LIBERTAS MATHEMATICA, VOL. XIV (1994)

ON A SINGULAR PERTURBATION PROBLEM WITH MIXED CONDITION

Taoufiq Benkiran

INTRODUCTION

In the square $\Omega =]0,\pi,[\times]0,\pi[$, we consider the following Dirichlet problem for the second order elliptic equation.

$$(P_\varepsilon) \qquad \begin{aligned} -\varepsilon^2\frac{\partial^2}{\partial x^2}u_\varepsilon - \frac{\partial^2}{\partial y^2}u_\varepsilon + u_\varepsilon = f_\varepsilon \quad &\text{in } \Omega, \\ u_\varepsilon(0,y) = u_\varepsilon(\pi,y) = 0 \quad &\text{for } 0<y<\pi, \\ \frac{\partial}{\partial y}\ u_\varepsilon(x,0) = \frac{\partial}{\partial y}u_\varepsilon(x,\pi) = 0 \quad &\text{for } 0<x<\pi, \end{aligned}$$

where f_ε is in the space $L^2(\Omega)$.

We begin this paper by proving that as f_ε converges to f in $L^2(\Omega)$, the solution of the problem (P_ε) converges in $L^2(\Omega)$ to the solution of the following problem:

$$(P_0) \qquad \begin{aligned} -\frac{\partial^2}{\partial y^2}u + u = f \quad &\text{in } \Omega, \\ \frac{\partial}{\partial y}u(x,0) = \frac{\partial}{\partial y}u(x,\pi) = 0 \quad &\text{for } 0<x<\pi \ . \end{aligned}$$

On the other hand, if $f_\varepsilon = f$ is sufficiently smooth, we construct the asymptotic expansion of the solution u_ε of the problem (P_ε).

1. Convergence of the solution of the problem (P_ε), if f_ε converges in $L^2(\Omega)$

In the square $\Omega =]0,\pi[\times]0,\pi[$, we consider the following Dirichlet problem for the second order elliptic equation

$$(P_\varepsilon) \qquad \begin{aligned} -\varepsilon^2\frac{\partial^2}{\partial x^2}u_\varepsilon \ - \frac{\partial^2}{\partial y^2}u_\varepsilon + u_\varepsilon = f_\varepsilon \quad &\text{in } \Omega, \\ u_\varepsilon(0,y) = u_\varepsilon(\pi,y) = 0 \quad &\text{for } 0<y<\pi, \\ \frac{\partial}{\partial y}u_\varepsilon(x,0) = \frac{\partial}{\partial y}u_\varepsilon(x,\pi) = 0 \quad &\text{for } 0<,x<\pi \end{aligned}$$

Proposition 1:

If u_ε is the solution of problem (P_ε) and if f_ε converges to f in $L^2(\Omega)$ then u_ε converges in $L^2(\Omega)$ to the solution of the boundary problem (P_0)

$$-\frac{\partial^2}{\partial y^2}u+u=f \quad \text{in } \Omega,$$

(P_0)

$$\frac{\partial}{\partial y}u(x,0)=\frac{\partial}{\partial y}u(x,\pi)=0 \quad \text{for } 0<x<\pi \ .$$

Proof:

Let's take $(\Psi_{n,m})$ an orthonormal basis of $L^2(\Omega)$ (see[8]):

(1.1) $\Psi_{n,m}(x,y)=(2/\pi)Sin(nx)Cos(my)$ and $\Psi_{n,0}(x,y)=\frac{\sqrt{2}}{\pi}Sin(nx)$.

Taking the scalar product of the two sides of (P_ϵ) and (P_0) by $\Psi_{n,m}$ and using the fact that $\frac{\partial}{\partial y}u_\epsilon(x,0)=\frac{\partial}{\partial y}u_\epsilon(x,\pi)=u_\epsilon(0,y)=u_\epsilon(\pi,y)=0$ and $\frac{\partial}{\partial y}u(x,0)=\frac{\partial}{\partial y}u(x,\pi)=0$ one has

(1.2) $$(\epsilon^2n^2+m^2+1)u_{n,m,\epsilon}=f_{n,m,\epsilon} \ ,$$

(1.3) $$(m^2+1)u_{n,m}=f_{n,m} \ ,$$

with

(1.4) $u_{n,m,\epsilon}=\int_\Omega u_\epsilon(x,y)\Psi_{n,m}(x,y)dxdy; \quad f_{n,m,\epsilon}=\int_\Omega f_\epsilon(x,y)\Psi_{n,m}(x,y)dxdy$

and

$$u_{n,m}=\int_\Omega u(x,y)\Psi_{n,m'}(x,y)dxdy; \quad f_{n,m}=\int_\Omega f(x,y)\Psi_{n,m}(x,y)dxdy \ .$$

Consequently, the solution of problems (P_ϵ) and (P_0) are given respectively by:

(1.5) $$u_\epsilon(x,y)=\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\left(\frac{f_{n,m,\epsilon}}{\mu_{n,m,\epsilon}}\right)\Psi_{n,m}(x,y)$$

and

(1.6) $$u(x,y)=\sum_{n\geq 1}\sum_{m\geq 0}\left(\frac{f_{n,m}}{\mu_{n,m,0}}\right)\Psi_{n,m}(x,y) \ ,$$

where:

(1.7) $$\mu_{n,m,\epsilon}=\epsilon^2n^2+m^2+1 \ .$$

We remark that for ever (n,m) in $IN^*\times IN$:

(1.8) $$\underset{\epsilon\to 0}{Lim}\ u_{n,m,\epsilon}=\underset{\epsilon\to 0}{Lim}\left(\frac{f_{n,m,\epsilon}}{\mu_{n,m,\epsilon}}\right)=\left(\frac{f_{n,m}}{\mu_{n,m,0}}\right)=u_{n,m}$$

This prove the weak convergence of u_ϵ to u in $L^2(\Omega)$.

On the other hand, we have:

(1.9) $$|u_{n,m\epsilon}|=\left|\frac{f_{n,m,\epsilon}}{\mu_{n,m,\epsilon}}\right|\leq C\left|\frac{f_{n,m}}{\mu_{n,m,0}}\right| \ ,$$

where C is a constant independent of ϵ and (n,m) (this can be easily seen form the convergence of f_ϵ to f in $L^2(\Omega)$).

So we are able to apply the Lebesgue dominated convergence theorem:

(1.10) $$\lim_{\epsilon\to 0}\|u_\epsilon\|^2 = \lim_{\epsilon\to 0}\sum_{n\geq 1}\sum_{m\geq 0}\left|\frac{f_{n,m,\epsilon}}{m_{n,m,\epsilon}}\right|^2 = \sum_{n\geq 1}\sum_{m\geq 0}\left|\frac{f_{n,m}}{\mu_{n,m,0}}\right|^2 = \|u\|^2,$$

which will complete the proof of the proposition 1([2]).

Remark 1:

If f_ϵ converges to f in $H^1(\Omega)$, without having $f(0,y)=f(\pi,y)=0$, then the (u_ϵ) solution doesn't converge in $H^1(\Omega)$. Indeed, in the (P_ϵ) and (P_0) problems, let's take

(1.11) $$f_\epsilon(x,y)=f(x,y)=g(x)h(y)\in H^1(\Omega) \text{ and } g \notin H^1{}_0(]0,\pi[).$$

Then the solution (u_ϵ) of (P_ϵ) converges in $L^2(\Omega)$ to the solution of (P_0), with

(1.12) $$u(x,y)=g(x)v(y) \quad ,$$

where v is the solution of the following boundary problem:

$$(P_1)\quad \begin{cases} -\dfrac{\partial^2}{\partial y^2}v+v=h & \text{for } 0<y<\pi \ , \\[2mm] \dfrac{\partial}{\partial y}v(0)=\dfrac{\partial}{\partial y}v(\pi)=0 \ . \end{cases}$$

As $g\notin H^1{}_0(]0,\pi[)$ implies that $u\notin E$, where

(1.13) $$E:=\{u\in H^1(\Omega);u(0,y)=u(\pi,y)=0\} \ .$$

Consequently, if u_ϵ converges in $H^1(\Omega)$, E being a closed subset of $H^1(\Omega)$, u must be in E, which isn't the case.

2. Convergence of the solution of the problem (P_ϵ), if $f_\epsilon=f$ is smooth

We consider again the boundary problems:

$$(P_\epsilon)_1\quad \begin{cases} -\epsilon^2\dfrac{\partial^2}{\partial x^2}u_\epsilon-\dfrac{\partial^2}{\partial y^2}u_\epsilon+u_\epsilon=f & \text{in } \Omega, \\ u_\epsilon(0,y)=u_\epsilon(\pi,y)=0 & \text{for } 0<y<\pi, \\ \dfrac{\partial}{\partial y}u_\epsilon(x,0)=\dfrac{\partial}{\partial y}u_\epsilon(x,\pi)=0 & \text{for } 0<x<\pi, \end{cases}$$

and

$$(P_0)_1\quad \begin{cases} -\dfrac{\partial^2}{\partial y^2}u+u=f & \text{in } \Omega, \\[2mm] \dfrac{\partial}{\partial y}u(x,0)=\dfrac{\partial}{\partial y}u(x,\pi)=0 & \text{for } 0<x<\pi, \end{cases}$$

where f belongs to $L^2(\Omega)$ and verifies

(H): For every y in $]0,\pi[$, the function $f(\cdot,y)\in C^{\infty}([0,\pi])$.

Proposition 2

If f verifies the hypothesis (H), and if u_ϵ and u are the respective solutions of $(P_\epsilon)_1$ and $(P_0)_1$ problems, then

$$\|u_\epsilon - u\| \leq C\sqrt{\epsilon} \ , \tag{2.1}$$

where $\| \ \|$ is the $L^2(\Omega)$ norm and C is a constant.

Proof:

Following the first section results, u_ϵ converges to u in $L^2(\Omega)$, so there is a boundary conditions loose near $\{0\}\times]0,\pi[$ and $\{\pi\}\times]0,\pi[$, and consequently u_ϵ can be written as

$$u_\epsilon = v_\epsilon + \omega_\epsilon + \chi_\epsilon \ , \tag{2.2}$$

where ω_ϵ and χ_ϵ are the limits layers phenomenas near $\{0\}\times]0,\pi[$ and $\{\pi\}\times]0,\pi[$ respectively.

To determine the expansion of v_ϵ, we use the recurrent relation exposed by A. Nayfeh [6]:

$$v_\epsilon(x,y) = v_0(x,y) + \epsilon^2 v_1(x,y) + \epsilon^4 v_2(x,y) + \cdots \tag{2.3}$$

Reporting the expansion in $(P_\epsilon)_1$ and gathering the terms in ϵ, we obtain:

$$\left(-\frac{\partial^2}{\partial y^2}v_0 + v_0\right) + \sum_{k\geq 1}\epsilon^{2k}\left(-\frac{\partial^2}{\partial y^2}v_k + v_k - \frac{\partial^2}{\partial x^2}v_{k-1}\right) = f \ . \tag{2.4}$$

Consequently

$$-\frac{\partial^2}{\partial y^2}v_0 + v_0 = f \quad \text{in } \Omega \ , \tag{2.5}$$

$$-\frac{\partial^2}{\partial y^2}v_k + v_k = \frac{\partial^2}{\partial x^2}v_{k-1} \quad \text{for } k\geq 1 \ . \tag{2.6}$$

To v_k we impose the conditions

$$\frac{\partial}{\partial y}v_k(x,0) = \frac{\partial}{\partial y}v_k(x,\pi) = 0 \qquad \text{for } k\in N, y\in]0,\pi[. \tag{2.7}$$

So, v_0 is the solution of the $(P_0)_1$ problem (i.e. $v_0 = u$).

The problems (2.6) and (2.7) admit unique solution following the fact that f verifies the hypothesis (H). On the other hand, we have for ω_ϵ:

$$\omega_\epsilon(x,y) = \omega_0(x,y,\epsilon) + \epsilon^2\omega_1(x,y,\epsilon) + \epsilon^4\omega_2(x,y,\epsilon) + \cdots \tag{2.8}$$

Here ω_ϵ doesn't occur near $[0]\times]0,\pi[$, and so, it is natural to look to (ω_k) as follows

(2.9) $$\omega_k(x,y,\epsilon)=\omega_k(t,y) \text{ with } t=\epsilon^{-1}x \ .$$

This variable change induces a change in the A_ϵ operator:

(2.10) $$A_\epsilon=-\frac{\partial^2}{\partial t^2}-\frac{\partial^2}{\partial y^2}+I_d \ ,$$

where I_d is the identity operator.

Let's apply the operator A_ϵ to the ω_ϵ expansion:

(2.11) $$A_\epsilon(\omega_0+\epsilon^2\omega_1+\epsilon^4\omega_2+\cdots)=0 \ ,$$

which implies

(2.12) $$-\frac{\partial^2}{\partial t^2}\omega_k-\frac{\partial^2}{\partial y^2}\omega_k+\omega_k=0 \quad \text{in } \Omega \text{ for } k\in N,$$

with the following boundary conditions:

(2.13) $$\frac{\partial}{\partial y}\omega_k(t,0)=\frac{\partial}{\partial y}\omega_k(t,\pi)=0 \quad \text{for } t\geq 0 \ ,$$
$$\omega_k(0,y)+v_k(0,y)=0 \quad \text{for } 0<y<\pi \ ,$$
$$\omega_k(t,y)\to 0 \qquad \text{when} \quad t\to+\infty \ .$$

Similarly, we have for χ_ϵ:

(2.14) $$\chi_\epsilon(x,y)=\chi_0(x,y,\epsilon)+\epsilon^2\chi_1(x,y,\epsilon)+\epsilon^4\chi_2(x,y,\epsilon)+\cdots \ ,$$

and operating the following variable change

(2.15) $$\chi_k(x,y,\epsilon)=\chi_k(s,y), s=\epsilon^{-1}(\pi-x) \ ,$$

the operator A_ϵ will be

(2.16) $$A_\epsilon=-\frac{\partial^2}{\partial s^2}-\frac{\partial^2}{\partial y^2}+I_d \ .$$

So, we find that

(2.17) $$-\frac{\partial^2}{\partial s^2}\chi_k-\frac{\partial^2}{\partial y^2}\chi_k+\chi_k=0 \quad \text{in } \Omega \text{ for } k\in N,$$

with

(2.18) $$\frac{\partial}{\partial y}\chi_k(s,0)=\frac{\partial}{\partial y}\chi_k(s,\pi)=0 \quad \text{for } s\geq 0 \ ,$$
$$\chi_k(0,y)+v_k(\pi,y)=0 \quad \text{for } 0<y<\pi \ .$$

For the case $k=0$, the solutions of the problems (2.12),(2.13) and (2.17), (2.18) are respectively:

(2.19) $$\omega_0(t,y)=-\sum_{p\geq 0}\left(\frac{f_p(0)}{1+p^2}\right)\phi_p(y)exp(-t\sqrt{1+p^2})$$

(2.20) $$\chi_0(s,y)=-\sum_{p\geq 0}\left(\frac{f_p(\pi)}{1+p^2}\right)\phi_p(y)exp(-s\sqrt{1+p^2})$$

where

(2.21) $$\phi_p(y)=\sqrt{\tfrac{2}{\pi}}\,Cos(py);\quad \phi_0(y)=\sqrt{\tfrac{1}{\pi}}\ \text{and}\ f_p(x):=\int_0^{\pi} f(x,y)\phi_p(y)dy\ .$$

Consequently

(2.21) $$u_\epsilon=(u+\chi_0+\omega_0)+\sum_{k\geq 1}\epsilon^{2k}(v_k+\chi_k+\omega_k)\ ,$$

which will complete the proof of the inequality (2.1).

<u>Remark 2</u>

If we consider the problem $(P_\epsilon)_1$, with f verifying the hypothesis (H), while $f(0,y)=f(\pi,y)=0$, for every p in $N, f_p(0)=f_p(\pi)=0$ and consequently the solutions of problems (2.12), (2.13) and (2.17), (2.18) for $k=0$ are identically zero, then the inequality (2.1) takes the form

(2.22) $$\| u_\epsilon - u \| \leq C\epsilon^2 .$$

<u>**References:**</u>

1. **T. BENKIRAN**, Comportement asymptotique d'un problème de perturbation singulière non classique, (to appear).

2. **T. BENKIRAN**, Etude d'un problème de perturbation singulière en une dimension, à paraitre.

3. **H. BRÉZIS**, Analyse Fonctionnelle, Théorie et Applications, Masson, Paris, 1983.

4. J.L. LIONS, Perturbation Singulière dans les problèmes aux limites et en Controle Optimal, Springer-Verlag, Berlin, 1973.

5. J.L. LIONS, **E. MAGENES**, Problèmes aux limites non homogènes, Dunod, Vol 1 èt 2, 1969.

6. **A. NAYFEH**, Perturbation methods. Wiley 1973.

7. **A. ZYGMUND**, Trigonometric Series, Vol (I) (II), Cambridge at the University Press, 1968.