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Unsymmetrical Distributions for Quality Control

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The use of the normal distribution for quality control is well known. This distribution cannot be used for evaluating non-symmetric collectives. For monitoring such processes can be used (among other) models as: folded normal distribution (one dimensional unsymmetry) and genaralized Rayleigh distribution (two dimensional unsymmetry). This paper concentrates the results of these distributions for use to process capability and quality control study.

Folded normal distribution, [1],[2],[5],[6],[8].Measurements are frequently recorded without their algebraic sign. As a consequence, the underlying distribution of measurements is replaced by a distribution of absolute measurements by folding the negative side of the distribution onto the positive side. When the underling distribution (of the algebraic values) is normal, the new distribution is called **folded normal distribution** and is obtained by folding the normal density

$$f(y)=\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{y-\mu}{\sigma}\right)^2},\ -\infty<y<\infty \tag{1}$$

at the Null-point. We get for $x=|y|$ the probability density function

$$f_1(x)=\frac{1}{\sigma\sqrt{2\pi}}\left[e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}+e^{-\frac{1}{2}\left(\frac{x+\mu}{\sigma}\right)^2}\right],\ x\geq 0 \tag{2}$$

and the distribution function

$$F_1(x)=\Phi\left(\frac{x-\mu}{\sigma}\right)+\Phi\left(\frac{x+\mu}{\sigma}\right)-1 \tag{3}$$

where Φ is the standardized form of the normal distribution function.

For $\mu=0$ and $\sigma=1$ we get

$f_1(x)=2\ \varphi(x),\ x\geq 0$

$F_1(x)=2\ (\Phi(x)-0.5)$

For the mean value and standard deviation we have

$$\mu_{f_1}=E[X]=\mu\left[\Phi\left(\frac{\mu}{\sigma}\right)-\Phi\left(-\frac{\mu}{\sigma}\right)\right]+2\frac{\sigma}{\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{\mu}{\sigma}\right)^2}$$

$$\sigma_{f_1}^{\;2} = E\left[(X - \mu_{f_1})^2\right] = \sigma^2 + \mu^2 - \mu_{f_1}^{\;2}$$

One can solve the equation

$$\sigma_{f_1}^{\;2} + \mu_{f_1}^{\;2} = \sigma^2 + \mu^2 \tag{4}$$

for estimates of μ and σ^2.
For μ_{f_1} and σ_{f_1} we use the unbiased statistics

$$\bar{x} = \frac{1}{n}\sum_{1}^{n} x_i \quad \text{and} \quad s^2 = \frac{1}{n-1}\sum_{1}^{n}(x_i - \bar{x})^2 \tag{5}$$

Figure 1 shows the connection between the folded normal distribution (b) and the underlying normal distribution (a). Table 1 gives statistical values for the folded normal distribution.

The c_{pk}-value will be calculated as follows

$$c_{pk} = \frac{1}{3}u_{1-p} \tag{6}$$

where p is part of the distribution outer tolerance and u the normal-quantile.

Example. Table 2 gives the measurements with and without algebraic sign. Figure 2 gives the evaluation of measurements with algebraic sign and Figure 3 the evaluation of absolute measurements.

Excentric Rayleigh distribution, [1],[2],[6],[9]. A two-dimensional unsymmetry is a vector v which is given by the components (x,y) or by (x_B, φ) where

$$x_B^{\;2} = x^2 + y^2 \quad \text{and} \quad \varphi = arctg(y/x)$$

To find the distribution of x_B we suppose first that x and y are normal distributed with μ=0 and $\sigma_x = \sigma_y = \sigma$.

The probability function of v is

$$\varphi_v = \varphi_x \cdot \varphi_y = \frac{1}{2\pi\sigma^2} e^{-\frac{1}{2}\frac{x^2+y^2}{\sigma^2}} = \frac{1}{2\pi\sigma^2} e^{-\frac{1}{2}\left(\frac{x_B}{\sigma}\right)^2}$$

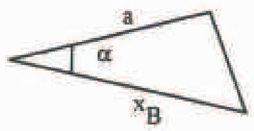
The probability density function of x_B is

$$f_2(x_B) = x_B \int_0^{2\pi} \varphi_v \, d\varphi = \frac{x_B}{\sigma^2} e^{-\frac{1}{2}\left(\frac{x_B}{\sigma}\right)^2} \tag{7}$$

and the distribution function

$$F_2(x_B) = 1 - e^{-\frac{1}{2}\left(\frac{x_B}{\sigma}\right)^2} \qquad (8)$$

For the case with excentricity



we have

$$\varphi_v = \frac{1}{2\pi\sigma^2} e^{-\frac{1}{2\sigma^2}(a^2 - x_B^2 - 2ax_B\cos\alpha)}$$

The density function for x_B is

$$f_2(x_B) = \frac{x_B}{2\pi\sigma^2} e^{-\frac{1}{2\sigma^2}(a^2 - x_B^2)} \int_0^{2\pi} e^{\frac{ax_B}{\sigma^2}\cos\alpha} d\alpha \qquad (9)$$

and the distribution function

$$F_2(x_B) = \frac{x_B}{2\pi\sigma^2} \int_0^{x_B} xe^{-\frac{1}{2\sigma^2}(a^2 - x_B^2)} \left(\int_0^{2\pi} e^{\frac{ax_B}{\sigma^2}\cos\alpha} d\alpha\right) dx \qquad (10)$$

For the mean value and the standard deviation we have

$$\mu_{f_2} = E[X] = \int_0^{\infty} x f_2(x) dx$$

$$\sigma_{f_2}^2 = E\left[(X - \mu_{f_2})^2\right] = a^2 + 2\sigma^2 - \mu_{f_2}^2$$

One can solve the equation

$$\sigma_{f_2}^2 + \mu_{f_2}^2 = a^2 + \sigma^2 \qquad (11)$$

to estimate μ and σ^2.

For μ_{f_2} and σ_{f_2} are valid the same estimates as for the folded normal distribution (5). Figure 4 shows excentric Rayleigh distributions for some values of a/σ. Table 3 gives the statistical values of the excentric Rayleigh distribution.

c_{pk}-value will be calculated in the same manner as for the folded normal distribution (6).

Example. Table 4 gives the (x,y)- and x_B-measurements. Figure 5 shows the corresponding distributions of the components. Figure 6 and 7 give the evaluations of x-and y-measurements as normal distributions and Figure 8 the evaluation of x_B-measurements as excentric Rayleigh-distribution.

Control Charts, [3],[4],[7]. To assert something about the capability of one process of such unsymmetric variable we need information about the stability of the process with regards to median and dispersion. For this we 'll define median- and range-charts for the folded normal distribution and for the excentric Rayleigh distribution. To define such charts we need the theory of order statistics. For $X_1, X_2, \ldots, X_n$ n independent variates, each with distribution function F(x), the density function of the k-order statistic $X_{(k)}$ is

$$\varphi_k(x) = \frac{n!}{(n-k)!(k-1)!} F^{k-1}(x)[1-F(x)]^{n-k} f(x) \tag{12}$$

and the distribution function

$$\Phi_k(x) = P(X_{(k)} <= x) \tag{13}$$

The joint density function of $X_{(r)}$ and $X_{(s)}$, $1 <= r <= s <= n$ is

$$\varphi_{r,s}(x,y) = C.F^{r-1}(x) f(x)[F(y)-F(x)]^{s-r-1} f(y)[1-F(y)]^{n-s} \tag{14}$$

with $C = \frac{n!}{(r-1)!(s-r-1)!(n-s)!}$

Median-Chart. For n=2k+1, $\tilde{x} = x_{(k+1)}$. The distribution function of $\tilde{x}$ is

$$F_{\tilde{x}}(x) = \Phi_{k+1}(x) = \sum_{i=k+1}^{n} \frac{n!}{i!(n-i)!} F^i(x)[1-F(x)]^{n-i} = 1 - \sum_{i=0}^{k} \frac{n!}{i!(n-i)!} F^i(x)[1-F(x)]^{n-i}$$

$$= 1 - H(q; n-k, k+1) = H(1-q; k+1, n-k) \tag{15}$$

q=1-F(x), p=1-q

F is the distribution function of the folded normal distribution or excentric Rayleigh distribution and H is the β-distribution function.

From the Equations

$$\alpha/2 = H(1-q; k+1, n-k)$$

$$1-\alpha/2 = H(1-q; k+1, n-k) \tag{16}$$

we get for α=0.05 and α=0.01 the limits of $\tilde{x}$-Chart.

For n=2k, $\tilde{x}=\frac{1}{2}(x_{(k)}+x_{(k+1)})$.

With v=$x_{(k)}+x_{(k+1)}$ we get w=$\tilde{x}$=v/2. The joint density function of $x_{(k)}$ and $x_{(k+1)}$ is

$$\varphi_{(x_{(k)},x_{(k+1)})}=\frac{(2k)!}{[(k-1)!]^2}F^{k-1}(x_{(k)})f(x_{(k)})[1-F(x_{(k+1)})]^{k-1}f(x_{(k+1)}) \quad (17)$$

and of $x_{(k)}$ and v

$$\varphi_{(x_{(k)},x_{(k+1)})}=\frac{(2k)!}{[(k-1)!]^2}F^{k-1}(x_{(k)})f(x_{(k)})[1-F(v-x_{(k)})]^{k-1}f(v-x_{(k)}) \quad (18)$$

With $x_{(k+1)}$ = v-$x_{(k)}$ >= $x_{(k)}$ the density function of v is

$$h_n(v)=\int_0^{v/2}\varphi(x_{(k)},v)dx_{(k)}$$

and the distribution function

$$H_n(v)=C\int_{y=2x}^{v}\int_{x=0}^{v/2}F^{k-1}(x)f(x)[1-F(y-x)]^{k-1}f(y-x)dxdy$$

where $C=\frac{(2k)!}{[(k-1)!]^2}$

With s=y-2x we get

$$H_n(v)=C\int_{s=0}^{v-2x}\int_{x=0}^{v/2}F^{k-1}(x)f(x)[1-F(s+x)]^{k-1}f(s+x)dxds=$$

$$-\frac{C}{k}\int_{x=0}^{v/2}F^{k-1}(x)f(x)[1-F(s+x)]^{k}\Big|_{s=0}^{s=v-2x}dx$$

To distinguish between v and w we'll write

$$H_n^v(z)=-\frac{C}{k}\int_{x=0}^{v/2}F^{k-1}(x)f(x)\{[1-F(z-x)]^k-[1-F(x)]^k\}dx \quad (19)$$

and for w=v/2

$$H_n^w(z)=P(w<=z)=P(v<=2z)=H_n^v(2z) \quad (20)$$

also

$$F_{\bar{x}}(z) = -\frac{C}{k}\int_0^{z} F^{k-1}(x) f(x)\left\{\left[1 - F(2z - x)\right]^k - \left[1 - F(x)\right]^k\right\} dx \qquad (21)$$

f and F are the density function and the distribution function of the folded normal distribution or the excentric Rayleigh distribution. The 0.005-, 0.025-,0.5-,0.975-,0.995-Quantiles give the standard limits and the standard meanline of the $\bar{x}$-chart. Tables 5-7 give the model and the coefficients for calculation of these quantiles. The multiplication with σ leads to the real limits.

Range-Chart. The distribution function of the range R=Max-Min=$x_{(n)}$-$x_{(1)}$ is

$$H_n(r) = \int_0^{\infty} f(x)\left[F(x+r) - F(x)\right]^{n-1} dx \qquad (22)$$

f und F are the density function and the distribution function of the folded normal distribution or excentric Rayleigh distribution. Tables 8 and 9 give the model and the coefficients for the calculation of the standard quantiles of the distribution function H. The multiplication with σ leads to the real limits. (For the complete list of the tables please contact the author).

Examples. The measurements from a folded normal distributed collective (μ=0.829, σ=1.113) are in the Table 10 and Fig. 9 shows the $\bar{x} - R$-chart. Table 11 gives the measurements from a excentric Rayleigh distributed collective (a=2.909, σ=3.232) and Fig. 10 the $\bar{x} - R$-chart.

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		u/Sigma 0.0	.5	1.0	1.5	2.0	2.5	3.0
%-Level up to	1	68.27	62.47	47.72	30.23	15.73	6.68	2.27
	2	95.45	92.70	84.00	69.12	50.00	30.85	15.87
	3	99.73	99.36	97.72	93.32	84.13	69.15	50.00
	4	99.99	99.98	99.86	99.38	97.72	93.32	84.13
Sigma*	5	~100	~100	99.99	99.98	99.87	99.38	97.72
	6			~100	~100	99.99	99.98	99.87
	7					~100	~100	99.99
	8							~100
Mean value		.7979	.8956	1.1666	1.5586	2.0170	2.5040	3.0000
50%		.6745	.7623	1.0507	1.5032	2.0001	2.4998	3.0000
95%		1.9604	2.1816	2.6465	3.1452	3.6452	4.1452	4.6452
99% Sigma*		2.5777	2.8437	3.3299	3.8296	4.3296	4.8296	5.3296
Mean value (%)		57.51	57.24	55.10	52.23	50.67	50.16	50.03

Table 1

		(Excentricity)/Sigma 0.0	.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0	4.5	5.0	5.5
%-Level up to	1	39.35	35.73	26.71	16.38	8.19	3.32	1.00	.28	.06	0.00	0.00	0.00
	2	86.47	83.00	73.09	57.63	39.65	23.21	11.33	4.53	1.47	.38	.08	.01
	3	98.89	98.22	95.63	89.62	78.56	62.38	43.25	25.56	12.59	5.00	1.66	.44
	4	99.96	99.93	99.71	98.92	96.59	91.00	80.35	64.24	44.97	26.00	13.29	5.40
Sigma*	5	~100	~100	99.99	99.96	99.78	99.00	96.93	91.62	81.26	65.31	45.99	27.56
	6			~100	~100	99.99	99.96	99.80	99.16	97.11	91.97	81.81	66.00
	7					~100	~100	99.99	99.96	99.82	99.20	97.23	92.21
	8							~100	~100	99.99	99.97	99.82	99.23
Mean value		1.2533	1.3304	1.5486	1.8749	2.2724	2.7112	3.1726	3.6463	4.1272	4.6126	5.1011	5.5917
50%		1.1774	1.2516	1.4755	1.8255	2.2458	2.6976	3.1652	3.6419	4.1244	4.6107	5.0997	5.5906
95%		2.4477	2.5923	2.9406	3.3687	3.8265	4.2979	4.7776	5.2623	5.7501	6.2401	6.7319	7.2240
99% Sigma*		3.0348	3.2069	3.5872	4.0302	4.4940	4.9704	5.4526	5.9388	6.4269	6.9182	7.4115	7.9058
Mean value (%)		54.41	54.32	53.54	52.19	51.12	50.56	50.30	50.18	50.11	50.08	50.06	50.04

Table 3

Values with sign		Absolute values	
Value	Frequency	Value	Frequency
9		0	xxx 3
8	x 1	1	xxxxxxx 7
7	xx 2	2	xxxxxxxxx 9
6	xxx 3	3	xxxxxxxxx 9
5	xxxxx 5	4	xxxxxxxx 8
4	xxxxxxxx 8	5	xxxxx 5
3	xxxxxxxxx 9	6	xxx 3
2	xxxxxxxx 8	7	xx 2
1	xxxxx 5	8	x 1
0	xxx 3	9	
-1	xx 2		
-2	x 1		
-3			
-4			

Table 2

x-value	y-value	x_p-value
-6.5	3.0	7.2
-7.5	.5	7.5
-8.5	2.5	8.9
-8.0	4.0	8.9
-4.0	3.5	5.3
-10.0	4.0	10.8
-5.5	1.5	5.7
-7.5	2.0	7.8
-6.0	2.0	6.3
-8.0	6.0	10.0
-6.0	4.0	7.2
-8.0	3.0	8.5
-7.0	3.0	7.6
-7.5	4.0	8.5
-8.0	2.0	8.3
-7.0	3.5	7.8
-9.0	1.0	9.1
-9.0	3.0	9.5
-6.5	1.0	6.6
-9.0	4.5	10.1

Table 4

Median-chart (folded normal distribution) n=2					
Modell : $(ax^b + c)^d$					
Quantile	a	b	c	d	
0,005	0,01136	4,35508	0,01451	0,63829	
0,025	0,03116	4,03217	0,02412	0,51118	
0,5	0,71814	2,23445	0,54863	0,50362	
0,975	1,85326	1,47231	2,45006	0,60016	
0,995	1,92374	1,34548	3,10017	0,64411	

Table 5

Median-chart (folded normal distribution)					
Modell : $ax^4 + bx^3 + cx^2 + dx + e$					
n	a	b	c	d	e
3	0,00000	0,05729	-0,02879	-0,01317	0,06037
4	-0,04350	0,27653	-0,33489	0,19107	0,08707
5	-0,04873	0,30505	-0,36816	0,18288	0,08986
6	-0,04489	0,25770	-0,22214	0,10915	0,14582
7	-0,04958	0,28347	0,26043	0,12364	0,14034
8	-0,04373	0,23437	-0,13831	0,06809	0,18464

Table 6: 0.005-Quantiles

Median-chart (excentric Rayleigh distribution)					
Modell : $ax^3 + bx^2 + cx + d$					
n	a	b	c	d	
2	-0,01688	0,24720	-0,22220	0,35020	
3	-0,02016	0,27620	-0,26970	0,36050	
4	-0,01953	0,25920	-0,15080	0,44070	
5	-0,02007	0,26310	-0,15030	0,43550	
6	-0,01972	0,25430	-0,09110	0,50440	
7	-0,01991	0,25540	-0,08900	0,50130	
8	-0,01958	0,24880	-0,04950	0,55290	

Table 7: 0.005-Quantiles

R-chart (folded normal distribution) $\mu/\sigma < 2$					
Modell : $ax^4 + bx^3 + cx^2 + dx + e$					
n	a	b	c	d	e
2	0,00004	-0,00137	0,00393	-0,00058	0,00447
3	0,00116	-0,02697	0,07193	-0,01334	0,06912
4	0,00657	-0,08733	0,20972	-0,04294	0,17914
5	0,01529	-0,16210	0,36532	-0,07730	0,29400
6	0,02393	-0,23154	0,50676	-0,10770	0,40020
7	0,03055	-0,28803	0,62373	-0,13117	0,49505
8	0,03435	-0,32789	0,71295	-0,14654	0,57904

Table 8: 0.005-Quantiles

R-chart (excentric Rayleigh distribution) $a/\sigma \leq 1$					
Modell : $ax^3 + bx^2 + cx + d$					
n	a	b	c	d	
2	-0.00048	0.00172	-0.00006	0.00564	
3	-0.00734	0.02673	-0.00105	0.08673	
4	-0.01877	0.06872	-0.00282	0.22159	
5	-0.03059	0.11205	-0.00477	0.35974	
6	-0.04133	0.15138	-0.00637	0.48609	
7	-0.05100	0.18636	-0.00781	0.59840	
8	-0.05955	0.21705	-0.00894	0.69780	

Table 9: 0.005-Quantiles

Measurements (folded normal distribution)													
i	1	2	3	4	5	6	7	8	9	10	11	12	13
1	0.454	0.474	0.442	1.15	1.344	2.628	0.12	1.155	1.952	1.8	0.382	2.211	2.378
2	0.145	0.151	0.141	0.377	0.441	0.725	0.038	0.379	0.611	0.574	0.121	0.655	0.692
3	0.322	0.152	0.783	1.566	2.608	1.533	0.998	1.277	1.643	0.604	2.352	0.178	0.926
4	0.28	0.655	2.136	0.186	1.078	2.273	1.113	0.82	1.166	0.929	0.064	2.039	0.749
5	1.863	1.832	2.619	2.525	1.116	1.755	1.427	2.853	0.398	2.589	1.807	0.986	1.279

Table 10

Measurements (excentric Rayleigh distribution)													
i	1	2	3	4	5	6	7	8	9	10			
1	1	5	2	3	6	2	3	2	6	3			
2	2	6	5	4	2	4	4	3	4	4			
3	4	3	7	6	5	6	1	7	8	7			
4	7	8	8	9.5	1	7	5	9	4	10			

Table 11

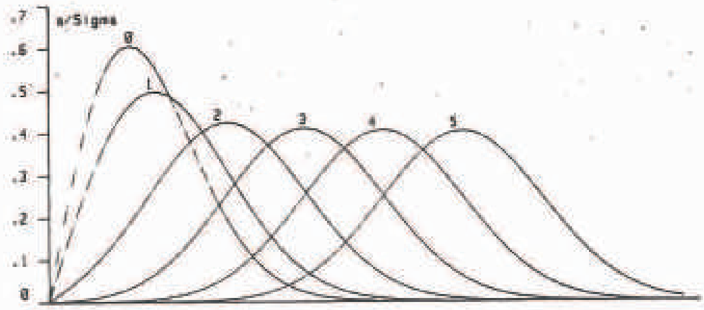


Figure 4. Exc. Rayleigh density for a/sigma-values

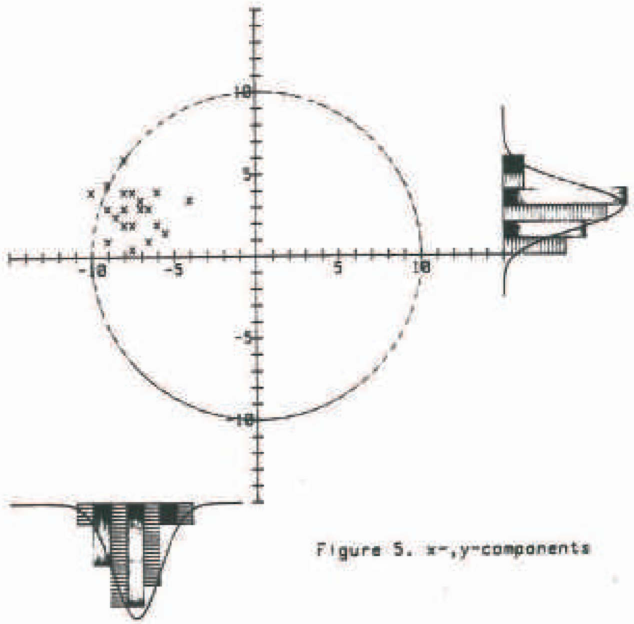
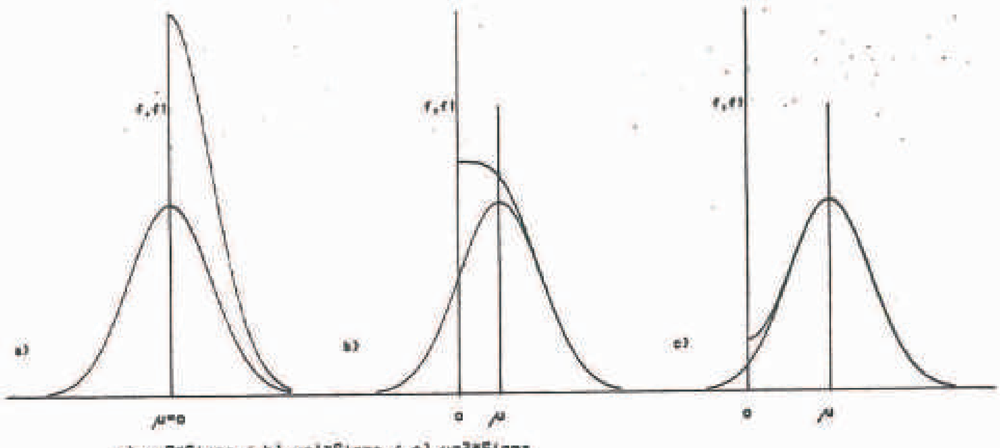
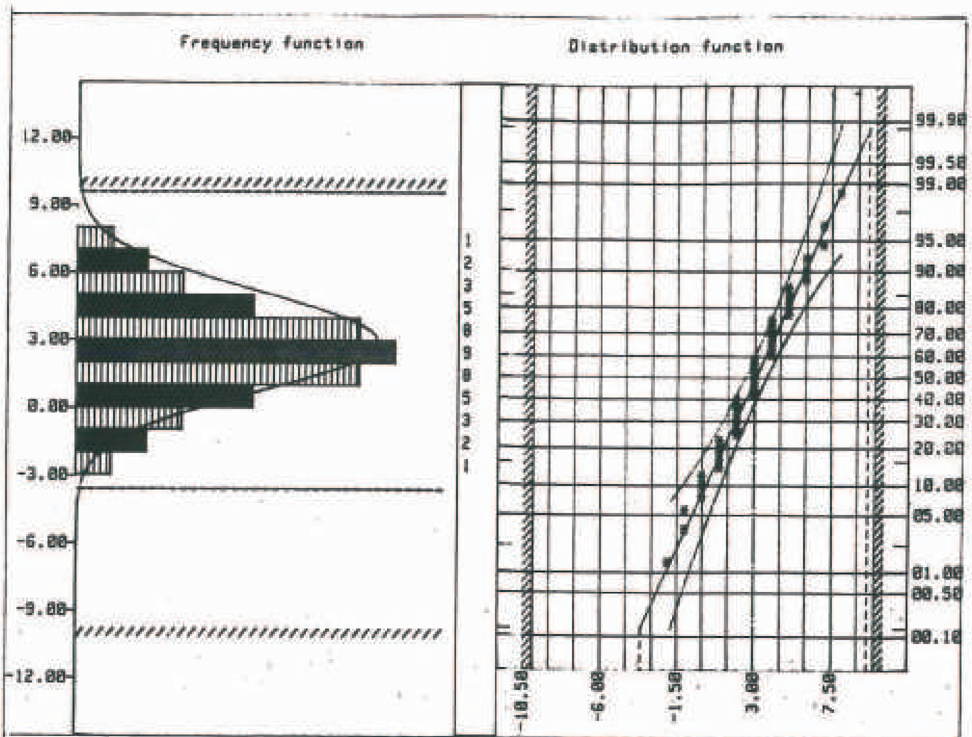


Figure 5. x-,y-components



a) μ=0*Sigma / b) μ=1*Sigma / c) μ=2*Sigma

Figure 1. Normal- &Folded Normal-distribution



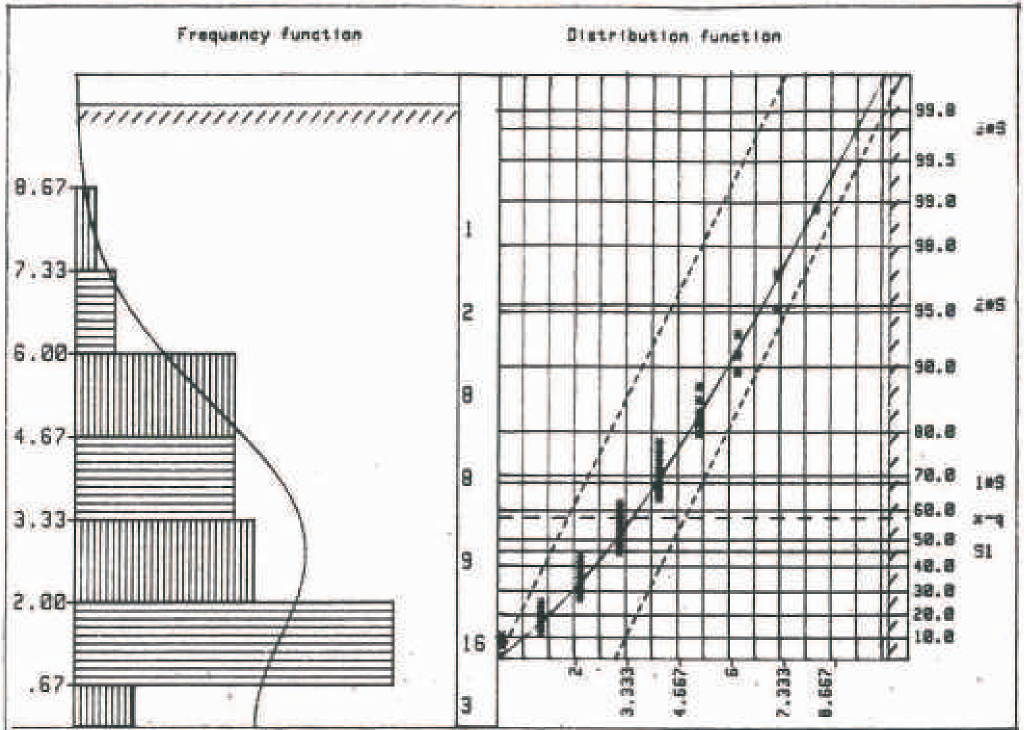
STATISTICAL CHARACTERISTIC VALUES

Titel: Measurements with sign
Normality accepted

Sample size:	47	Outlier:	0
Lower tolerance limit:	-10.000	Upper tolerance limit:	10.000
Meanvalue	3.000	Standard deviation	2.207
6*s dispersion	13.240	out of tolerance %	.076
Exceeding LTL %	0.000	Exceeding UTL %	.076
Cp - Factor	1.511	Cpk-Factor	1.057

95%-Confidence Meanvalue: 2.352 <= Mittelw<= 3.648
Stand.dev: 1.834 <= Sigma <= 2.772

Figure 2. Normal Evaluation



STATISTICAL CHARACTERISTIC VALUES

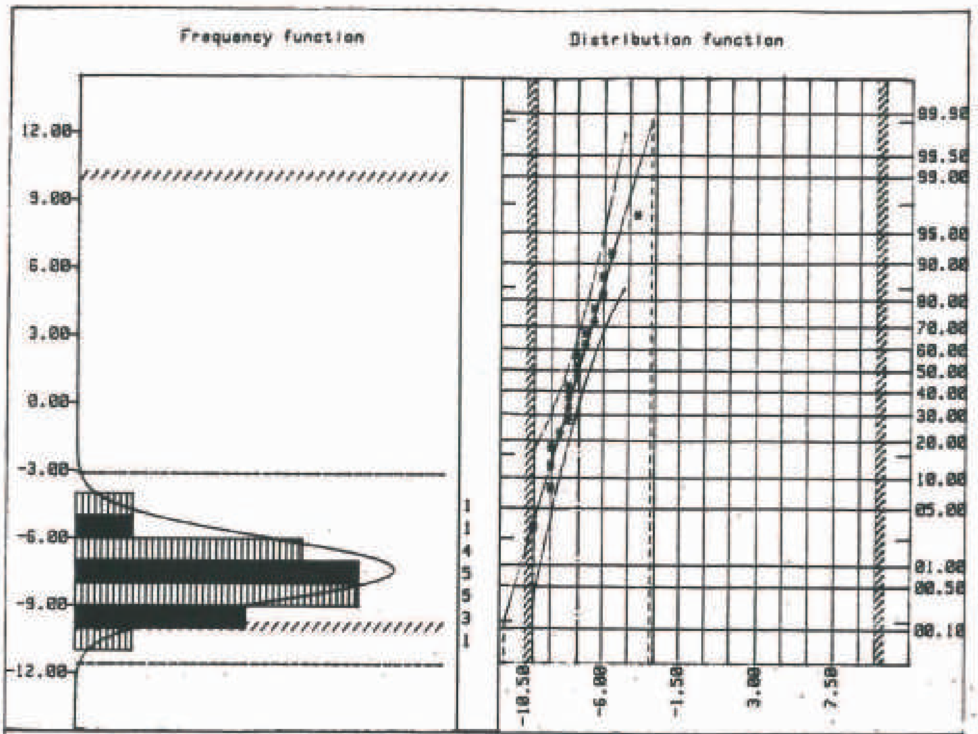
Folded normal distribution

Titel: Measurements without sign

Sample size 47 Upper tolerance 10.000

Folded distribution		Normal distribution
Meanvalue	3.170	2.987
Standard deviation	1.348	2.219
Exceending tolerance	.079 %	
99 % - Quantile	8.156	
99.73 % - Quantile	9.170	
Cpk	1.054	

Figure 3. Folded Normal Evaluation



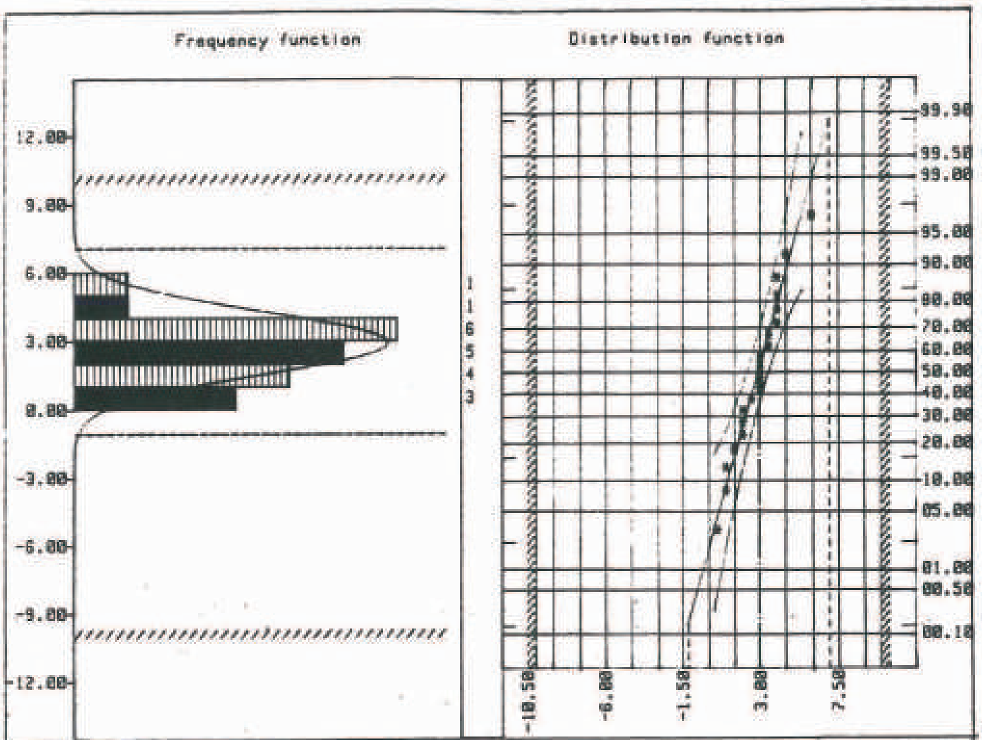
STATISTICAL CHARACTERISTIC VALUES

Titel: Coaxiality / x-Axis
Normality accepted

Sample size:	20	Outlier:	0
Lower tolerance limit:	-10.000	Upper tolerance limit:	10.000
Meanvalue	-7.425	Standard deviation	1.417
6*s dispersion	8.501	out of tolerance %	3.457
Exceeding LTL %	3.457	Exceeding UTL %	0.000
Cp - Factor	2.353	Cpk-Factor	.606

95%-Confidence Meanvalue: -8.088 <= Mittelw<= -6.762
Stand.dev: 1.077 <= Sigma <= 2.069

Figure 6. Evaluation of x-component

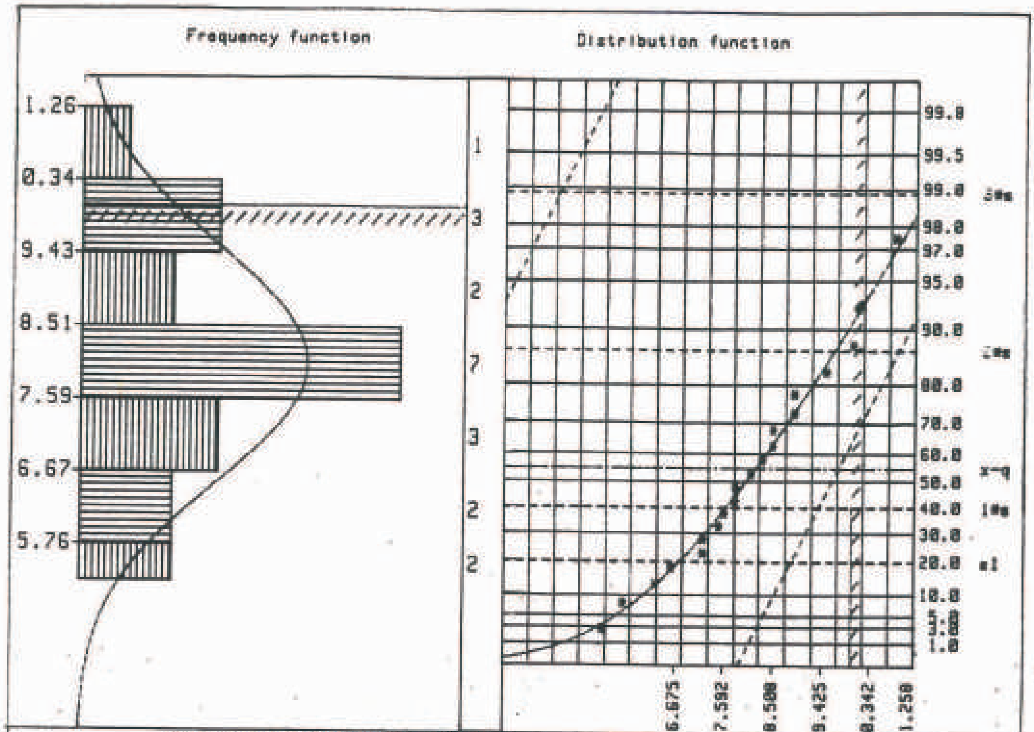


STATISTICAL CHARACTERISTIC VALUES

Titel: Coaxiality / y-Axis
Normality accepted

Sample size:	20	Outlier:	0
Lower tolerance limit:	-10.000	Upper tolerance limit:	10.000
Meanvalue	2.900	Standard deviation	1.363
6*s dispersion	8.170	out of tolerance %	0.000
Exceeding LTL %	0.000	Exceeding UTL %	0.000
Cp - Factor	2.446	Cpk-Factor	1.736
95%-Confidence	Meanvalue: 2.262 <= Mittelw<= 3.538		
	Stand.dev: 1.037 <= Sigma <= 1.991		

Figure 7. Evaluation of y-component



STATISTICAL CHARACTERISTIC VALUES

Excentric Rayleigh distribution

Titel: Coaxiality

Sample size 20 Upper tolerance 10.000

Rayleigh distribution			Normal distribution
Meanvalue/Excentricity	8.030		7.884
Standard deviation/Sigma	1.449		1.488
Exceeding tolerance	9.035	%	
99 % - Quantile	11.466		
99.73 % - Quantile	12.135		
Cpk	.446		

Figure 8. Excentric Rayleigh evaluation

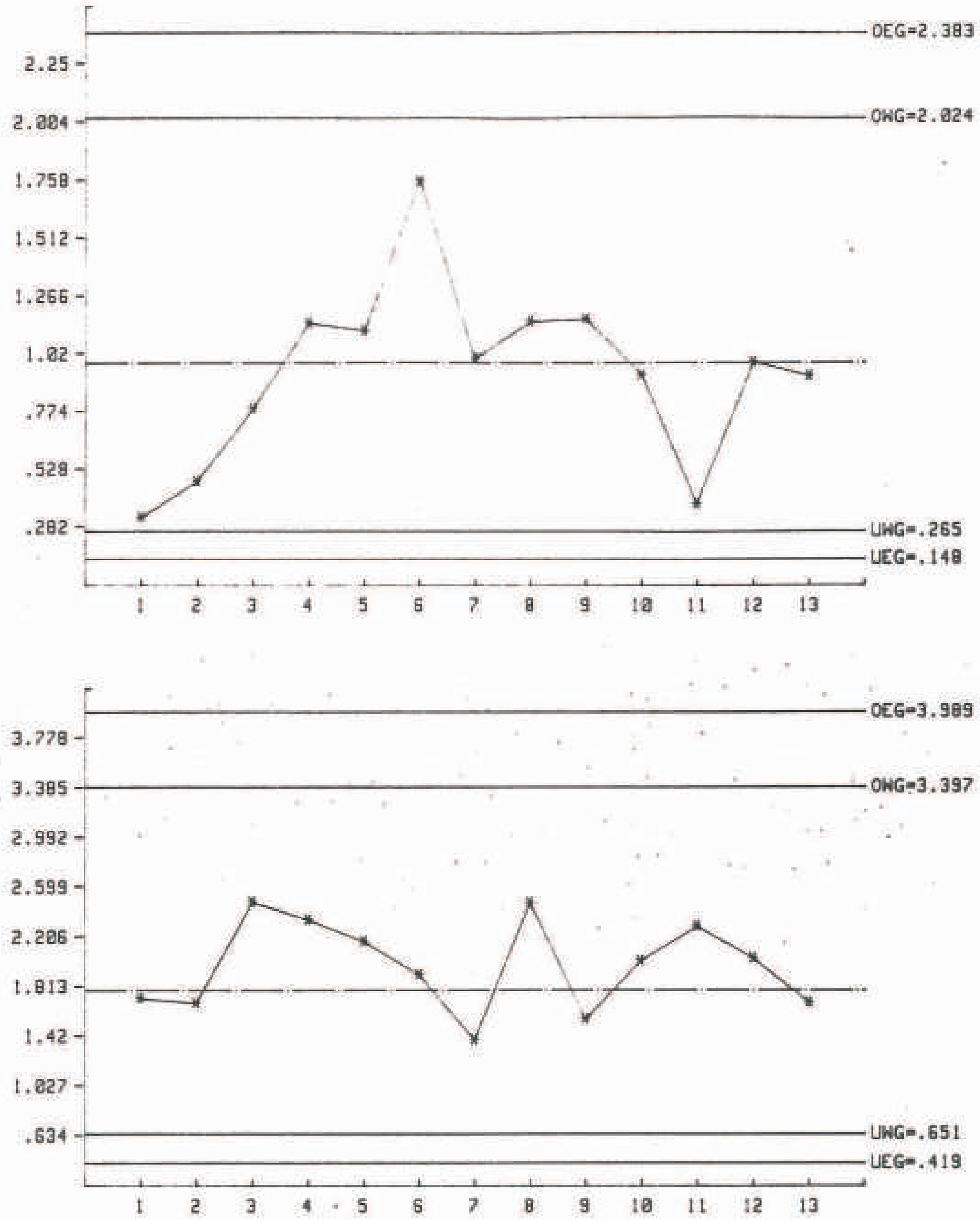


Figure 9. $\bar{x}$-R-Chart (Folded Normal distribution)

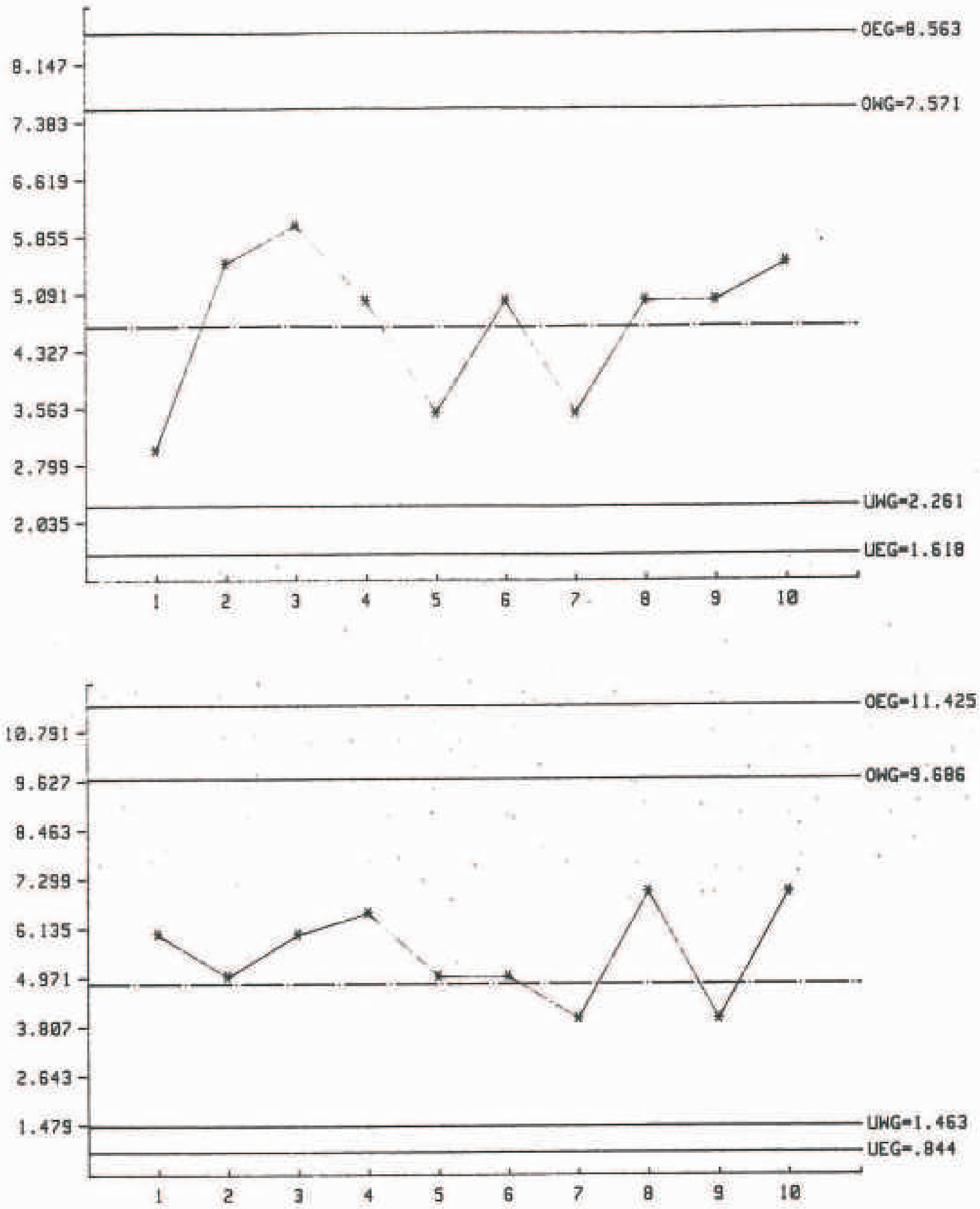


Figure 18. $\tilde{x}$-R-Chart (Exc. Rayleigh distribution)