, and finally, that in some conditions, the solutions of the equation (2) make a compact family of approximate solutions in $C([0,T], R^n)$, or $L^p([0,T], R^n)$ $\quad 1 \leq p \leq \infty$ for the solution of the equation (1).

In this note we try to prove a similar result for stochastic Volterra equations.

Usually, a growth condition and a Lipschitz condition on the coefficients ensure a strong unique solution for differential and integral stochastic equations.

A growth condition, and the continuity of the coefficients with respect to x, ensure the existence (without uniqueness) of a weak solution for stochastic equations.

By using the method employed by C. Corduneanu, the Schauder fixed point theorem, we can prove the existence (without uniqueness) of a strong solution for a Volterra stochastic equation without drift. We shall work in the real case, because in R^n, the problem may be treated in the same way.

We consider the stochastic Volterra equation

$$x(t) = \int_0^t k\Big(t, s, x(s)\Big)\, dw(s) \quad , \tag{3}$$

where $w(t)$ is a real Wiener process with respect to an increasing flow $(F_t)_{t \geq 0}$ of complete σ-algebras on a complete probability space (Ω, F, P), and $k(t, s, x)$ is a function, $k \cdot \Omega \times R \to R$, where $\Delta = \Big\{(t, s), \quad 0 \leq s \leq t \leq T\Big\}$ and R is the real line.

The integral of (3) is an Itō integral with respect to the Wiener process $w(t)$.

Let $L^2(\Omega, F, P)$ be the space of real random variables x, such that $Ex^2(\omega) < \infty$, where Ex was denoted by the expectation of a random variable. A real stochastic process (random function) $x(t)$ is called progressively measurable, if for every $t \geq 0$, $x(t, \omega)$ is F_t-measurable.

Definition 1 *A solution of (3) is understood to be a continuous progressively measurable process $x(t, \omega)$ satisfying (3) with probability 1, for all $t \geq 0$, at once, t belonging to some interval $[0, T_0] \in [0, T]$.*

A solution $x(t)$ of (3) is called strong if for all $t \in [0, T_0]$, $F_0 < T$: the process $x(t)$ is measurable with respect to F_t^w, the P completion of the σ-algebras,

$\sigma\Big\{w_s, \quad 0 \leq s \leq t\Big\}$.

We also consider the one-dimensional singularly perturbed stochastic equation

$$\varepsilon \dot{x}_\varepsilon(t) = -x_\varepsilon(t) + \int_0^t k\Big(t, s, x_\varepsilon(s)\Big)\, dw(s)\,, \qquad \varepsilon > 0 \tag{4}$$

with the initial condition

$$x_\varepsilon(0) = x_0 \qquad x_0 \in R \tag{5}$$

A solution of (4) and (5) will be a stochastic process satisfying the equations (4) and (5), with probability 1 for all $t \geq 0$ at once, t belonging to some interval, and having the properties mentioned in Definition 1.

We denote by $L^2(\Omega, T) = L^2\big([0,T]\big), L^2\big(\Omega, F, P\big)$, the Hilbert space of all functions $x = x(t,\omega)$, $t \in [0,T]$, with values $x(t)$ being real random variables in $L^2\big(\Omega, F, P\big)$, which fulfills the following assumptions:

$$\text{for every } t \in [0,T]\,, \quad x(t) \text{ is } F_t^w \text{ measurable}\,. \tag{6}$$

$$\int_0^T Ex^2(t)\, dt < \infty\,, \text{ the integral with respect to } t \tag{7}$$

being Lebesgue.

The topology of $L^2\big(\Omega, T\big)$ is generated by the norm

$$\| x \|_2 = \left(\int_0^T Ex^2(t,\omega)\, dt\right)^{1/2}$$

the integral being taken with respect to the Lebesgue measure.

We have the following:

Theorem 1 *Suppose*

a) the function $k(t,s,x)$ *is measurable with respect to all* $(t,s) \in D$, *and continuous with respect to* $x \in R$.

b) For all $(t,s) \in \Delta$

$$|K(t,s,x)|^2 \le K_2|x|^2 + K_1\,, \quad \textit{for all } x \in R$$

c) $\lim_{h\to 0} \int_0^T \Big(\int_0^t |k(t+h,s,x) - k(t,s,x)|^2\, ds\Big)\, dt = 0$ *, for all* $x \in R$

Then, the equation (3) has a strong solution on an interval $[0,T_0]$, $0 < T_0 \le T$, *where* T_0 *does not depend on* ω.

Proof: Let U be the operator from $L^2(\Omega,T)$ to $L^2(\Omega,T)$ defined by

$$Ux = \int_0^t K\Big(t,s,x(s)\Big)\, dw(s)$$

We show first that U is a linear compact operator from $L^2(\Omega,T)$ to $L^2(\Omega,T)$.

We denote by S the unit sphere in $L^2(\Omega,T)$.

We see, that for every x belonging to S, we get

$$\begin{aligned}
\| Ux(t+h) - Ux(t) \|_2 &= \int_0^T E\Big(Ux(t+h) - Ux(t)\Big)^2 dt \\
&= \int_0^T E\left(\int_0^{t+h} k\Big(t+h,s,x(s)\Big)\, dw(s) - \int_0^t k\Big(t,s,x(s)\, dw(s)\Big)\right)^2 dt \\
&\le 2\int_0^T E\left(\int_t^{t+h} k\Big(t+h,s,x(s)\Big)\, dw(s)\right)^2
\end{aligned}$$

$$+ \ 2\int_0^T E\Big[\int_0^t \Big(k(t+h,s,x(s)) - k\big(t,s,x(s)\big)\, dw(s)\Big]^2$$

$$= \ 2\int_0^T \int_t^{t+h} Ek\big(t+h,s,x(s)\big)^2 ds + 2\int_0^T \int_0^t E\Big(k\big(t+h,s,x(s)\big)$$

$$- \ k\big(t,s,x(s)\big)\Big)^2 dsdt \tag{8}$$

The condition b) of the theorem ensures the existence of the expectation of $k\big(t,s,x(s)\big)$, when $x \in S$, and the Lebesgue integrability of $Ek^2\big(t,s,x(s)\big)$. We applied also the properties of the Itō integral.

The conditions a), b) and c) of the theorem result in

$$\lim_{h\to 0} \int_0^T E\big(Ux(t+h) - Ux(t)\big)^2 dt = 0 \quad , \tag{9}$$

uniformly with respect to $x \in S$.

The relation (9) and the condition b) of Theorem 1 leads to the compactness of U, as an operator from $L^2(T,\Omega)$ to $L^2(T,\Omega)$. On behalf of condition b), we get that the unit sphere S is taken into itself by the operator U, provided we substitute to the space $L^2(\Omega,T)$, the corresponding space of restrictions of its functions to a subinterval $[0,T_0]$, $T_0 < T$.

Now, by using the Schauder fixed point theorem [5] in $L^2(\Omega,T)$, we obtain the existence of a solution for the equation (3), belonging to the space $L^2(\Omega,T_0)$. The properties of the Itō integral imply that the solutions are continuous with probability 1 and so, for every $t \in [0,T_0]$, (3) is satisfied w.p.1. The continuity w.p.1 of the process $x(t)$ results in the fact that (3) is verified w.p.1 for all $t \in [0,T_0]$ at once. The way as we defined $L^2(\Omega,T)$, containing all the functions F_t^w measurable, implies that $x(t)$ is a strong solution on $[0,T_0]$.

Before studying the solution of (4) and (5) we need some auxiliary results.

Let us consider the family of functions $\phi(t,\varepsilon) = \varepsilon^{-1}\exp\left(-\varepsilon^{-1}t\right)$, for $t \in R_+$, $\varepsilon > 0$, and $\phi(t,\varepsilon) = 0$ on the negative half axis.

The following properties of the family of functions defined above are obvious:

α) $\int_0^t \phi(s,\varepsilon)\,ds = 1 - \exp\left(-\varepsilon^{-1}t\right) \qquad t \in R^+$.

β) $\lim_{\varepsilon\to 0}\int_0^T \phi(s,\varepsilon)\,ds = 1$, for any $T > 0$.

γ) $\lim_{\varepsilon\to 0}\int_\delta^T \phi(s,\varepsilon)\,ds = 0$, for $0 < \delta < T$ (see[2] , [6]).

Due to the fact that the family of functions $\left\{\phi(t,\varepsilon),\quad \varepsilon > 0\right\}$ satisfies the conditions α) , β), γ), we can say that this family provides an *approximate identity* for the convolution on any interval $[0,T]$, $T > 0$ (see [2] , [6]).

The perturbed equations (4) and (5) may be rewritten in the equivalent form

$$x_\varepsilon(t) = e^{-\frac{t}{\varepsilon}}\,x_0 + \int_0^t e^{-\frac{(t-s)}{\varepsilon}} \int_0^S \sigma\left(s,u,x(u)\right)dw(u)\,ds \quad . \tag{10a}$$

or using the above notations

$$x_\varepsilon(t) = e^{-\frac{t}{\varepsilon}}\,x_0 + \int_0^t \phi_\varepsilon(t-s)\,Ux(s)\,ds \tag{10b}$$

Theorem 2 *The perturbed equation (4) with the initial condition (5) has a strong solution on interval $[0,T_\varepsilon]$, $[T_\varepsilon \leq T$, if the conditions a), b), c) of the Theorem 1 are fulfilled.*

Proof: We notice that $e^{-\frac{t}{\varepsilon}}\,x_0$ converges to 0 in the norm of $L^2\left([0,T],R\right)$, as $\varepsilon > 0$, for any $x_0 \in R$, so $e^{-\frac{t}{\varepsilon}}$ converges to 0, also in the norm of $L^2(\Omega,T)$. Hence, without loss of generality we can suppose that for any $x_0 \in R$, $\epsilon^{-\frac{1}{2}} x_0$ belongs to the unit sphere S of $L^2(\Omega, T)$, for some ε.

For a given $\varepsilon > 0$, the operator V_ε is defined as

$$\left(V_\varepsilon x\right)(t) = \int_0^t \phi_\varepsilon(t-s)(Ux)(s)\,ds \quad , \qquad t \in [0, T] \quad , \tag{11}$$

and is linear, compact, as an operator in $L\left(L^2(\Omega, T), L^2(\Omega, T)\right)$.

In order to see that it suffices to point out that the convolution operator involved in (11) is continuous from $L^2\left([0, T], R\right)$ in $L^2\left([0, T], R\right)$. Therefore, the convolution product appearing in (11) defines a continuous, compact operator in $L^2(\Omega, T)$.

On behalf of condition (9) and the properties α), β) and γ) of function $\phi_\varepsilon(t-s)$, one sees also that a ball centered in $\epsilon^{-\frac{1}{2}} x_0$ is taken into itself by the operator V, provided we substitute to the space $L^2(\Omega, T)$ the corresponding space of restrictions of its functions to a subinterval $[0, T_\varepsilon]$. Moreover, the intervals where the solutions x_ε are defined do not depend on ε, so we consider the solutions x_ε of Eqs. (4) and (5) defined on the same interval, without loss of generality, denoted by $[0, T_0]$.

Taking into account the property α) of the approximate identity we obtain for $x_\varepsilon \in S$:

$$\begin{aligned}
\| V x_\varepsilon \|_2 &= \int_0^T E\left[\int_0^t \phi_\varepsilon(t-s) \int_0^s k\left(s, u, x_\varepsilon(u)\right) dw(u)\, ds\right]^2 dt \\
&= \int_0^T \int_0^t \int_0^t \phi_\varepsilon(t-s)\phi_\varepsilon(t-v)\, E\left(\int_0^s k\left(s, u, x_\varepsilon(u)\, dw_0 \cdot \int_0^v k\left(v, u, x_\varepsilon(u)\, dw_u\right)\right) dsdvdt \\
&= \int_0^T \int_0^t \int_0^t \phi_\varepsilon(t-s)\phi_\varepsilon(t-v) \cdot \int_0^{\min(s,v)} Ek\left(s, u, x_\varepsilon(u)\right) k(v, u, x_\varepsilon(u)\, dudsdvdt
\end{aligned}$$

$$\leq \int_0^T \int_0^t \int_0^t \phi_\varepsilon(t-s)\phi_\varepsilon(t-v) \cdot \int_0^{\min(s,v)} \left(K_1 + K_2 E x_\varepsilon^2(u)\right) du ds dv dt$$

$$\leq C \int_0^T \Big(\int_0^t \phi_\varepsilon(t-s)\, ds \cdot \int_0^t \phi_\varepsilon(t-v)\, dv \Big)\, dt \leq C\,, \quad \text{for all } \varepsilon \geq 0 \quad .$$

where by C we denoted a constant (see [1]). So $x_\varepsilon \in S$, for all $\varepsilon > 0$, $t \in [0, T_0]$.

Since S is closed and convex, we obtain by Schauder's fixed point theorem, the existence of a solution x_ε on $[0, T_0]$ for all $\varepsilon > 0$.

The properties of the Itō integral and function $\phi_\varepsilon(t)$ imply that each of the solutions $x_\varepsilon(t)$ is continuous with probability 1 on $[0, T_0]$. The continuity results in the fact that x_ε verifies the equation (10) for all $t \in [0, T_0]$ at once, w.p.1, so $x_\varepsilon(t)$ satisfies all the requirements stated in Definition 1. Theorem 2 is proved.

Since, in general, the solution of equation (3) is not unique, we cannot expect $x_\varepsilon(t)$ to converge towards a limit as ε tends to 0. This seems to be very natural, if we also keep in mind that the equations (4) and (5) do not necessarily enjoy uniqueness.

Actually, we shall prove that the set of solutions of (4) and (5), for $\varepsilon > 0$, constitutes a compact set in $L^2(\Omega, T_0)$, even in the case where we consider all the solutions of (4) and (5) corresponding to a given ε, with the chosen initial value.

We denote by $A = \left\{x_\varepsilon, \quad \varepsilon > 0\right\}$ the set of all solutions of (4) and (5), and for the simplicity of notations, we do not mark the differences between the solutions corresponding to the same ε.

We now prove

Theorem 3 *The set A is relatively compact in the topology of $L^2(\Omega, T_0)$ and the limits of its convergent subsequences are solutions of equation (3).*

Actually,

$$\sup_{\varepsilon\in A} \| x_\varepsilon \|_2 = \sup_{\varepsilon\in A} \left(\int_0^{T_0} E\,|x_\varepsilon(s)|^2\, ds \right)^{1/2} < \infty \quad . \tag{12}$$

because $x_\varepsilon(s)$ for all $\varepsilon > 0$, belongs to the unit sphere. The second condition is also satisfied.

$$x_\varepsilon(t+h) - x_\varepsilon(t) = \left(e^{\frac{-(t+h)}{\varepsilon}} x_0 - e^{-\frac{t}{\varepsilon}} x_0 \right) + \int_0^{t+h} \phi(t+h-s) \int_0^s k\big(s, u, x_\varepsilon(u)\big)\, dw(u) ds$$

$$- \int_0^t \phi_\varepsilon(t-s) \int_0^s k\big(s, u, x_\varepsilon(u)\big)\, dw(u) ds$$

$$= \left(e^{\frac{-(t+h)}{\varepsilon}} - e^{-\frac{t}{\varepsilon}} \right) x_0 + \int_0^t \left(\phi_\varepsilon(t+h-s) - \phi(t-s)\right) \int_0^s k\big(s, u, x_\varepsilon(u)\big)\, dw(u) ds$$

$$+ \int_t^{t+h} \phi_\varepsilon(t+h-s) \int_0^s k\big(s, u, x_\varepsilon(u)\big)\, dw(u) ds \quad .$$

Taking into account that all $x_\varepsilon(t)$ belong to the unit sphere S and condition (b) of Theorem 1 on k, we obtain

$$\| x_\varepsilon(t+h) - x_\varepsilon(t) \|_2 \le \int_0^{T_0} \left(e^{\frac{-(t+h)}{\varepsilon}} - e^{-\frac{t}{\varepsilon}} \right)^2 x_0^2\, dt$$

$$+ \int_0^{T_0} E \left(\int_0^t \big(\phi_\varepsilon(t+h-s) - \phi_\varepsilon(t-s) \big) \int_0^s k\big(s, u, x_\varepsilon(u)\big)\, dw(u) ds \right)^2$$

$$+ \int_0^{T} E \left(\int_t^{t+h} \phi_\varepsilon(t+h-s) \cdot \int_0^s k\big(s, u, x_\varepsilon(u)\big)\, dw(u) ds \right)^2 = B(h)$$

$$+ C(h) + D(h)$$

Obviously $\lim_{h\to 0} B(h) = 0$, uniformly with respect to ε.

$$C(h) \;=\; \int_0^{T_0} E\Bigg[\int_0^t \int_0^t \big(\phi_\varepsilon(t+h-s) - \phi_\varepsilon(t-s) \big) \big(\phi_\varepsilon(t+h-v) - \phi_\varepsilon(t-v) \big)$$

$$\cdot \left(\int_0^s k\big(s, u, x_\varepsilon(u)\big)\, dw(u) \cdot \int_0^v k\big(v, u, x_\varepsilon(u)\big)\, dw(u) \right) du ds \Bigg] dt$$

$$= \int_0^{T_0}\int_0^t\int_0^t (\phi_\varepsilon(t+h-s)-\phi_\varepsilon(t-s))(\phi_\varepsilon(t+h-v)-\phi_\varepsilon(t-v))$$

$$\cdot \int_0^{\min(s,v)} Ek(s,u,x_\varepsilon(u)\cdot k\Big(v,u,x_\varepsilon(u)\Big)\,dudsdvdt \qquad (13)$$

We used the properties of the Itō integral and the Fabini theorem. So, taking into account $x_\varepsilon \in S$, and the growth condition b), we obtain

$$C(h) \le \int_0^{T_0}\int_0^t\int_0^t (\phi_\varepsilon(t+h-s)-\phi_\varepsilon(t-s))(\phi_\varepsilon(t+h-v)-\phi_\varepsilon(t-v)) \qquad (14)$$

$$\cdot\int_0^{\min(s,v)}\Big(K_1+K_2Ex_\varepsilon^2(0)\Big)\,dudsdvdt \le C_1\int_0^{T_0}\Big(\int_0^t \phi_\varepsilon(t+h-s)-\phi_\varepsilon(t-s)\,ds\Big)^2 dt$$

A simple calculation of the last integral leads to $\lim_{h\to 0} C(h)=0$, uniformly with respect to ε.

Similarly, we prove that

$$\lim_{h\to 0} D(h) \le \lim_{h\to 0} C_2\int_0^{T_0}\Big(\int_t^{t+h}\phi_\varepsilon(t+h-s)\,ds\Big)^2 dt = 0 \quad , \qquad (15)$$

uniformly with respect to ε. Hence, the set of solutions A is relatively compact in $L^2(\Omega, T_0)$.

Now, we shall prove that the limits of convergent subsequences of the set A are solutions of equation (3).

Let $\{\varepsilon_n\}$ be a sequence of positive numbers such that $\varepsilon_n \to 0$, on $n\to\infty$.

For each n, let $x_{\varepsilon_n}(t)$ be a solution of equation (10) with the specified initial condition, defined on the interval $[0, T_0]$ and such that $\| x_{\varepsilon_n} \|_2 \le 1$ ($x_{\varepsilon_n}\in S$).

This fact means that $x_{\varepsilon_n}(t)$ is continuous w.p.1 and satisfies w.p.1 the equation

$$x_{\varepsilon_n}(t) = e^{-\frac{t}{\varepsilon_n}}x_0 + \int_0^t \phi(t-s,\varepsilon_n)\int_0^s k\Big(s,u,x_{\varepsilon_n}(u)\Big)\,dw(u)ds \qquad (16)$$

We consider now a subsequence of x_{ε_n}, denoted for the simplicity of notation, also by x_{ε_n} convergent in $L^2(\Omega, T_0)$ to x. The convergence of $x_{\varepsilon_n}(t,w)$ w.p.1 for almost all $t \in [0, T_0]$, and this implies that $x_{\varepsilon_n}(t)$ converges in probability for every $t \in [0, T_0]$ to $x(t)$.

The continuity of k with respect to x implies that $\int_0^s k(s,u,x_{\varepsilon_n}(s)\,dw(s)$ converges in probability to $\int_0^s k(s,u,x(u)\,dw(u)$ for every $s \in [0, T_0]$.

Now we can write

$$\int_0^t \phi(t-s,\varepsilon_n)\int_0^s k\big(s,u,x_{\varepsilon_n}(u)\big)\,ds$$

$$= \int_0^t \phi(t-s,\varepsilon_n)\left(\int_0^s k\big(s,u,x_{\varepsilon_n}(u)\big)\,dw(u) - \int_0^s k\big(s,u,x(u)\big)\,dw(u)\right)\,ds$$

$$+ \int_0^t \phi(t-s,\varepsilon_n)\left(\int_0^s k\big(s,u,x(u)\big)\,dw(u)\right)\,ds$$

$$= B^1_{\varepsilon_n}(t) + B^2_{\varepsilon_n}(t)$$

Because $\phi(t,\varepsilon)$ provides an approximate identity for the convolution ([see [6]), we have

$$\lim_{\varepsilon_n \to 0} B^2_{\varepsilon_n}(t) = \int_0^t k\big(t,s,x(s)\big)\,dw(s)\,, \quad \text{w.p.1, for every } t \in [0, T_0]$$

A simple calculation leads to the fact that $B^1_{\varepsilon_n}$ converges to 0, when $\varepsilon_n \to 0$ in $L^2(\Omega, T_0)$. This implies $\lim_{\varepsilon_n \to 0} B^1_{\varepsilon_n}(t) = 0$, in probability, for every $t \in [0, T_0]$. So, taking the limit in probability for $\varepsilon_n \to 0$, in the equation (16), we obtain that $x(t,u)$, the limit of $\{x_{\varepsilon_n}\}$ satisfies with probability 1 the equation (3) for every $t \in [0, T_0]$.

$x(t,\omega)$ satisfying the equation (3), w.p.1, has a continuous version due to the continuity of the Itō integral. So, we obtain the process $x(t,\omega)$, satisfies w.p.1 equation (3) for all $t \in [0, T_0]$. $x(t,\omega)$ is F_t^w measurable, because $x_{\varepsilon_n}(t,\omega)$ is F_t^w measurable, so conformably to the Definition (1) $x(t,\omega)$ is a solution of (3), and the theorem is proved.

Remark:

1. The problem could also be treated in $C\Big([0,T],\, L^2(\Omega, F, P)\Big)$ but the result would be the same because we need only the convergence in probability of $x_n(s)$ for every $s \in [0, T_0]$, for the convergence of the Itō integral.

2. The problem could be solved in $\mathbb{R}^m$ in the same way.

3. We could extend the problem without difficulty to a Volterra operator with stochastic kernel,

$$Vx(t) = \int k\Big(t, s, x(s,\omega), w\Big)\, dw(\omega) \quad ,$$

that is to the study of the equation

$$x(t,\omega) = f(t,\omega) + \int_0^t k\Big(t, s, x(s,\omega)\omega\Big)\, dw(s) \tag{17}$$

4. The solutions which we obtained are strong, but they are defined on a smaller interval than initial. The interval of existence does not depend on ω.

Acknowledgement

I would like to thank Professor Corduneanu for his fruitful ideas and marvelous ways of solving a problem. His excellent ideas and technical tools could be extended and lead to real interesting and new problems in the domain of stochastic equations.

I would also like to express my gratitude for Professor Corduneanu's continuous support and encouragement, from the beginning, to work in the field of mathematics.

References

[1] L. Arnold: "Stochastic Differentialgleichungen Theorie und Anwendung", R. Oldenbourg Verlag, Munchen, Wien 1973.

[2] C. Corduneanu, A Singular Perturbed Approach to Abstract Volterra Equations in "Nonlinear Analysis and Applications", edited by Lecture Notes in Pure and Applied Mathematics, Vol. 109.

[3] C. Corduneanu, "Principles of Differential and Integral Equations", Chelsea Publ. Col., New York.

[4] R. E. Edwards, "Fourier Series, A Modern Introduction", Vol. I, Ed. Springer-Verlag, New-York, 1979.

[5] A. Haimovici, "Ecuatii diferentiale si integrale (in Romanian). Edit. Didoctica si Pedagogica, Bucuresti 1965.

[6] C. Sadosky, "Interpolation of Spaces and Singular Integrals", M. Dekker, New-York, 1979.

[7] K. Yoshida, "Functional Analysis", Springer-Verlag, 1965.