LIBERTAS MATHEMATICA, VOL. XIV (1994)

ON THE STRUCTURE OF QUASI - CAYLEY GRAPHS - DECOMPOSITION THEOREMS

St. Antohe and E. Olaru

The paper refers to quasi - Cayley graphs which generalize Cayley graphs [3]. Latticeal properties of quasi - Cayley graphs are first analyzed. Three decomposition theorems follow. At the end, as applications, we study some properties of the categorical product of graphs.

1. Introduction

In this paper, we consider only simple graphs, i.e. undirected, without multiple edges and loops. For such a graph Γ we denote by $V(\Gamma)$ the vertex set and by $E(\Gamma)$ the edge set and we write $\Gamma = (V(\Gamma), E(\Gamma))$ or briefly $\Gamma = (V, E)$. For $(x, y) \in E(\Gamma)$, we also use $x \sim y$.

There are many generalization of Cayley graphs (see e. g. Biggs [3]). Another class of graphs associated with a group has been introduced by Antohe, [2], as follows:

(1.1) __Definition__ Let G be a multiplicative group (with identity 1) and A a subset of G with the properties:

(i) $1 \notin A$;

(ii) $x \in A \Rightarrow x^{-1} \in A$.

The graph $\Gamma := \Gamma_G(A)$ with $V(\Gamma) = G$ and $E(\Gamma) = \{(x, y) : (x, y) \in G \times G$ and $xy^{-1} \in A\}$ is called the <u>quasi</u> - <u>Cayley</u> graph of (the group) G with respect to A.

If G is finite and A is a set of generators for G, then $\Gamma_G(A)$ is the Cayley graph (Biggs, [3], p. 106).

(1.2) __Remark__ For group G and $A \subset G$ the following statements are equivalent:

(a) $A \subseteq A^{-1}$;

(b) $A = A^{-1}$,

where A^{-1} is the inverse of A (that is $A^{-1} = \{x^{-1} : x \in A\}$).

(1.3) __Definition__ A set A of a group G having the properties:

$$1 \notin A \quad \text{and} \quad A = A^{-1} \qquad (1)$$

will be called a __self__ - __invertible set__ of G.

(1.4) __Examples__

1) Let G be a finite group of order n and H a subgroup of order m. The set H - {1} is evidently self - invertible in G and the quasi - Cayley graph $\Gamma_G(H - \{1\})$ is a disjunct reunion of complete graphs K_m; in particular $\Gamma_G(G - \{1\}) = K_n$.

Indeed, if we denote as usually $Hx = \{hx : h \in H\}$, then :

$$(x, y) \in E(\Gamma_G(H - \{1\})) \Leftrightarrow xy^{-1} \in H \Leftrightarrow y \in Hx,$$

that is Hx generates the complete graph K_m.

2) Let $G = \langle a \rangle = \{a^k : k \in Z\}$ be a cyclic group. The set $A = \{a, a^{-1}\}$ is self invertible in G and in the Cayley graph $\Gamma = \Gamma_G(A)$ for any two distinct verticies $x = a^p$, $y = a^q$ we have:

$$(x, y) \in E(\Gamma) \Leftrightarrow a^{|p-q|} = a$$

If G is infinite, then $p \neq q$ implies $a^p \neq a^q$ and so $|p - q| = 1$, that is $E(\Gamma) = \{(a^p, a^{p+1}) : p \in Z\}$. If G is finite of order n, then:

$$(x, y) \in E(\Gamma) \Leftrightarrow |p - q| = 1 \ (\text{mod } n),$$

i.e. $\Gamma = C_n$. Analogously, if G is infinite, then we denote $\Gamma_G(A)$ by C_∞. These graphs will play a special role in this paper. Sometimes $K_2 = C_2$ may be considered to be a cycle graph.

(1.5) For a group G we denote the family of self - invertible sets of G by *A*(G) and C(G) will represent the set of all corresponding quasi - Cayley graphs.

The sets *A*(G) and C(G) have an atomic boolean algebraic structure, and they are complete and isomorphic.

We present the structure of the atom graphs in C(G) and prove that any quasi - Cayley graph is a reunion of Cayley graphs and this decomposition is unique (theorem 4.1). Also, we give decomposition theorems of quasi - Cayley graphs with respect to intersection and categorical product of graphs (theorems 4.2 and 4.3). Some properties of categorical product and applications are shown.

2. Latticeal properties

Let G be a group and A be a self - invertible set in G. We denote $\mathcal{A}(G) = \{A : A \subset G, 1 \notin A, A = A^{-1}\}$, and $C(G) = \{\Gamma_G(A) : A \in \mathcal{A}(G)\}$.

(2.1) Proposition The set $\mathcal{A}(G)$ of all self - invertible sets in G is an atomic boolean algebra, complete with respect to intersection and reunion.

Proof

If we denote with $\mathcal{P}(G - \{1\})$ the set of all subsets of G - {1}, then $\mathcal{A}(G) \subseteq \mathcal{P}(G - \{1\})$. Let us consider $A, B \in \mathcal{A}(G)$. It is clear that $A \cap B$, $A \cup B \in \mathcal{A}(G)$, so that $(\mathcal{A}(G), \cap, \cup)$ is a sublattice of $\mathcal{P}(G - \{1\})$. Therefore $\mathcal{A}(G)$ is a distributive lattice, having Φ as a zero element and G - {1} as the unit element.

If $A \in \mathcal{A}(G)$, then $A' = (G - \{1\}) - A \in \mathcal{A}(G)$. So A' is the complementary set of A in the lattice $\mathcal{A}(G)$.

Let us consider $(A_i)_{i \in I}$ a family of elements from $\mathcal{A}(G)$, that is $A_i^{-1} = A_i$, $1 \notin A_i$, for any $i \in I$. Hence

$$1 \notin \bigcup_{i \in I} A_i \text{ and } 1 \notin \bigcap_{i \in I} A_i$$

we have:

$$A_i \subseteq \bigcup_{i \in I} A_i \Rightarrow A_i^{-1} \subseteq \Big(\bigcup_{i \in I} A_i\Big)^{-1} \Rightarrow A_i \subseteq \Big(\bigcup_{i \in I} A_i\Big)^{-1} \Rightarrow \bigcup_{i \in I} A_i \subseteq \Big(\bigcup_{i \in I} A_i\Big)^{-1}$$

and

$$\bigcap_{i \in I} A_i \subseteq A_i \Rightarrow \Big(\bigcap_{i \in I} A_i\Big)^{-1} \subseteq A_i^{-1} \Rightarrow \Big(\bigcap_{i \in I} A_i\Big)^{-1} \subseteq A_i \Rightarrow \Big(\bigcap_{i \in I} A_i\Big)^{-1} \subseteq \bigcap_{i \in I} A_i \Rightarrow$$

$$\bigcap_{i \in I} A_i \subseteq \Big(\bigcap_{i \in I} A_i\Big)^{-1}.$$

According to (1.2), we have $\bigcup_{i\in I} A_i$, $\bigcap_{i\in I} A_i \in A(G)$, therefore Boole algebra $A(G)$ is complete.

Let us consider $A \in A(G)$, $A \neq \Phi$ and $a \in A$. If $a^2 = 1$, then $a = a^{-1}$ and so $\{a\}$ is an atom. For any $A \in A(G)$ there is the atom $\{a\}$ or $\{a, a^{-1}\}$ included in A, therefore the boolean algebra $A(G)$ is also atomic.

(2.2) Observation The atom elements of latice $A(G)$ are minimal sets in $A(G) - \{\Phi\}$ of shape $\{a\}$, if $a^2 = 1$ and $\{a, a^{-1}\}$, if $a^2 \neq 1$.

In the following propositions we shall present some properties of quasi - Cayley graphs.

(2.3) Proposition If G is a group, then the following statements are true:

i) if $A \in A(G)$, and $A' = (G - \{1\}) - A$ is the complementary set of A in $G - \{1\}$, then $A' \in A(G)$ and $\Gamma_G(A')$ is the complementary graph of $\Gamma_G(A)$, that is: $\Gamma_G(A') = \overline{\Gamma_G(A)}$;

ii) if $A_1, A_2 \in A(G)$, and $A_1 \subset A_2 \Rightarrow \Gamma_G(A_1) \subset \Gamma_G(A_2)$;

iii) if H is a subgroup of G and $A \in A(H)$, then $\Gamma_H(A) \subset \Gamma_G(A)$ and $\Gamma_H(A) = [H]\Gamma_G(A)$.

Proof

i) The implications $(x, y) \in E(\Gamma_G(A')) \Leftrightarrow xy^{-1} \in A' \Leftrightarrow xy^{-1} \notin A$ prove the equality.

ii) The two graphs have the same vertex set and $(x, y) \in E(\Gamma_G(A_1)) \Leftrightarrow xy^{-1} \in A_1 \Rightarrow xy^{-1} \in A_2 \Leftrightarrow (x, y) \in E(\Gamma_G(A_2))$.

iii) The inclusion is evident. If $x, y \in H$ and $x \sim y$ in the graph $\Gamma_H(A)$, we get that the two verticies are adiacent in $\Gamma_G(A)$ too since xy^{-1} is unique; consequently, $\Gamma_H(A)$ is generated by the vertex set H in $\Gamma_G(A)$.

We shall use $\cap$ and $\cup$ respectively for the intersection and the reunion of graphs.

(2.4) **Proposition** If $(H_i)_{i\in I}$ is a family of subgroups of a group G and $A_i \in \mathcal{A}(H_i)$, $i \in I$, then:

j) $\Gamma_{\bigcap_{i\in I} H_i}(\bigcap_{i\in I} A_i) = \bigcap_{i\in I} \Gamma_{H_i}(A_i)$,

particulary

$$\Gamma_G(\bigcap_{i\in I} A_i) = \bigcap_{i\in I} \Gamma_G(A_i);$$

jj) $\Gamma_G(\bigcup_{i\in I} A_i) = \bigcup_{i\in I} \Gamma_G(A_i)$.

Proof

j) The two graphs have the same vertex set and

$$(x, y) \in E(\Gamma_{\bigcap_{i\in I} H_i}(\bigcap_{i\in I} A_i)) \Leftrightarrow x, y \in \bigcap_{i\in I} H_i, xy^{-1} \in \bigcap_{i\in I} A_i \Leftrightarrow$$
$$(\forall i)(i\in I, x, y \in H_i, xy^{-1} \in A_i) \Leftrightarrow (\forall i)(i\in I, (x, y) \in E(\Gamma_{H_i}(A_i)) \Leftrightarrow$$
$$(x, y) \in \bigcap_{i\in I} \Gamma_{H_i}(A_i);$$

jj) The implications

$$(x, y) \in E(\Gamma_G(\bigcup_{i\in I} A_i)) \Leftrightarrow xy^{-1} \in \bigcup_{i\in I} A_i \Leftrightarrow (\exists k)(k\in I, xy^{-1} \in A_k) \Leftrightarrow$$
$$(\exists k)(k\in I, (x, y) \in E(\Gamma_G(A_k))) \Leftrightarrow (x, y) \in E(\bigcup_{i\in I} \Gamma_G(A_i))$$

prove the equality.

(2.5) **Proposition** The set C(G) of quasi - Cayley graphs is a boolean algebra with respect to graphs reunion and intersection.

Proof

From (2.4) we deduce that the intersection and the reunion af any graph family C(G) is also a graph belonging to C(G). In addition, $(C(G), \cap, \cup)$ is a distributive lattice with the discrete graph $\Gamma_G(\Phi)$ as zero element, the unit element being the complete graph $\Gamma_G(G - \{1\})$.

If $\Gamma_G(A) \in C(G)$ and $A' = (G - \{1\}) - A$, then $\Gamma_G(A) \cap \Gamma_G(A') = \Gamma_G(\Phi)$ and $\Gamma_G(A) \cup \Gamma_G(A') = \Gamma_G(G - \{1\})$, so that $\Gamma_G(A')$ is the complement of element $\Gamma_G(A)$ in the lattice $C(G)$, and according to (2.3), we get

$\Gamma_G(A') = \overline{\Gamma_G(A)}$.

(2.6) <u>Theorem</u> The Boole algebras $\mathcal{A}(G)$ and $C(G)$ are isomorphic.

<u>Proof</u>

We define application f: $\mathcal{A}(G) \to C(G)$ by $f(A) = \Gamma_G(A)$. Obviously f is surjective. Let us prove that f is injective.

Let $A_1, A_2 \in \mathcal{A}(G)$ be so that $f(A_1) = f(A_2)$. We obtain $\Gamma_G(A_1) = \Gamma_G(A_2)$. The two graphs have the same vertex set. If $x \in A_1$, it results $(1, x) \in E(\Gamma_G(A_1)) = E(\Gamma_G(A_2))$ hence $x\, 1^{-1} = x \in A_2$, that is $A_1 \subset A_2$. Similarly, $A_2 \subset A_1$. Consequently, we get $A_1 = A_2$, so f is bijective.

The relations $f(A_1 \cup A_2) = \Gamma_G(A_1) \cup \Gamma_G(A_2) = f(A_1) \cup (A_2)$ and

$f(A') = \Gamma_G(A') = \overline{\Gamma_G(A)} = \overline{f(A)}$ show that f is a Boole algebra isomorphism.

(2.7) <u>Corollary</u> The Boole algebra $C(G)$ is complete and atomic, and the atom graphs are: $\Gamma_G(\{a\})$, if $a^2 = 1$ or $\Gamma_G(\{a,a^{-1}\})$ if $a^2 \neq 1$, where $a \in G$.

3. Atom graphs

The atoms graphs will play a key role in establishing the structure of a quasi - Cayley graph.

(3.1) <u>Theorem</u> The atom graphs of $C(G)$ algebra have the following structure:

1) $\Gamma_G(\{a\})$ is the graph whose all connected components are isomorphic to K_2;

2) $\Gamma_G(\{a,a^{-1}\})$ are the graphs whose connected components are isomorphic to chordless cycles C_p, C_∞ (p = ord a).

<u>Proof</u>

Consider $a \in G$, $a \neq 1$. We may have $a^2 = 1$ or $a^2 \neq 1$.

1) If $a^2 = 1$, then $a^{-1} = a$ and for $x, y \in V(\Gamma) = G$, $(x, y) \in E(\Gamma) \Leftrightarrow xy^{-1} = a \Leftrightarrow y = ax$. Since ax is unique in G, it results that x is adiacent only to ax in Γ. Therefore, the connected components of graph $\Gamma_G(\{a\})$ are isomorphic to K_2.

2) If $a^2 \neq 1$, the cyclic subgroup $<a>$ has a higher order than 2 or it is infinite. Let us consider the partition $\{<a>x : x \in G\}$ determined by the subgroup $<a>$. If a^ix, $a^jx \in <a>x$, then

$$a^ix \sim a^jx \Leftrightarrow (a^ix)(a^jx)^{-1} = a^{i-j} \in \{a,a^{-1}\} \Leftrightarrow |i-j| = 1 .$$

Hence vertex set $<a>x$ generates a chordless cycle of order p in $\Gamma_G(\{a,a^{-1}\})$ and a cycle C_∞ if any $<a>$ is an infinite subgroup. Let us prove that there are no edges to join verticies of different cycles.

Let $<a>x \cap <a>y = \Phi$ and $a^ix \in <a>x$, $a^jy \in <a>y$. If any $a^ix \sim a^jy$, then

$$(a^ix)(a^jy)^{-1} \in (\{a,a^{-1}\}) \Leftrightarrow a^i(xy^{-1})a^j \in \{a,a^{-1}\} \Rightarrow xy^{-1} = a^{j-i+1} \text{ or } xy^{-1} = a^{j-i+1} \Rightarrow$$

$$x \in <a>y \Leftrightarrow <a>x = <a>y$$

(3.2) <u>Corollary</u> If $|G| = n$ and ord $a = n$, then $\Gamma_G(\{a,a^{-1}\}) = C_n$ (by C_2 mean K_2).

4. Decomposition

Using the isomorphism between lattices $\mathcal{A}(G)$ and $C(G)$, we obtain:

(4.1) <u>Theorem</u> Any quasi - Cayley graph can be uniquely written as a reunion of Cayley graphs.

Proof

Let $\Gamma_G(A)$ be a quasi - Cayley graph. In the lattice $\mathcal{A}(G)$ A is uniquely written as a reunion of atoms:

$$A = \big(\bigcup_{a^2=1} \{a\} \big) \cup \big(\bigcup_{a^2\neq 1} \{a, a^{-1}\} \big) .$$

Since f is an isomorphism (2.6), we have

$$\Gamma_G(A) = f(A) = f\big(\bigcup_{a^2=1} \{a\} \big) \cup f\big(\bigcup_{a^2\neq 1} \{a, a^{-1}\} \big) =$$
$$\big(\bigcup_{a^2=1} f(\{a\}) \big) \cup \big(\bigcup_{a^2\neq 1} f(\{a, a^{-1}\}) \big)$$

so that

$$\Gamma_G(A) = \big(\bigcup_{a^2=1} \Gamma_G(\{a\}) \big) \cup \big(\bigcup_{a^2\neq 1} \Gamma_G(\{a, a^{-1}\}) \big) \qquad (2)$$

and the atom graphs $\Gamma_G(\{a\})$ and $\Gamma_G(\{a, a^{-1}\})$ are reunions of Cayley graphs (see (3.1)).

(4.2) **Theorem** Any quasi - Cayley graph may be uniquely written as an intersection of graphs complementary to atom graphs.

Proof

Let $\Gamma_G(B)$ be a quasi - Cayley graph with $B \in \mathcal{A}(G)$. Since $\mathcal{A}(G)$ is a Boole algebra, there is an $A \in \mathcal{A}(G)$, so that $B = (G - \{1\}) - A = A'$ and, according to (2.3), we have

$$\Gamma_G(B) = \Gamma_G(A') = \overline{\Gamma_G(A)} \qquad (3)$$

The graph $\Gamma_G(A)$ has a type (2) decomposition, according to the previous theorem, that is

$$\Gamma_G(A) = \bigcup_{i\in I} \Gamma_G(A_i) \qquad (4)$$

where $\Gamma_G(A_i)$ are atom graphs in C(G).

From (3) and (4) we obtain

$$\Gamma_G(B) = \bigcap_{i\in I}\overline{\Gamma_G(A)}$$

The following theorem refers to some Cayley - graph decomposition in relation to the categorical product of graphs.

(4.3) <u>Theorem</u> For any generated finite abelian group G, there is a generator set $A \in \mathcal{A}(G)$, such that the Cayley graph $\Gamma_G(A)$ is written as a categorical product of Cayley graphs.

<u>Proof</u>

To prove this theorem we shall present several notions and supplementary results.

Let $G_1, G_2, \dots, G_n$ be arbitrary groups. The set

$$G = \{(a_1, a_2, \dots, a_n) : a_i \in G_i\}$$

is a group with respect to the operation

$$(a_1, a_2, \dots, a_n)(b_1, b_2, \dots, b_n) = (a_1 b_1, a_2 b_2, \dots, a_n b_n)$$

called the <u>direct</u> <u>product</u> <u>of</u> <u>groups</u> $G_1, G_2, \dots, G_n$ and denoted by $\overset{n}{\underset{i=1}{\times}} G_i$;

$(1, 1, \dots, 1)$ is the identity element.

The graph $\Gamma_i = (V_i, E_i)$, $i = 1, 2, \dots, n$, being given we denote by

$\Gamma_1 \otimes \Gamma_2 \otimes \dots \otimes \Gamma_n$ or $\bigotimes_{i=1}^{n} \Gamma_i$ the graph defined by

$$V\left(\bigotimes_{i=1}^{n} \Gamma_i\right) = \bigcup_{i=1}^{n} V(\Gamma_i) = \{(x_1, x_2, \dots, x_n) : x_i \in V(\Gamma_i)\}$$

and $(x_1, x_2, \dots, x_n) \sim (y_1, y_2, \dots, y_n) \Leftrightarrow x_i \sim y_i$, $i = 1, 2, \dots, n$.

This product is called <u>categorical</u> (or <u>cardinal</u> or <u>tensorial</u>). Some of the properties of this product are studied in [1].

(4.4) <u>Proposition</u> Let $G_1, G_2, \dots, G_n$ be arbitrary groups, and $A_i \in \mathcal{A}(G_i)$, $i = 1, 2, \dots, n$. Then the cardinal product $\prod_{i=1}^{n} A_i = \{(a_1, a_2, \dots, a_n) : a_i \in A_i\}$ belongs to the lattice $\mathcal{A}\left(\overset{n}{\underset{i=1}{\times}} G_i\right)$ and

$$\Gamma_{\mathop{\times}\limits_{i=1}^{n} G_i}\left(\prod_{i=1}^{n} A_i\right) = \bigotimes_{i=1}^{n} \Gamma_{G_i}(A_i) \tag{5}$$

Proof

Clearly, $(1, 1, \ldots, 1) \notin \prod_{i=1}^{n} A_i$ and $(a_1, a_2, \ldots, a_n) \in \prod_{i=1}^{n} A_i$ implies $(a_1^{-1}, a_2^{-1}, \ldots, a_n^{-1}) \in \prod_{i=1}^{n} A_i$, since $A_i \in \mathcal{A}(G_i)$, $i = 1, 2, \ldots, n$.

The graphs involved in (5) have the same vertex set $\mathop{\times}\limits_{i=1}^{n} G_i$. Let $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$ be two elements in the group $\mathop{\times}\limits_{i=1}^{n} G_i$.

The implications

$$(x, y) \in E\left(\Gamma_{\mathop{\times}\limits_{i=1}^{n} G_i}\left(\prod_{i=1}^{n} A_i\right)\right) \Leftrightarrow (\forall i)\ (x_i\, y_i^{-1} \in A_i) \Leftrightarrow$$

$$(\forall i)\ ((x_i, y_i) \in E(\Gamma_{G_i}(A_i)) \Leftrightarrow (x, y) \in E\left(\bigotimes_{i=1}^{n} \Gamma_{G_i}(A_i)\right)$$

proves the equality (5).

The next theorem refers to the structure of finitely generated abelian groups (see [6]).

(4.5) **Proposition** Let G be a finitely generated abelian group with n - generators. There will be elements $a_1, a_2, \ldots, a_n \in G$ such that

$$G = \langle a_1 \rangle \times \langle a_2 \rangle \times \ldots \times \langle a_n \rangle = \mathop{\times}\limits_{i=1}^{n} \langle a_i \rangle \tag{6}$$

If G is finite, all cyclic subgroups $\langle a_i \rangle$ are finite. Let $A_i = \{a_i, a_i^{-1}\}$ be a self - invertible set in the cyclic group $\langle a_i \rangle$, $i = 1, 2, \ldots, n$ ($a_i^2 = 1$ is included).

The set $A = \prod_{i=1}^{n} \{a_i, a_i^{-1}\}$ belongs to the lattice $\mathcal{A}(G)$ and according to (5) we have

$$\Gamma_G(A) = \Gamma_{\mathop{\times}\limits_{i=1}^{n} \langle a_i \rangle}\left(\prod_{i=1}^{n} (a_i, a_i^{-1})\right) = \bigotimes_{i=1}^{n} \Gamma_{\langle a_i \rangle}(\{a_i, a_i^{-1}\})$$

5. Applications

From [6] we shall use the following result:

(5.1) Proposition Let H and K be two cyclic groups of m - order, n - order respectively. The group $H \times K$ is cyclic if and only if m and n are prime to each other, that is $(m, n) = 1$.

We consider the additive group Z_m and Z_n, where

$Z_m = \{\hat{0}, \hat{1}, ..., \widehat{(m-1)}\}$, $Z_n = \{\bar{0}, \bar{1}, ..., \overline{(n-1)}\}$. These groups are cyclic:

$$Z_m = <\hat{1}>, Z_n = <\bar{1}>.$$

(5.2) Corollarly The direct product group $Z_m \times Z_n$ is a cyclic group if and only if $(m, n) = 1$. If $(m, n) = 1$ then $Z_m \times Z_n$ and Z_{mn} are isomorphic.

Next, using some previous results we shall deduce several properties of the categorical product of graphs.

(5.3) Proposition Let C_m and C_n, $m, n > 1$ be two cycles. Then $C_m \otimes C_n$ is a cycle if and only if one factor is isomorphic to K_2, while the other is an odd cycle.

Proof

If $m = 2$ from $(2, n) = 1$ we obtain n as a odd $n = 2k+1$. Let the additive groups $Z_2 = \{\hat{0}, \hat{1}\}$ and $Z_{2k+1} = \{\bar{0}, \bar{1}, ..., \overline{(2k+1)}\}$ with $A = \{\hat{1}\}$, $B = \{\bar{1}, \overline{(2k)}\}$ as self - invertible sets; then we have $C_2 = K_2 = \Gamma_{Z_2}(A)$, $C_{2k+1} = \Gamma_{Z_{2k+1}}(B)$. The corollary (5.2) ensures that the group $Z_2 \times Z_{2k+1}$ is cyclic. According to (4.4), the set $A \times B = \{(1, 1), (1, 2k)\}$ is self - invertible, atom in the lattice $A(Z_2 \times Z_{2k+1})$ and it generates the group $Z_2 \times Z_{2k+1}$.

Consequently

$$\Gamma_{Z_2\times Z_{2k+1}}(A \times B) = \Gamma_{Z_2}(A)\otimes\Gamma_{Z_{2k+1}}(B)$$

and we obtain

$$C_2\otimes C_{2k+1} = C_{2(2k+1)}.$$

Reciprocally, suppose that $C_m \otimes C_n = C_{mn}$. We may write $C_m = \Gamma_{Z_m}(A)$, $C_n = \Gamma_{Z_n}(B)$ where $A \in A(Z_m)$, $B \in A(Z_n)$ are atom elements. It results that $|A| \leq 2$, $|A| \leq 2$.

If $|A| = 1$, $A = \{\hat{1}\}$, then $C_m = \Gamma_{Z_m}(\{\hat{1}\})$ is a cycle only when $m = 2$. The group $Z_2 \times Z_n$ is cyclic if and only if $(2, n) = 1$, that is n is odd number and so the graph $C_n = \Gamma_{Z_n}(\bar{1}, \overline{(n-1)})$ is an odd cycle.

If $|A| = 2$, we have $A = \{\hat{1}, (m \hat{-} 1)\}$ and $B = \{\bar{1}, \overline{(n-1)}\}$. The set $A \times B$ is self - invertible in the group $Z_m \times Z_n$. According to (3.2), since the element $(\hat{1}, \bar{1})$ is of order mn, the graph $\Gamma_{Z_m\times Z_n}(A \times B)$ is a cycle only if it is atom in $A(Z_m \times Z_n)$, which implies $n = 2$. From $(m, 2) = 1$ we obtain that m is an odd integer.

(5.4) Proposition Let C_m and C_n be with $m, n > 2$. If $(m, n) = 1$, then

$$C_m \otimes C_n = C_{mn} \cup C_{mn} = 2\,C_{mn} \tag{7}$$

Proof

Let consider the additive groups Z_m and Z_n. If $(m, n) = 1$, then the group $Z_m \times Z_n$ is cyclic. If we take the atoms $A = \{\hat{1}, (m \hat{-} 1)\}$ in $A(Z_m)$ and $B = \{\bar{1}, \overline{(n-1)}\}$ in $A(Z_n)$, then $C_m = \Gamma_{Z_m}(A)$, $C_n = \Gamma_{Z_n}(B)$ and $A \times B \in A(Z_m \times Z_n)$.

If we denote $P_1 = \{(\hat{1}, \bar{1}), ((m \hat{-} 1), \overline{(n-1)})\}$ and $P_2 = \{(\hat{1}, \overline{(n-1)}), ((m \hat{-} 1), \bar{1})\}$, then $A \times B = P_1 \cup P_2$ and we obtain

$$\Gamma_{Z_m\times Z_n}(A \times B) = \Gamma_{Z_m}(A) \otimes \Gamma_{Z_n}(B) = \Gamma_{Z_m\times Z_n}(P_1) \cup \Gamma_{Z_m\times Z_n}(P_2) \tag{8}$$

The sets P_1 and P_2 are atoms in $A(Z_m \times Z_n)$, having elements of mn order, so that $\Gamma_{Z_m \times Z_n}(P_1) = C_{mn}$ and $\Gamma_{Z_m \times Z_n}(P_2) = C_{mn}$. Substituting in (8) we obtain (7).

(5.5) **Proposition** If n is an even integer, then

$$K_2 \otimes C_n = 2\,C_n \tag{9}$$

holds.

Proof

Let consider the additive groups $Z_n = \{\hat{0}, \hat{1}\}$ and $Z_n = \{\bar{0}, \bar{1}, ..., \overline{(n-1)}\}$. If n = 2q, then $(2q)\,\bar{1} = 0$. The cyclic subgroup $H = \langle \{\hat{1}, \bar{1}\} \rangle$ has the order n, since in $Z_2 \times Z_n$ we have $(2q)\,(\hat{1}, \bar{1}) = (\hat{0}, \bar{0})$. If $A = \{\hat{1}\} \in A(Z_2)$ and $B = \{\bar{1}, \overline{(n-1)}\} \in A(Z_n)$, then $A \times B$ is an atom in the lattice $A(Z_2 \times Z_n)$.

According to (3.1) we obtain

$$\Gamma_{Z_2 \times Z_n}(A \times B) = C_n \cup C_n = 2\,C_n \tag{10}$$

In addition we have

$$\Gamma_{Z_2 \times Z_n}(A \times B) = \Gamma_{Z_2}(A) \otimes \Gamma_{Z_n}(B) = K_2 \otimes C_n \tag{11}$$

Finally, (10) and (11) ensure that (9) is true.

REFERENCES

1. St. Antohe, Properties of some Graph Categories, Buletin Univ. of Galati, 1986, fasc. II, 35-39
2. St. Antohe, Quasi - Cayley Graphs, Analele Univ. of Galati, 1989, fasc. II, 28-32
3. N. L. Biggs, Algebraic graph theory, Cambridge, Univ. Press, 1974
4. M. Farzan, D. Waller, Kronecker products and Local Joints of Graphs, Can. J. Math., Vol. XXIX, No. 2, 1977, 255-269
5. M. Malita, M. Malita, Bazele inteligentei artificiale, Editura Tehnica, Bucuresti, 1987
6. D. Popescu, C. Vraciu, Elemente de teoria grupurilor finite, Editura Stiintifica si Enciclopedica, Bucuresti, 1986