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ON WEAKLY BOOLEAN GROUP RINGS

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M.I. Ursul in [1] has analyzed weakly Boolean rings in the topological sense. In this note we apply the algebraic aspect of it to group rings and obtain conditions under which a group ring is a weakly Boolean ring.

Definition 1. A group ring KG (or RG) of a group G over a field K (or over a ring R) is said to be weakly Boolean if for every $x \in RG$ (or KG) there exists a natural number $n(\alpha) > 1$ with $x^{n(\alpha)} = x$.

Lemma 2. Every finite field F_p of characteristic p, p a prime p>0 is weakly Boolean.

Proof. Since we are given F_p is a finite field we have for every $x \in F_p$ there exists a natural integer $n > 1$ with $x^n = x$. for otherwise it would contradict the fact F_p is finite. Hence F_p is Weakly Boolean.

Remark. From the above lemma we see the class of weakly Boolean rings is non empty.

Example 1. Let $Z_2 = (0,1)$ be a field of characteristic two and $G = \langle g | g^3 = 1 \rangle$ clearly $Z_2G = (0, 1, g, g^2, 1+g, 1+g^2, g+g^2, 1+g+g^2)$ is weakly Boolean.

Example 2. Let $G = \langle g | g^2 = 1 \rangle$ and $Z_2 = (0,1)$. Clearly Z_2G is not weakly Boolean.

Theorem 3. Let $Z_2 = (0,1)$ and $G = \langle g | g^{2n} = 1 \rangle$ then the group ring Z_2G is not weakly Boolean.

Proof. Take $1+g^n \in Z_2G$; Clearly $(1+g^n)^2 = 0$. Thus Z_2G is not weakly Boolean.

Theorem 4. A finite ring R is weakly Boolean if and only if R has no nilpotent elements.

Proof. If R has no nilpotent elements using the fact R is a finite ring we have for every $x \in R$ there exists an integer n such that $x^n = x$; thus R is weakly Boolean. Conversely given R is weakly Boolean we have for every $x \in R$ $x^n = x$ so R cannot contain nilpotent elements.

Theorem 5. Let $Z_2 = (0,1)$ and $G = \langle g \mid g^n = 1 \rangle$, n an odd number. Then Z_2G is weakly Boolean.

Proof. Clearly Z_2G has no non zero nilpotent elements as $(2,n) = 1$. Thus by theorem 4 Z_2G is weakly Boolean.

Theorem 6. Let G be a torsion free group and K any field. KG the group ring of G over K is not weakly Boolean.

Proof. Clearly $G = G.1 \subseteq KG$; since G is torsion free for every $g \in G$ we have $g^n \neq 1$ for any finite integer n. Hence KG is not weakly Boolean.

Theorem 7. Let $Z_2 = (0,1)$ and S_n be the symmetric group of degree n. Clearly Z_2S_n is not weakly Boolean.

Proof. Take $\alpha = 1 + \begin{pmatrix} 1 & 2 & 3 & \ldots & i & \ldots & j & \ldots & n \\ 1 & 2 & 3 & \ldots & j & \ldots & i & \ldots & n \end{pmatrix}$

where $1,2,3,\ldots,\hat{i},\ldots,\hat{j},\ldots,n$ where ^ denotes the element which is permuted. Clearly $\alpha^2 = 0$ Hence $Z_2 S_n$ is not weakly Boolean.

Theorem 8. Let $Z_p = (0, 1, \ldots, p-1)$ where p is a prime > 2. $Z_p S_n$ is not weakly Boolean. $(p<n$, $p/n)$.

Proof. Take $\alpha = 1 + \alpha_1 + \ldots + \alpha_{p-1}$ clearly S_n contains a subgroup of order p; where $\alpha_1, \ldots, \alpha_{p-1}$ elements are from S_n forming a subgroup of order p. Hence $\alpha^2=0$. Thus Z_p S_n is not weakly Boolean.

Problem. If in the above theorem (p, n) =1 or p> n and n/p but (p, n) $\neq$ 1; we are not able to conclude anything about Z_p S_n.

Remark 1. If we take S_n such that n is infinite and Z_2 = (0,1). Clearly Z_2S_n is not weakly Boolean.

Remark 2. Let Z_p = (0,1,...,p-1) and G be a cyclic group of order p. Then Z_pG is not weakly Boolean.

Proof. Take $\alpha = 1+g+\ldots+g^{p-1}$, clearly $\alpha^2 = 0$. Hence Z_pG cannot be weakly Boolean.

Problem. If in the above remark take G to be a cyclic group of order q with (p,q) = 1. Is Z_pG weakly Boolean?

Theorem 9. Let G be a infinite group in which every element is of finite order and Z_p be the field of integers modulo p, p a prime Z_pG is not weakly Boolean if G has an element of order p.

Proof. Let $g \in G$ with $g^p = 1$ then $\alpha = 1+g+\ldots+g^{p-1}$ is such that $\alpha^2 = 0$. Thus Z_pG is not weakly Boolean.

Reference

[1] Ursul, N.I., Connectivity in weakly Boolean Topological rings, IZV, Akad, Nauk, Moldor, SSR Ser. Fiz-Tekhn Mat. Nauk. 79, No.1 17-21, (1989).