

SOME OPEN PROBLEMS IN FIXED POINT THEORY  
BY MEANS OF FIXED POINT STRUCTURES

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ABSTRACT. The classical results of Darbo and Sadovskii proved to be a fruitful source of inspiration for discoveries about fixed point theory. Some abstract version of these results are given in [42], [45], [46], [48] and [50]. In this paper we present some open problems. These problems are formulated in the terms of the fixed point structures.

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1. INTRODUCTION. Fixed point theory is currently a very active branch of mathematics, and there is a good reason to believe that it will become even more active in the future (see [54], [37]-[40], [20], [16], [30], [56], [2], [6], [7], [9], [11], [13], [24], ...). In the present paper we present a new direction of research in the fixed point theory: fixed point structure theory.

The notion "fixed point structure", which we gave in 1986 (see [42], [45], [46], [48] and [50]), is a generalization of some notion as "topological space with fixed point property" (Brouwer, Schauder, Tychonov, ... (see [20], [21], [23], [25], [30], [38],

[11], [54], [56],...)), "ordered set with fixed point property" (Tarski, Bourbaki, Birkhoff,... (see [37], [31], [40], [51], [54],...)), "mapping with fixed point property on family of set" (Jones, de Blasi (see [45], [46], [47],...)), "object with fixed point property" (Lawvere, Lambek, Rus,... (see [40], [41], [1], [8], [32], [37], [55],...)), "fixed point structure" (Muenzenberger-Smithson, Precup,... (see [33], [35], [36])). In the papers [42]-[46], [48], [50], we use the technique of the fixed point structure to give some new fixed point theorems for single and multivalued mappings. In this paper we present some open problems. These problems are formulated in the terms of the fixed point structures.

Throughout this paper we follow terminologies and notations in [45]. Here are some of them

- (1)  $P(X) := \langle Y \subset X \mid Y \neq \emptyset \rangle$ ,  
 $M(Y) := \langle f \mid f: Y \rightarrow Y \text{ a mapping} \rangle$ .
- (2) For  $f: X \rightarrow X$ ,  
 $I(f) := \langle A \in P(X) \mid f(A) \subset A \rangle$ ,  
 $F_f := \langle x \in X \mid f(x) = x \rangle$ .
- (3) For  $f, g: X \rightarrow Y$ ,  
 $C(f, g) := \langle x \in X \mid f(x) = g(x) \rangle$ .
- (4) For  $(X, d)$  a metric space and  $f: X \rightarrow X$ ,  
 $P_b(X) := \langle A \in P(X) \mid \delta(A) < +\infty \rangle$ ,  
 $P_{cl}(X) := \langle A \in P(X) \mid A = \bar{A} \rangle$ ,  
 $P_{cp}(X) := \langle A \in P(X) \mid A \text{ a compact set} \rangle$ ,  
 $I_b(f) := \langle A \in I(f) \mid \delta(A) < +\infty \rangle$ ,  
 $I_{cl}(f) := \langle A \in I(f) \mid A = \bar{A} \rangle$ ,  
 $I_{cp}(f) := \langle A \in I(f) \mid A \text{ a compact set} \rangle$ .
- (5) For  $(X, \|\cdot\|)$  a Banach space and  $f: X \rightarrow X$ ,  
 $P_{cv}(X) := \langle A \in P(X) \mid A \text{ a convex subset} \rangle$ ,

$$I_{cv}(f) := \langle A \in I(f) \mid A \text{ a convex subset} \rangle.$$

2. FIXED POINT STRUCTURE. For the convenience of the reader, we will recall some fundamental notions which appear in the fixed point structure theory (see [42], [44], [45], [46], [48]). We begin with some definitions and examples.

Definition 2.1. A triple  $(X, S, M)$  is a fixed point structure (briefly, f.p.s.) if

- (i)  $X$  is a nonempty set,  $S \subset P(X)$ ,  $S \neq \emptyset$ ;
- (ii)  $M: P(X) \rightarrow \bigcup_{Y \in P(X)} M(Y)$ ,  $Y \rightarrow M(Y) \subset M(Y)$ , is a multi-valued mapping, such that, if  $Z \subset Y$ ,  $Z \neq \emptyset$ , then  $M(Z) \supset \langle f|_Z \mid f \in M(Y), Z \in I(f) \rangle$ ;
- (iii) Every  $Y \in S$  has the fixed point property (briefly, f.p.p.) with respect to  $M(Y)$ .

Remark 2.1. Let  $X$  be a nonempty set and  $M(Y) \subset M(Y)$  for  $Y \subset X$ . The triple  $(X, S, M)$  which satisfies (i) and (ii) in Definition 2.1, we call weak fixed point structure (briefly, w.f.p.s.).

Now some examples of f.p.s.

Example 2.1. The trivial f.p.s.  $X$  is a nonempty set,  $S = \langle \{x\} \mid x \in X \rangle$  and  $M(Y) = M(Y)$ .

Example 2.2. The f.p.s. of contractions.  $(X, d)$  is a complete metric space,  $S = P_{cl}(X)$  and  $M(Y) = \langle f: Y \rightarrow Y \mid f \text{ a contraction} \rangle$ .

Example 2.3. Fixed point structure of Brouwer-Schauder-Tychonov.  $X$  is a locally convex linear topological space,

$$S = P_{cp,cv}(X) \text{ and } M(Y) = C(Y, Y).$$

Example 2.4. Browder's fixed point structure.  $X$  is a Hilbert space,  $S = P_{b,cl,cv}(X)$  and  $M(Y) = \{f: Y \rightarrow Y \mid f \text{ is nonexpansive}\}$ .

Definition 2.2. Let  $X$  be a nonempty set,  $Z \subset P(X)$  and  $Z \neq \emptyset$ . A mapping  $\theta: Z \rightarrow \mathbb{R}_+$  has the intersection property if  $Y_n \in Z$ ,  $Y_{n+1} \subset Y_n$ ,  $n \in \mathbb{N}$  and  $\lim_{n \rightarrow \infty} \theta(Y_n) = 0$  implies  $Y_\infty := \bigcap_{n \in \mathbb{N}} Y_n \neq \emptyset$ ,  $Y_\infty \in Z$  and  $\theta(Y_\infty) = 0$ .

Example 2.5. Let  $(X, d)$  be a complete metric space and  $Z = P_{b,cl}(X)$ . Then  $\theta = \delta$  is a mapping with intersection property.

Example 2.6. Let  $(X, d)$  be a complete metric space and  $Z = P_{b,cl}(X)$ . Then

a)  $\theta = \alpha_K$  (Kuratowski's measure of noncompactness) is a mapping with intersection property.

b)  $\theta = \alpha_H$  (Hausdorff's measure of noncompactness) is a mapping with intersection property.

Example 2.7 [44]. Let  $(X, d, \mathcal{W})$  be a convex metric space with the property (C). Then the mapping  $\beta_{EL}: P_{b,cl}(X) \rightarrow \mathbb{R}_+$ ,  $A \rightarrow \sup\{d(a, A) \mid a \in \text{co } A\}$  is a mapping with intersection property.

Definition 2.3. Let  $(X, S, M)$  be a fixed point structure,  $\theta: Z \rightarrow \mathbb{R}_+$  ( $S \subset Z \subset P(X)$ ) and  $\eta: P(X) \rightarrow P(X)$ . The pair  $(\theta, \eta)$  is compatible with  $(X, S, M)$  if

(i)  $\eta$  is a closure operator,  $S \subset \eta(Z) \subset Z$ , and  $\theta(\eta(Y)) = \theta(Y)$  for all  $Y \in Z$ ;

(ii)  $\{Y \mid \eta(Y) = Y, Y \subset X\} \cap \{Y \in Z \mid \theta(Y) = 0\} \subset S$ .

Example 2.8.  $(X, S, M)$  is the Schauder's f.p.s.,  $Z = P_b(X)$ ,  $\theta = \alpha_K$  and  $\eta(A) = \overline{co} A$ .

Example 2.9. Let  $(X, d)$  be a complete metric space,  $S = P_{cp}(X)$ ,  $M(Y) = \langle f: Y \rightarrow Y \mid f \text{ is a contractive mapping} \rangle$ ,  $Z = P_b(X)$ ,  $\theta = \alpha_K$  and  $\eta(A) = \bar{A}$ .

Definition 2.4. Let  $X$  be a nonempty set,  $Z \subset P(X)$ ,  $\theta: Z \rightarrow R_+$  and  $\rho: R_+ \rightarrow R_+$  a comparison function. a mapping  $f: Y \rightarrow Y$  ( $Y \subset X$ ) is a  $(\theta, \rho)$ -contraction if

- (i)  $A \in P(Y) \cap Z$  implies  $f(A) \in Z$ ,
- (ii)  $\theta(f(A)) \leq \rho(\theta(A))$ , for all  $A \in P(Y) \cap Z \cap I(f)$ .

Definition 2.5. A mapping  $f: Y \rightarrow Y$  is  $\theta$ -condensing if

- (i)  $A \in P(Y) \cap Z$  implies  $f(A) \in Z$ ,
- (ii)  $\theta(f(A)) < \theta(A)$ , for all  $A \in P(Y) \cap Z \cap I(f)$  such that  $\theta(A) \neq 0$ .

Remark 2.2. For some examples of  $(\theta, \rho)$ -contraction and  $\theta$ -condensing mappings see [46], [48], [2], [6], [16], [18], [19].

3. INVARIANT SUBSETS AND FIXED POINTS. Our list of open problems begins with

Problem 1. Let  $(X, S, M)$  be a f.p.s.  $Y \subset X$  and  $f: Y \rightarrow Y$ . The problem is to investigate the existence of an invariant subset of  $f$ , which is in  $S$ .

In this connection we have the following preliminary results

Theorem 3.1. Let  $(X, S, M)$  be a fixed point structures and  $(\theta, \eta)$  ( $\theta: Z \rightarrow R_+$ ,  $\eta: P(X) \rightarrow P(X)$ ) a compatible pair with  $(X, S, M)$ . Let  $Y \in \eta(Z)$  and  $f \in M(Y)$ . We suppose that

- (i)  $\theta|_{\eta(Z)}$  has the intersection property,
- (ii)  $f$  is a  $(\theta, \varphi)$ -contraction.

Then

- (a)  $I(f) \cap S \neq \emptyset$ ,
- (b)  $F_f \neq \emptyset$ ,
- (c) If  $F_f \in Z$ , then  $\theta(F_f) = 0$ .

Proof. (a) + (b). From the definition of a  $(\theta, \varphi)$ -contraction,  $Y \in \eta(Z)$  implies  $f(Y) \in S$ . Let  $Y_1 = \eta(f(Y))$ , ...,  $Y_n = \eta(f(Y_{n-1}))$ ,  $n \in N$ . It is clear that  $Y_{n+1} \subset Y_n$ ,  $Y_n \in F_\eta$  and  $Y_n \in I(f)$ . Let  $Y_\infty = \bigcap_{n \in N} Y_n$ . We have

$$\theta(Y_n) \leq \varphi(\theta(Y_{n-1})) \leq \dots \leq \varphi^n(\theta(Y)) \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Since  $\theta: \eta(Z) \rightarrow R_+$  is a mapping with the intersection property, we have that  $Y_\infty \neq \emptyset$ ,  $Y_\infty \in \eta(Z)$  and  $\theta(Y_\infty) = 0$ . These imply that  $Y_\infty \in I(f) \cap S$ . We have also that  $f|_{Y_\infty} \in M(Y_\infty)$ . Since  $(X, S, M)$  is a f.p.s. we have  $F_f \neq \emptyset$ .

- (c). From  $f(F_f) = F_f$ , we have  $\theta(F_f) = 0$ .

Theorem 3.2. Let  $(X, S, M)$  be a f.p.s. and  $(\theta, \eta)$  ( $\theta: Z \rightarrow R_+$ ) a compatible pair with  $(X, S, M)$ . Let  $Y \in \eta(Z)$  and  $f \in M(Y)$ . We suppose that

- (i)  $A \in Z$ ,  $x \in X$  imply  $A \cup \{x\} \in Z$  and  $\theta(A \cup \{x\}) = \theta(A)$ ,
- (ii)  $f$  is  $\theta$ -condensing mapping.

Then

- (a)  $I(f) \cap S \neq \emptyset$ ,
- (b)  $F_f \neq \emptyset$ ,

(c) If  $F_f \in Z$ , then  $\theta(F_f) = 0$ .

Proof. (a) + (b). We need the following results

Lemma 3.1 (see [45]). Let  $X$  be a nonempty set,  $\eta: PCX \rightarrow PCX$  a closure operator,  $Y \in F_\eta$  and  $f: Y \rightarrow Y$  a mapping. Let  $A \subset Y$  be a nonempty subset of  $Y$ . Then there exists  $A_0 \subset Y$ , such that

$$A_0 \supset A, A_0 \in F_\eta \cap I(f) \text{ and } \eta(f(A_0) \cup A_0) = A_0.$$

Now we prove (a) + (b). Let  $a \in Y$  and  $A = \langle a \rangle$ . Then by the above lemma, there exists  $A_0 \in F_\eta \cap I(f)$  such that  $\eta(f(A_0) \cup A_0) = A_0$ . But  $\theta$  is  $\theta$ -condensing. Thus we have  $\theta(\eta(f(A_0) \cup A_0) \cup \langle a \rangle) = \theta(f(A_0) \cup \langle a \rangle) = \theta(f(A_0)) = \theta(A_0)$ . This implies  $\theta(A_0) = 0$ . Thus we have that  $A_0 \in I(f) \cap S$  and  $f|_{A_0} \in M(A_0)$ . Since  $(X, S, M)$  is a f.p.s., we have  $F_f \neq \emptyset$ .

(c). From  $f(F_f) = F_f$ , we have  $\theta(F_f) = 0$ .

Remark 3.1. For more details about these results see [46] and [48].

Remark 3.2. For the Problem 1 in the case of the trivial f.p.s. see [47]. See also [48].

Remark 3.3. For the Problem 1 in the case of the Schauder's f.p.s. see [2], [4], [24].

4. COINCIDENCE THEORY. The following statement is a longstanding conjecture in the fixed point theory

Schauder's conjecture (see [10], [17], [27], [34], [49], [52]). Let  $Y$  be a closed, bounded convex set in a Banach space and

$f: Y \rightarrow Y$  continuous mapping. Assume there exists an integer  $n_0 \geq 1$  such that  $f^{n_0}$  is compact. Then  $f$  has a fixed point.

More general we have

Horn's conjecture (see [27], [52], [49]). Let  $Y$  be a compact convex subset of  $X$  and let  $f, g: Y \rightarrow X$  be commuting continuous mappings. Then  $C(f, g) \neq \emptyset$ .

These give rise to

**Problem 2.** Which are the f.p.s.,  $(X, S, M)$  with the following property

$Y \in S, f, g \in M(Y), f \circ g = g \circ f$  imply that  $C(f, g) \neq \emptyset$ ?

**Problem 3.** Let  $(X, S, M)$  be a fixed point structures,  $Y \in S, f, g \in M(Y)$ . We suppose that  $f \circ g = g \circ f$ . Establish conditions on  $f$  and  $g$  which imply that  $C(f, g) \neq \emptyset$ .

The following results are related to these problems.

**Theorem 4.1** (Horn [27]). If  $f, g \in C([a, b], [a, b])$  and  $f \circ g = g \circ f$  then  $C(f, g) \neq \emptyset$ .

This theorem implies that the Brouwer's f.p.s. on  $R$  is a solution of the Problem 2.

**Theorem 4.2** (Rus [49]). The following statements are equivalent:

(i) (Horn's conjecture) Let  $X$  be a Banach space,  $Y \subset X$  a compact convex subset of  $X$  and  $f, g: Y \rightarrow Y$  be commuting continuous

mappings. Then this pair of mappings has at least a coincidence point.

(ii) (Rus' conjecture) Let  $X$  be a Banach space,  $Y \subset X$  a bounded closed convex subset of  $X$  and  $f, g: Y \rightarrow Y$  be commuting continuous mappings. We suppose that

$$\alpha_K(f(A) \cup g(A)) < \alpha_K(A), \quad \forall A \in I(f) \cap I(g), \text{ such that } \alpha_K(A) \neq 0.$$

Then  $C(f, g) \neq \emptyset$ .

Proof. It is clear that (ii)  $\Rightarrow$  (i). Thus we prove that (i)  $\Rightarrow$  (ii). The proof follows from

Lemma 4.1. Let  $X$  be a nonempty set,  $\eta: P(X) \rightarrow P(X)$  a closure operator,  $Y \in F_\eta$  and  $f, g: Y \rightarrow Y$  such that  $f \circ g = g \circ f$ . Let  $A_1 \subset Y$ ,  $A_1 \neq \emptyset$ . Then there exists  $A_0 \subset Y$  such that  $A_0 \supset A_1$ ,  $A_0 \in F_\eta \cap I(f) \cap I(g)$  and  $\eta(f(A_0) \cup g(A_0) \cup A_1) = A_0$ .

Indeed, let  $A_1 = F_f$ . From a fixed point theorem for  $\alpha_K$ -condensing mappings (see [41]) we have  $F_f \neq \emptyset$ . From Lemma 4.1, (with  $\eta = \text{co}$ ), there exists  $A_0$  such that  $A_0 \supset A_1$ ,  $A_0 \in F_{\text{co}} \cap I(f) \cap I(g)$  and  $\eta(f(A_0) \cup g(A_0) \cup A_1) = A_0$ . From these we have  $\alpha_K(A_0) = 0$ . Thus  $A_0 \in F_{\text{co}} \cap Z_{\alpha_K}$ . Now the pair  $f, g: A_0 \rightarrow A_0$  satisfies the conditions from Horn's conjecture.

Remark 4.1. For other results in connection with the Problem 2 and Problem 3, see [26], [49], [52] and [55].

5. COMMON FIXED POINTS. There exist a lot of results on common fixed point theory (see [20], [30], [37]-[39], [53], [54], [56] and the papers cited therein). Some new problems, in this theory, can be formulated in the terms of the f.p.s. We start with the following problem:

**Problem 4.** Which are the f.p.s.  $(X, S, M)$ , with the following property

$$(c) Y \in S, f, g \in M(Y), f \circ g = g \circ f \Rightarrow F_f \cap F_g \neq \emptyset ?$$

By definition a f.p.s. which satisfies the condition (c) is called f.p.s. with the common fixed point property (briefly, c.f.p.p.).

Now, some examples of f.p.s. with c.f.p.p.

**Example 5.1 (Huneke).**  $X = \mathbb{R}$ ,  $S = \langle \text{co}(a, b) \mid a, b \in \mathbb{R} \rangle$  and  $M(Y) = \langle f \in C(Y, Y) \mid f|_J \text{ is full continuous for all subinterval } J \subset Y \rangle$ .

**Example 5.2.** Let  $X$  be a locally convex topological space,  $S = P_{cp, cv}^{(X)}$  and  $M(Y) = \langle f \in C(Y, Y) \mid f \text{ is an affin mapping} \rangle$ .

**Example 5.3 (Huneke).** The Brouwer f.p.s. on  $\mathbb{R}$  is not a f.p.s. with c.f.p.p.

**Example 5.4 (Tarski).**  $(X, \mathcal{D})$  is a complete lattice,  $S = \langle Y \subset X \mid (Y, \mathcal{D}) \text{ is a complete lattice} \rangle$  and  $M(Y) = \langle f: Y \rightarrow Y \mid f \text{ is an increasing mapping} \rangle$ .

More general, we have

**Lemma 5.1.** Let  $(X, S, M)$  be a f.p.s. If  $Y \in S$ ,  $f \in M(Y)$  imply  $\text{card } F_f = 1$ , then  $(X, S, M)$  is with the c.f.p.p.

**Lemma 5.2.** Let  $(X, S, M)$  be a fixed point structure. If  $Y \in S$ ,  $f \in M(Y)$  imply  $F_f \in S$ , then  $(X, S, M)$  is with the c.f.p.p.

Proof. Let  $Y \in S$  and  $f, g \in M(Y)$  such that  $f \circ g = g \circ f$ .  
Then  $F_f \in I(g)$ . This implies  $F_f \cap F_g \neq \emptyset$ .  
Now we consider

**Problem 5.** Let  $(X, S, M)$  be a f.p.s.,  $(\theta, \eta)$  ( $\theta: Z \rightarrow R_+$ ,  
 $S \subset Z \subset P(X)$ ) a compatible pair with  $(X, S, M)$ ,  $Y \in \eta(Z)$  and  
 $f, g \in M(Y)$ . We suppose that

$$(i) \quad f \circ g = g \circ f,$$

$$(ii) \quad (X, S, M) \text{ is a f.p.s. with c.f.p.p.}$$

Establish conditions on  $f$  and  $g$  which imply that  $F_f \cap F_g \neq \emptyset$ .

In what follow we present some partial results for the  
Problem 5. We have

**Theorem 5.1.** Let  $(X, S, M)$  be a f.p.s. with the c.f.p.p., and  
 $(\theta, \eta)$  ( $\theta: Z \rightarrow R_+$ ) a compatible pair with  $(X, S, M)$ . Let  $Y \in \eta(Z)$  and  
 $f, g \in M(Y)$ . We suppose that

$$(i) \quad \theta|_{\eta(Z)} \text{ has the intersection property,}$$

$$(ii) \quad f \circ g = g \circ f,$$

$$(iii) \quad \text{there exists a comparison function } \varphi: R_+ \rightarrow R_+ \text{ such that}$$

$$\theta(f(A) \cup g(A)) \leq \varphi(\theta(A)), \text{ for all } A \in I(f) \cap I(g) \cap Z.$$

Then

$$F_f \cap F_g \neq \emptyset.$$

Proof. First we remark that  $f, g: Y \rightarrow Y$  and  $f(Y) \cup g(Y) \subset AcY$   
imply that  $A \in I(f) \cap I(g)$ . Now let  $Y_1 := \eta(f(Y) \cup g(Y)), \dots,$   
 $Y_{n+1} := \eta(f(Y_n) \cup g(Y_n)), n \in N$ . It is clear that  $Y_n \in I(f) \cap I(g)$ .  
On the other hand we have

$$\begin{aligned} \theta(Y_{n+1}) &= \theta(\eta(f(Y_n) \cup g(Y_n))) = \theta(f(Y_n) \cup g(Y_n)) \leq \\ &\leq \rho(\theta(Y_n)) \leq \dots \leq \rho^{n+1}(\theta(Y)) \rightarrow 0 \text{ as } n \rightarrow \infty. \end{aligned}$$

From (i) it follows that  $Y_\infty := \bigcap_{n \in \mathbb{N}} Y_n \neq \emptyset$  and  $\theta(Y_\infty) = 0$ . We also have  $\eta(Y_\infty) = Y_\infty$  and  $Y_\infty \in I(f) \cap I(g)$ . These imply  $Y_\infty \in S$ . But  $(X, S, M)$  is a f.p.s. with c.f.p.p. Hence,  $F_f \cap F_g \neq \emptyset$ .

**Remark 5.1.** A pair,  $f, g$ , which satisfies the condition (iii) in the Theorem 5.1 is called  $(\theta, \rho)$ -contraction pair.

From the Theorem 5.1 we have

**Theorem 5.2.** Let  $X$  be a strictly convex Banach space,  $Y \in P_{b,cl,cv}(X)$  and  $f, g: Y \rightarrow Y$  two nonexpansive mappings. We suppose that

- (i)  $f \circ g = g \circ f$ ,
- (ii) the pair  $(f, g)$  is an  $(\alpha_K, \rho)$ -contraction pair.

Then

$$F_f \cap F_g \neq \emptyset.$$

**Proof.** Let  $S := P_{cp,cv}(X)$  and  $M(Y) := \{f: Y \rightarrow Y \mid f \text{ is a nonexpansive mapping}\}$ . Then  $(X, S, M)$  is a f.p.s. with c.f.p.p. and the pair  $(\alpha_K, \overline{co})$  is a compatible pair with  $(X, S, M)$ . The proof follows from the Theorem 5.1.

**Theorem 5.3.** Let  $(X, S, M)$  be a f.p.s. with the c.f.p.p. and  $(\theta, \eta)$   $(\theta: Z \rightarrow \mathbb{R}_+)$  a compatible pair with  $(X, S, M)$ . Let  $Y \in \eta(Z)$  and  $f, g \in M(Y)$ . We suppose that

- (i)  $x \in Y, A \in Z$  imply  $A \cup \{x\} \in Z$  and  $\theta(A \cup \{x\}) = \theta(A)$ ,
- (ii)  $f \circ g = g \circ f$ ,
- (iii)  $\theta(f(A) \cup g(A)) < \theta(A)$ , for all  $A \in I(f) \cap I(g) \cap Z$ .

such that  $\theta(A) \neq 0$ .

Then

$$F_f \cap F_g \neq \emptyset.$$

Proof. The proof follows from Lemma 4.1. See the proof of the Theorem 4.2.

Remark 5.2. A pair,  $f, g$ , which satisfies the condition (iii) in the Theorem 5.3, is called  $\theta$ -condensing pair.

Remark 5.3. For other results in connection with the Problem 4 and Problem 5 see [16], [20], [24], [30], [37], [39], [47], [54] and [56].

6. INVOLUTIONS AND FIXED POINTS. Let  $X$  be a nonempty set. A mapping  $f: X \rightarrow X$  is an involution if there exists  $n \in \mathbb{N}$ ,  $n \geq 1$ , such that  $f^n = 1_X$ . Very little is known about the fixed points of the involutions (see [22], [37], [38], [39], ...). In this connection the following problem may be of interest.

Problem 6. Let  $(X, S, M)$  be a f.p.s. and  $f \in M(X)$ , an involution of order  $n$ . When does there exist  $Y \subset X$  such that  $Y \cup f(Y) \cup \dots \cup f^{n-1}(Y) \in S$ ?

For example, let  $(\mathbb{R}, S, M)$  be the Brouwer f.p.s. on  $\mathbb{R}$ , i.e.  $S = \{co(a, b) \mid a, b \in \mathbb{R}\}$ , and  $M(Y) = C(Y, Y)$ . In this case, if  $Y = co(x, f(x))$ ,  $x \in \mathbb{R}$ , then  $Y \cup f(Y) \cup \dots \cup f^{n-1}(Y) \in S$ . On the other hand  $Y \cup f(Y) \cup \dots \cup f^{n-1}(Y) \in I(f)$ . Thus we have the following results: Every continuous involution,  $f: \mathbb{R} \rightarrow \mathbb{R}$ , has at least a fixed point.

7. RETRACTIBLE MAPPINGS. Let  $X$  be a nonempty set and  $Y \subset X$ . A mapping  $\rho: X \rightarrow Y$  is called a retraction of  $X$  onto  $Y$  if  $\rho|_Y = 1_Y$ . A mapping  $f: Y \rightarrow X$  is retractible by means of a retraction  $\rho: X \rightarrow Y$  if  $F_f = F_{\rho \circ f}$ . Here are some examples.

Example 7.1 (Poincaré, Bohl, Leray-Schauder). Let  $X$  be a Banach space and  $Y = \bar{B}(0; R) \subset X$ . If  $f: \bar{B}(0; R) \rightarrow X$  is such that  $\|x\| = R$ ,  $f(x) = \lambda x$  implies  $\lambda \leq 1$ , then  $f$  is retractible onto  $\bar{B}(0; R)$  with respect to the radial retraction

$$\rho: X \rightarrow \bar{B}(0; R), x \rightarrow \begin{cases} x & \text{if } x \in \bar{B}(0; R) \\ \frac{R}{\|x\|} x & \text{if } x \in X \setminus \bar{B}(0; R). \end{cases}$$

Example 7.2 (Altman). Let  $X$  be a Banach and  $f: X \rightarrow X$  a quasibounded mapping with  $|f| < 1$ . Then there exists  $R > 0$  such that  $f|_{\bar{B}(0; R)}: \bar{B}(0; R) \rightarrow X$  is retractible with respect to the radial retraction,  $\rho: X \rightarrow \bar{B}(0; R)$ .

Problem 7. Let  $(X, S, M)$  be a f.p.s. and  $(\theta, \eta)$  ( $\theta: Z \rightarrow \mathbb{R}_+$ ) a compatible pair with  $(X, S, M)$ . Let  $Y \in \eta(Z)$  and  $f: Y \rightarrow X$  a mapping. When does there exist a retraction  $\rho: X \rightarrow Y$  such that  $\rho \circ f \in M(Y)$ ?

In connection with this problem we have (see also [12], [31], [43], [46], [48], ...):

Theorem 7.1. Let  $(X, S, M)$  be a f.p.s. and  $(\theta, \eta)$  a compatible pair with  $(X, S, M)$ . Let  $Y \in \eta(Z)$ ,  $f: Y \rightarrow X$  a mapping and  $\rho: X \rightarrow Y$  a retraction. We suppose that

- (i)  $\theta|_{\eta(Z)}$  is a mapping with the intersection property,

- (ii)  $f$  is a strong  $(\theta, \rho)$ -contraction,
  - (iii)  $f$  is retractible onto  $Y$  by  $\rho$  and  $\rho \circ f \in \mathcal{M}(Y)$ ,
  - (iv)  $\rho$  is  $(\theta, \alpha)$ -Lipschitz,
  - (v) the function  $\alpha\rho$  is a comparison function.
- Then,  $F_f \neq \emptyset$  and if  $F_f \in \mathcal{Z}$ , then  $\theta(F_f) = 0$ .

Proof. From the conditions (ii), (iv) and (v), the mapping  $\rho \circ f: Y \rightarrow Y$  is a strong  $(\theta, \alpha\rho)$ -contraction, i.e.,

$$\theta(\rho \circ f)(A) \leq \alpha\rho(\theta(A)), \text{ for all } A \in \mathcal{P}(Y) \cap \mathcal{Z}.$$

By the Theorem 3.1 it follows that  $F_{\rho \circ f} \neq \emptyset$ . From the condition (iii) it follows that  $F_f \neq \emptyset$ . From  $f(F_f) = F_f$  and the condition (ii), we have  $\theta(F_f) = 0$ .

From the Theorem 7.1, we have

Theorem 7.2. Let  $X$  a Banach space and  $f: \overline{B}(0; R) \rightarrow X$  a continuous mapping. We suppose that

- (i)  $f$  is a strong  $(\alpha_K, \rho)$ -contraction,
  - (ii)  $f$  is a retractible by means of radial retraction.
- Then  $F_f \neq \emptyset$  and  $\alpha_K(F_f) = 0$ .

Proof. We take  $(X, S, M)$  the Schauder's f.p.s.,  $\theta = \alpha_K$  and  $\eta(Y) = \overline{\text{co}} Y$ .

Theorem 7.3. Let  $(X, S, M)$  be a fixed point structure and  $(\theta, \eta)$  a compatible pair with  $(X, S, M)$ . Let  $Y \in \eta(\mathcal{Z})$ ,  $f: Y \rightarrow X$  a mapping and  $\rho: X \rightarrow Y$  a retraction.

We suppose that

- (i)  $A \in \mathcal{Z}$ ,  $x \in Y$  imply  $A \cup \{x\} \in \mathcal{Z}$  and  $\theta(A \cup \{x\}) = \theta(A)$ ,
- (ii)  $f$  is a strong  $\theta$ -condensing mapping,
- (iii)  $f$  is retractible onto  $Y$  by  $\rho$  and  $\rho \circ f \in \mathcal{M}(Y)$ ,

(iv)  $\rho$  is a strong  $(\theta, 1)$ -contraction.

Then  $F_f \neq \emptyset$  and if  $F_f \in Z$ , then  $\theta(F_f) = 0$ .

Proof. From (ii) and (iv), the mapping  $\rho \circ f: Y \rightarrow Y$  is a strong  $\theta$ -condensing. By the Theorem 3.2,  $F_{\rho \circ f} \neq \emptyset$ . From the retractibility of  $f$  it follows that  $F_f \neq \emptyset$ . From  $f(F_f) = F_f$ , we have  $\theta(F_f) = 0$ .

Remark 7.1. For some results in connection with the above results see: [2], [6], [12], [16], [20], [43], [46], [48].

8. CATEGORICAL POINT OF VIEW. Let  $\mathcal{C}$  be a category and  $A, B \in \text{ob } \mathcal{C}$ . A morphism  $f \in (A, A)_{\mathcal{C}}$  has the f.p.p. if there exists a morphism  $g \in (B, A)_{\mathcal{C}}$  such that  $f \circ g = g \circ f$ . An object  $A \in \text{ob } \mathcal{C}$  has the f.p.p. if each  $f \in (A, A)_{\mathcal{C}}$  has the f.p.p. (see [31], [33], [34]).

Problem 8. Study the Problem 1-7 from a categorical point of view.

Remark 8.1. For some results in this direction see [1], [25], [26], [32], [33], [34] and [48].

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