LIBERTAS MATHEMATICA, VOL. XIV (1994)

REMARKS ON VOLTERRA INTEGRAL EQUATIONS OF THE SECOND KIND

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Theorems on the existence and uniqueness problems for Volterra integral equations of the second kind are classical results of analysis. In the present note we make comments about the elementary facts presented in the book of P. Linz [1] for the linear equation

$$f(t)=g(t)+\int_0^t k(t,s)f(s)ds, \tag{1}$$

as well as for a nonlinear generalization of this equation.

In [1] there are two theorems related to this equation. Theorem 3.1 which under the standard continuity conditions on k and g establishes by a contraction mapping argument the existence, uniqueness and the uniform convergence of the Picard iterations for equation (1). The second theorem of [1], Theorem 3.2, under less restrictive conditions on k, establishes the existence and uniqueness of a continuous solution of equation (1). In this case more powerfull method of continuation is used, the convergence of the Picard iteration is not asserted.

In the present note we formulate one theorem which covers both theorems mentioned above. We keep notations very close to those of [1] to make comparison of our results easy.

We denote by $C(J,R)$ the space of all real valued and continuous functions defined on the interval $J=[0,T], T>0$.

Define

$$f_n(t)=g(t)+\int_0^t k(t,s)f_{n-1}(s)ds \ , \ \ f_0(t)=g(t), \ \ n=1,2,\cdots,$$

$$\phi_n(t)=f_n(t)-f_{n-1}(t) \ , \ \ \phi_0(t)=g(t), \ \ n=1,2\cdots \ .$$

It is clear that $f_n(t)$ is the partial sum of the series

$$\sum_{i=0}^{\infty}\phi_i(t) \tag{2}$$

Our theorem reads

Theorem 1. Assume that:

(i) $g\in C(J,R)$,

(ii) for every $h\in C(J,R)$ and all $0\le p\le q\le t$ the integrals

$$\int_p^q k(t,s)h(s)ds, \quad \int_0^t k(t,s)h(s)ds \tag{3}$$

are continuous functions of $t\in J$,

(iii) $k(t,\cdot)$ is absolutely integrable (with respect to s) for all $t\in J$,

(iv) there exist a natural number N, and points $0=T_0<T_1<T_2\cdots<T_N=T$ and continuous functions $\omega_i\in C(J_i,R), \omega_i(t)>0, J_i=[T_{i-1}T_i]$ such that for all $i=1,2,\cdots,N$ and all $t\in J_i$,

$$\int_{T_{i-1}}^{t} |k(t,s)|\omega_i(s)ds\le\alpha\omega_i(t) \tag{4}$$

where $\alpha\in[0,1)$ is independent of t and i,

(v) for every $t\in[0,T]$

$$\lim_{\delta\to 0+}\int_t^{t+\delta} |k(t+\delta,s)|ds=0\ . \tag{5}$$

Then equation (1) has a unique solution f^* which is continuous in J. The Picard iterations converge to f^* uniformly in J_1.

Proof. The approach is a combination of the contraction and continuation methods, with properly chosen weighting which employs ω_i appearing in condition (iv). Consider first the interval $[0,T_1]$.

We have

$$\phi_n(t)=\int_0^t k(t,s)\phi_{n-1}(s)ds,\ \ f_n(t)=\sum_{i=0}^{n}\phi_i(t)$$

and by condition (4)

$$\begin{aligned}|\phi_n(t)| &\le \int_0^t |k(t,s)|\omega_1(s)\omega_1^{-1}(s)|\phi_{n-1}(s)|ds\\ &\le \alpha \max_{0\le s\le T_1}[\omega_1^{-1}(s)|\phi_{n-1}(s)|]\omega_1(t).\end{aligned}$$

Using this, by a simple induction we find

$$\max_{0 \le s \le T_1}[\omega_1^{-1}(s)\,|\phi_n(s)|] \le \alpha^n \max_{0 \le s \le T_1}[\omega_1^{-1}(s)\,|g(s)|] \ . \tag{6}$$

Since $\alpha < 1$ we see that series (2) is dominated by the convergent series

$$\max_{0 \le s \le T_1}[\omega_1^{-1}(s)\,|g(s)|] \cdot \max_{0 \le s \le T_1}\omega_1(s)\sum_{i=0}^{\infty}\alpha^i$$

and converges uniformly in $[0,T_1]$ to some f^* which is obviously a unique solution of equation (1). The rest of our reasoning remains quite similar to that of [1] what means that having established existence and uniqueness we proceed to the next interval J_2 where the weighting function ω_2 is employed. After repeating this process N times the proof of the Theorem is completed.

Let us make some observations:

When all ω_i are constant, for instance equal one, then we get the case described in P. Linz's book [1];

When there is an integrable function M such that

$$|k(t,s)| \le M(s), 0 \le s \le t \le T \tag{7}$$

then condition (iv) of the Theorem is fulfilled with $N = 1, \alpha = 0.5$ and

$$\omega_1(t) = exp(-2\int_0^t M(s)ds) \quad , \quad t \in J \ .$$

It is obvious that (7) holds if k is continuous, in this case one can take $M = max[\,|k(t,s)|\,, 0 \le s \le t \le T]$. Now the proof of the Theorem requires only one step and the convergence of the Picard iterations takes place in the whole interval J.

The answer to the question concerning the convergence of the Picard iterations in a more general case is given in the following

Theorem 2. If the conditions of Theorem 1 are fulfilled and condition (ii) holds also for k replaced by $|k|$, then the squence $\{f_n\}$ of the Picard iterations converges uniformly in J.

Proof. Consider the equation

$$z(t) = 1 + \lambda\int_0^t |k(t,s)|\,z(s)ds, \qquad t \in J, \tag{8}$$

with some $\lambda > 1$, such that $\lambda\alpha < 1$.

By Theorem 1 there is a contonuous and positive solution ω of equation (8).

We have

$$\int_0^t |k(t,s)|\omega(s)ds \leq \lambda^{-1}\omega(t), \qquad t \in J$$

and

$$|\phi_n(t)| \leq \int_0^t |k(t,s)|\omega(s)\omega^{-1}(s)|\phi_{n-1}(s)|ds$$
$$\leq \max_{0 \leq s \leq T}[\omega^{-1}(s)|\phi_{n-1}(s)|]\int_0^t |k(t,s)|\omega(s)ds$$
$$\leq \lambda^{-1} \max_{0 \leq s \leq T}[\omega^{-1}(s)|\phi_{n-1}(s)|]\omega(t) \ .$$

This inequality yields the relation

$$|\phi_n(t)| \leq \lambda^{-n} \max_{0 \leq s \leq T}[\omega^{-1}(s)|g(s)|]\omega(t) \quad , \quad t \in J \ .$$

Since $\lambda > 1$ we conclude that the series (2) converges uniformly in J.

Under the conditions of Theorem 1 the uniform convergence of the sequence $\{f_n\}$ can be again proved by the use of the combination of contraction and continuation methods. Indeed, by the proof of this theorem we know that the convergence takes place in the interval J_1.

Put

$$\rho_n(t) = f^*(t) - f_n(t) \quad , \quad t \in J \ .$$

Then for $t \in J_2$ we have

$$\rho_{n+1}(t) = \int_0^t k(t,s)\rho_n(s)ds$$

$$= \int_0^{T_1} k(t,s)\rho_n(s)ds + \int_{T_1}^t k(t,s)\rho_n(s)ds.$$

Hence using condition (iv) of Theorem 1 we get

$$\omega_2^{-1}(t)|\rho_{n+1}(t)| \leq \omega_2^{-1}(t)|\int_0^{T_1} k(t,s)\rho_n(s)ds|$$
$$+\alpha \max_{T_1 \leq s \leq T_2}[\omega_2^{-1}(s)|\rho_n(s)|] \ .$$

This inequality can be meant as the following one

$$v_{n+1} \leq a_n + \alpha \ v_n, \tag{6'}$$

with the corresponding meaning of v_{n+1}, a_n and v_n. Because of the uniform convergence of $\{f_n\}$ in $[0,T_1]$ the sequence $\{a_n\}$ converges to zero. This together with inequality (6') and condition $\alpha \in [0,1)$ yields the convergence of the sequence $\{v_n\}$ to zero. Because of the meaning of v_n we conclude the uniform convergence of $\{f_n\}$ in the interval J_2. After repeating this process N-1 times we conclude the uniform convergence of $\{f_n\}$ in the whole interval J.

Finally, observe that by the same way of reasoning we can generalize Theorem 3.13 and Theorem 4.8 of [1]. Our theorem for nonlinear equation

$$f(t) = g(t) + \int_0^t p(t,s)K(t,s,f(s))ds \ , \quad t \in J, \tag{9}$$

reads:

Theorem 3. Assume that:

(i) $g \in C(J,R)$,

(ii) K is real valued, continuous function defined for $0 \le s \le t \le T, u \in R$,

(iii) the Lipschitz condition

$$| K(t,s,u) - K(t,s,v) | \le L \,| u - v |$$

is satisfied for $0 \le s \le t \le T, u, v \in R$,

(iv) the function Lp satisfies conditions (ii) - (v) of Theorem 1 with $K(t,s,h(s))$ instead of $h(s)$. Then equation (10) has a unique continuous solution on J. The solution is the limit of the Picard iterations, with the convergence uniform on the interval J.

The generalization of the result for systems of equations is also obvious.

References

[1] P. Linz, *Analytical and Numerical Methods for Volterra Equations*, SIAM Studies in Applied Mathematics, SIAM, Philadelphia, 1985.