LIBERTAS MATHEMATICA, VOL. XIV (1994)

# ON A THEOREM OF LIOUVILLE

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In connection with the well known Wilson Theorem, $(p-1)!+1\equiv 0 \pmod{p}$, Liouville [1] raised the problem of determining all primes $p$ and natural numbers $m$ satisfying the equation $(p-1)!+1=p^m$.

A generalization of Wilson theorem says that

$$(k-1)!(p-k)!+(-1)^{k+1}\equiv 0 \ (mod p)$$

for any prime $p$ and $1\leq k\leq p$. In connection with this generalization we shall prove

**Theorem.** The equation

$$(k-1)!(p-k)!+(-1)^{k+1}=p^m \tag{1}$$

has no solutions in primes $p>7$ and natural numbers $m$ and $k, 1\leq k\leq p$ .

Proof. Notice first that for even $k$, Eq.(1) cannot be satisfied with $p>5$. For, as $2<(p-1)/2$ and $p-1=2\cdot\frac{p-1}{2}\,|\,(k-1)!(p-k)!$, it follows that $p-1\,|\,p^m+1=p^m-1+2$, i.e. $p-1\,|\,2$, whence either $p=2$ or $p=3$.

Thus $k$ must be odd and solving (1) amounts to solving

$$(k-1)!(p-k)!=p^m-1 \tag{2}$$

Now, we distinguish two cases according to $p$ either of the form $3q+1$, or of the form $3q-1$.

a) Let $p=3q+1$. For $p>7$, therefore $p>10$, as $(p-1)/3$ and $(p-1)/2$ are natural numbers and $2<3<(p-1)/3<(p-1)/2$, we have

$$(p-1)^2=2\cdot 3\cdot\frac{p-1}{3}\cdot\frac{p-1}{2}\,\Big|\left(\frac{p-1}{2}\right)!\,\Big|\,(k-1)!(p-k)! \ ,$$

implying together with (2) that

$$(p-1)^2\,|\,p^m-1 \tag{3}$$

whence

$$p-1\,|\,p^{m-1}+p^{m-2}+\cdots+p+1=(p^{m-1}-1)+(p^{m-2}-1)+\cdots+(p-1)+m \ .$$

That is $p-1\,|\,m$, whence $m\geq p-1$ and

$$p^{m-1} > (p-1)^{p-1} > (p-1)! \geq (k-1)!(p-k)! \ . \tag{4}$$

Consequently Eq. (2) has no solution for $p > 7$.

b) Let $p = 3q+1$. If $p-k \neq k-1$, one of these two number is greater than $(p+1)/2$. As $(p+1)/2$ and $(p+1)/3$ are natural numbers and since for $p > 7$ we have $2 < 3 < (p+1)/3 < (p+1)/2$ and

$$(p+1)^2 = 2 \cdot 3 \cdot \frac{p+1}{3} \cdot \frac{p+1}{2} \mid \left(\frac{p+1}{2}\right)! \mid (k-1)!(p-k)! \ ,$$

on account of Eq. (2) we deduce that

$$(p+1)^2 \mid p^m - 1 \ . \tag{5}$$

Notice that $p+1 \mid p^m - 1 = (p+1-1)^m - 1$ implies that $m$ is even. Then (5) can be written as

$$(p+1)^2 \mid (p+1-1)^m - 1 = M(p+1)^2 - m(p+1) \tag{6}$$

For (6) to hold one should have $p+1 \mid m$, whence $m \geq p+1$, thus implying inequalities (4). Consequently, Eq. (2) has no solution for $p > 7$.

Finally, if $p-k = k-1$, that is $k = (p+1)/2$, Eq. (2) becomes

$$\left[\left(\frac{p-1}{2}\right)!\right]^2 = p^m - 1 \ , \tag{7}$$

For $p > 7$, since $p-1 \mid \left(\frac{p-1}{2}\right)!$, it follows from (7) that $(p-1)^2 \mid p^m - 1$, i.e., the relationship (2) we have already discussed. The proof is therefore complete.

Remark. For $2 \leq p \leq 7$ the solutions of Eq. (1) are given by

$$p = 2;\ 0!1 + 1 = 2,\ p = 3;\ 0!2! + 1 = 3 \ ,$$
$$p = 5;\ 0!4! + 1 = 5^2,\ 1!3! - 1 = 5,\ 2!2! + 1 = 5,\ p = 7;\ 2!4! + 1 = 7^2.$$

## Reference

[1] J. Liouville, Sur l'équation $1 \cdot 2 \cdot 3 \cdots (p-1) + 1 = p^m$, J. Math. Pures Appl. (2) 1 (1856), 351-352