

ON HARDY TYPE DISCRETE INEQUALITIES

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**Abstract.** In the present paper we establish two new discrete inequalities of the Hardy type by using a fairly elementary analysis.

**1. Introduction.**

In 1920, G. H. Hardy [3, 4] proved the following inequality:

If  $p > 1$ ,  $a_n \geq 0$  and  $A_n = a_1 + \dots + a_n$ , then

$$(1) \quad \sum_{n=1}^{\infty} \left(\frac{A_n}{n}\right)^p < \left(\frac{p}{p-1}\right)^p \sum_{n=1}^{\infty} a_n^p$$

unless all the  $a$  are 0. The constant is best possible.

The inequality (1) is now known in the literature as Hardy's inequality. Hardy deduced (1) from the corresponding inequality for integrals and was unable to fix the constant on the right side in (1). The exact constant in (1) was determined by Landau in [7]. A number of alternative proofs and various generalizations of Hardy's inequality (1) have been given by several investigators, see [1-10], and the references therein. The main purpose of the present paper is to establish two new inequalities which claim their origin to the Hardy's inequality given in (1). Our proofs are simple and based on the idea used by Elliot in [2] to obtain the alternative proof of the Hardy's inequality given in (1).

**2. Main results.**

In what follows, we denote by  $N$  the set of natural numbers. Throughout, we assume that all the sums exist on the respective domains of definition and agree that the value of any function  $u(n)$  for  $n=0$  is zero, as well as the value of any function  $u(m,n)$  for  $m=0$  or  $n=0$  is zero.

Our first result is given in the following theorem:

**Theorem 1.** If  $p > 1$  is a constant,  $a(n) \geq 0$  for  $n \in N$  and

$$(2) \quad A(n) = \frac{1}{n} \sum_{m_1=1}^n \frac{1}{m_1} \sum_{m_2=1}^{m_1} \frac{1}{m_2} \cdots \sum_{m_{r-1}=1}^{m_{r-2}} \frac{1}{m_{r-1}} \sum_{m_r=1}^{m_{r-1}} a(m_r)$$

for  $n \in N$  with  $m_0 = n$ , then

$$(3) \quad \sum_{n=1}^{\infty} A^p(n) \leq \left(\frac{p}{p-1}\right)^{rp} \sum_{n=1}^{\infty} a^p(n).$$

The inequality holds in (3) if  $a(n) = 0$  for  $n \in N$ .

*Proof.* If  $a(n)$  is null, then (3) is trivially true. Let us suppose that  $a(n) > 0$  for all  $n \in N$ . Let  $M \geq 1$  be any integer and define

$$(4) \quad S_M = \sum_{n=1}^M A^p(n).$$

Writing  $\alpha_1(n) = A(n)$  and using the elementary inequality

$$(5) \quad u^{k+1} + kv^{k+1} \geq (k+1)uv^k, \quad u, v \geq 0, k \geq 1,$$

we observe that

$$\begin{aligned} (6) \quad & \alpha_1^p(n) - \left(\frac{p}{p-1}\right) \frac{1}{n} \left\{ \sum_{m_2=1}^n \frac{1}{m_2} \sum_{m_3=1}^{m_2} \frac{1}{m_3} \cdots \sum_{m_{r-1}=1}^{m_{r-2}} \frac{1}{m_{r-1}} \sum_{m_r=1}^{m_{r-1}} a(m_r) \right\} \alpha_1^{p-1}(n) \\ & = \alpha_1^p(n) - \left(\frac{p}{p-1}\right) \{n\alpha_1(n) - (n-1)\alpha_1(n-1)\} \alpha_1^{p-1}(n) \\ & = \left\{1 - \left(\frac{p}{p-1}\right)n\right\} \alpha_1^p(n) + \left(\frac{p}{p-1}\right)(n-1)\alpha_1(n-1) \alpha_1^{p-1}(n) \\ & \leq \left\{1 - \left(\frac{p}{p-1}\right)n\right\} \alpha_1^p(n) + \left(\frac{p}{p-1}\right)(n-1) \frac{1}{p} \{ \alpha_1^p(n-1) + (p-1)\alpha_1^p(n) \} \\ & = \left(\frac{1}{p-1}\right) \{ (n-1)\alpha_1^p(n-1) - n\alpha_1^p(n) \}. \end{aligned}$$

By substituting  $n=1, \dots, M$  in (6) and adding the inequalities we see that

$$(7) \quad \sum_{n=1}^M \alpha_1^p(n) - \left(\frac{p}{p-1}\right) \sum_{n=1}^M \frac{1}{n} \left\{ \sum_{m_2=1}^n \frac{1}{m_2} \sum_{m_3=1}^{m_2} \frac{1}{m_3} \cdots \sum_{m_{r-1}=1}^{m_{r-2}} \frac{1}{m_{r-1}} \sum_{m_r=1}^{m_{r-1}} a(m_r) \right\} \alpha_1^{p-1}(n) \\ \leq -\left(\frac{1}{p-1}\right) M \alpha_1^p(M) \leq 0$$

From (7) and using the Hölder's inequality with indices  $p, \frac{p}{p-1}$  we get

$$(8) \quad \sum_{n=1}^M \alpha_1^p(n) \leq \left(\frac{p}{p-1}\right) \sum_{n=1}^M \frac{1}{n} \left\{ \sum_{m_2=1}^n \frac{1}{m_2} \sum_{m_3=1}^{m_2} \frac{1}{m_3} \cdots \sum_{m_{r-1}=1}^{m_{r-2}} \frac{1}{m_{r-1}} \sum_{m_r=1}^{m_{r-1}} a(m_r) \right\} \alpha_1^{p-1}(n) \\ \leq \left(\frac{p}{p-1}\right) \left\{ \sum_{n=1}^M \left\{ \frac{1}{n} \sum_{m_2=1}^n \frac{1}{m_2} \sum_{m_3=1}^{m_2} \frac{1}{m_3} \cdots \sum_{m_{r-1}=1}^{m_{r-2}} \frac{1}{m_{r-1}} \sum_{m_r=1}^{m_{r-1}} a(m_r) \right\}^p \right\}^{\frac{1}{p}} \left\{ \sum_{n=1}^M \alpha_1^p(n) \right\}^{\frac{p-1}{p}}$$

Dividing both sides of (8) by the last factor on the right and rising the

result to the  $p$ -th power we get .

$$(9) \quad \sum_{n=1}^M \alpha_1^p(n) \leq \left(\frac{p}{p-1}\right)^p \sum_{n=1}^M \left\{ \frac{1}{n} \sum_{m_2=1}^n \frac{1}{m_2} \sum_{m_3=1}^{m_2} \frac{1}{m_3} \dots \sum_{m_{r-1}=1}^{m_{r-2}} \frac{1}{m_{r-1}} \sum_{m_r=1}^{m_{r-1}} a(m_r) \right\}^p$$

From (4) and (9) we observe that

$$(10) \quad S_M \leq \left(\frac{p}{p-1}\right)^p \sum_{n=1}^M \alpha_2^p(n)$$

where

$$(11) \quad \alpha_2(n) = \frac{1}{n} \sum_{m_2=1}^n \frac{1}{m_2} \sum_{m_3=1}^{m_2} \frac{1}{m_3} \dots \sum_{m_{r-1}=1}^{m_{r-2}} \frac{1}{m_{r-1}} \sum_{m_r=1}^{m_{r-1}} a(m_r).$$

From (11) and using the inequality (5) and following the same arguments as in the proof of inequality (6) we get

$$(12) \quad \alpha_2^p(n) - \left(\frac{p}{p-1}\right)^p \left\{ \sum_{m_3=1}^n \frac{1}{m_3} \sum_{m_4=1}^{m_3} \frac{1}{m_4} \dots \sum_{m_{r-1}=1}^{m_{r-2}} \frac{1}{m_{r-1}} \sum_{m_r=1}^{m_{r-1}} a(m_r) \right\} \alpha_2^{p-1}(n) \leq \left(\frac{1}{p-1}\right)^p \{ (n-1)\alpha_2^p(n-1) - n\alpha_2^p(n) \}.$$

Now by following exactly the same arguments below the inequality (6) up to the inequality (9) we obtain

$$(13) \quad \sum_{n=1}^M \alpha_2^p(n) \leq \left(\frac{p}{p-1}\right)^p \sum_{n=1}^M \left\{ \frac{1}{n} \sum_{m_3=1}^n \frac{1}{m_3} \sum_{m_4=1}^{m_3} \frac{1}{m_4} \dots \sum_{m_{r-1}=1}^{m_{r-2}} \frac{1}{m_{r-1}} \sum_{m_r=1}^{m_{r-1}} a(m_r) \right\}^p.$$

From (10) and (13) we observe that

$$(14) \quad S_M \leq \left(\frac{p}{p-1}\right)^{2p} \sum_{n=1}^M \alpha_3^p(n)$$

where

$$(15) \quad \alpha_3(n) = \frac{1}{n} \sum_{m_3=1}^n \frac{1}{m_3} \sum_{m_4=1}^{m_3} \frac{1}{m_4} \dots \sum_{m_{r-1}=1}^{m_{r-2}} \frac{1}{m_{r-1}} \sum_{m_r=1}^{m_{r-1}} a(m_r)$$

Continuing in this way, we finally get

$$(16) \quad S_M \leq \left(\frac{p}{p-1}\right)^{rp} \sum_{n=1}^M \alpha^p(n).$$

By letting  $M$  tend to infinity in (16) we get the required inequality in (3). The proof of the theorem is complete.

**Remark 1.** In the special case, if we take  $r=1$  and denote  $A(n)$  and  $a(n)$  by  $A_n$  and  $a_n$  respectively, then the inequality established in Theorem 1 reduces to the following inequality

$$(17) \quad \sum_{n=1}^{\infty} A_n^p \leq \left(\frac{p}{p-1}\right)^p \sum_{n=1}^{\infty} a_n^p.$$

We note that the inequality (17) is a slight variant of the Hardy's

inequality given in (1).

Our next result deals with the Hardy type discrete inequality in two independent variables.

**Theorem 2.** If  $p > 1$  is a constant,  $b(m, n) \geq 0$  for  $m, n \in N$  and

$$(18) \quad B(m, n) = \frac{1}{mn} \sum_{s=1}^m \sum_{t=1}^n \frac{1}{st} \sum_{x=1}^s \sum_{y=1}^t b(x, y)$$

for  $m, n \in N$ , then

$$(19) \quad \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} B^p(m, n) \leq \left(\frac{p}{p-1}\right)^{4p} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} b^p(m, n).$$

The inequality holds in (19) if  $b(m, n) = 0$  for  $m, n \in N$ .

*Proof.* If  $b(m, n)$  is null, then (19) is trivially true. Let us suppose that  $b(m, n) > 0$  for all  $m, n \in N$ . Let  $M \geq 1$ ,  $L \geq 1$  be any integers and define

$$(20) \quad S_{ML} = \sum_{m=1}^M \sum_{n=1}^L B^p(m, n).$$

From (20) we observe that

$$(21) \quad S_{ML} = \sum_{m=1}^M m^{-p} \sum_{n=1}^L \alpha_1^p(m, n)$$

where

$$(22) \quad \alpha_1(m, n) = \frac{1}{n} \sum_{t=1}^n \frac{1}{t} \sum_{s=1}^m \frac{1}{s} \sum_{x=1}^s \sum_{y=1}^t b(x, y).$$

From (22) and using the inequality (5) we observe that

$$\begin{aligned} (23) \quad \alpha_1^p(m, n) - \left(\frac{p}{p-1}\right) \frac{1}{n} \left\{ \sum_{s=1}^m \frac{1}{s} \sum_{x=1}^s \sum_{y=1}^t b(x, y) \right\} \alpha_1^p(m, n) \\ = \alpha_1^p(m, n) - \left(\frac{p}{p-1}\right) \{ n \alpha_1(m, n) - (n-1) \alpha_1(m, n-1) \} \alpha_1^{p-1}(m, n) \\ = \left\{ 1 - \left(\frac{p}{p-1}\right) n \right\} \alpha_1^p(m, n) + \left(\frac{p}{p-1}\right) (n-1) \alpha_1(m, n-1) \alpha_1^{p-1}(m, n) \\ \leq \left\{ 1 - \left(\frac{p}{p-1}\right) n \right\} \alpha_1^p(m, n) + \left(\frac{p}{p-1}\right) (n-1) \frac{1}{p} \{ \alpha_1^p(m, n-1) + (p-1) \alpha_1^p(m, n) \} \\ = \left(\frac{p}{p-1}\right) \{ (n-1) \alpha_1^p(m, n-1) - n \alpha_1^p(m, n) \}. \end{aligned}$$

Now keeping  $m$  fixed in (23) and letting  $n = 1, \dots, L$  and adding the inequalities we have

$$(24) \quad \sum_{n=1}^L \alpha_1^p(m, n) - \left(\frac{p}{p-1}\right) \sum_{n=1}^L \frac{1}{n} \left\{ \sum_{s=1}^m \frac{1}{s} \sum_{x=1}^s \sum_{y=1}^n b(x, y) \right\} \alpha_1^p(m, n)$$

$$\begin{aligned} &\leq \left(\frac{1}{p-1}\right) \sum_{n=1}^L \{(n-1)\alpha_1^p(m, n-1) - n\alpha_1^p(m, n)\} \\ &= -\left(\frac{1}{p-1}\right) L\alpha_1^p(m, L) \leq 0. \end{aligned}$$

From (24) and using the Hölder's inequality with indices  $p, \frac{p}{p-1}$  we observe that

$$(25) \quad \sum_{n=1}^L \alpha_1^p(m, n) \leq \left(\frac{p}{p-1}\right) \sum_{n=1}^L \frac{1}{n} \left\{ \sum_{x=1}^m \frac{1}{x} \sum_{y=1}^n b(x, y) \right\} \alpha_1^{p-1}(m, n)$$

$$\leq \left(\frac{p}{p-1}\right) \left\{ \sum_{n=1}^L \left\{ \frac{1}{n} \left\{ \sum_{x=1}^m \frac{1}{x} \sum_{y=1}^n b(x, y) \right\} \right\}^p \right\}^{\frac{1}{p}} \left\{ \sum_{n=1}^L \alpha_1^p(m, n) \right\}^{\frac{p-1}{p}}$$

Dividing both sides of (25) by the last factor on the right side and raising the result to the  $p$ -th power we get

$$(26) \quad \sum_{n=1}^L \alpha_1^p(m, n) \leq \left(\frac{p}{p-1}\right)^p \sum_{n=1}^L \left\{ \frac{1}{n} \left\{ \sum_{x=1}^m \frac{1}{x} \sum_{y=1}^n b(x, y) \right\} \right\}^p.$$

From (21) and (26) we observe that

$$(27) \quad S_{ML} \leq \left(\frac{p}{p-1}\right)^p \sum_{n=1}^L n^{-p} \sum_{s=1}^M \alpha_2^p(m, n)$$

where

$$(28) \quad \alpha_2(m, n) = \frac{1}{m} \sum_{x=1}^m \frac{1}{x} \sum_{y=1}^n b(x, y).$$

From (28) and using the inequality (5) it is easy to observe that

$$(29) \quad \alpha_2^p(m, n) - \left(\frac{p}{p-1}\right) \frac{1}{m} \left\{ \sum_{x=1}^m \sum_{y=1}^n b(x, y) \right\} \alpha_2^{p-1}(m, n) \\ \leq \left(\frac{1}{p-1}\right) \{(m-1)\alpha_2^p(m-1, n) - m\alpha_2^p(m, n)\}$$

Keeping  $n$  fixed in (29) and letting  $m = 1, \dots, M$  and adding the inequalities we have

$$(30) \quad \sum_{m=1}^M \alpha_2^p(m, n) - \left(\frac{p}{p-1}\right) \sum_{m=1}^M \frac{1}{m} \left\{ \sum_{x=1}^m \sum_{y=1}^n b(x, y) \right\} \alpha_2^{p-1}(m, n) \\ \leq \left(\frac{p}{p-1}\right) \sum_{m=1}^M \{(m-1)\alpha_2^p(m-1, n) - m\alpha_2^p(m, n)\} \\ = -\left(\frac{1}{p-1}\right) M\alpha_2^p(M, n) \leq 0.$$

From (30) and by following the same procedure below (24) up to (26) we get

$$(31) \quad \sum_{m=1}^M \alpha_2^p(m, n) \leq \left(\frac{p}{p-1}\right)^p \sum_{m=1}^M \left\{ \frac{1}{m} \sum_{x=1}^m \sum_{y=1}^n b(x, y) \right\}^p$$

From (27) and (31) we observe that

$$(32) \quad S_{ML} \leq \left(\frac{p}{p-1}\right)^{2p} \sum_{m=1}^M m^{-p} \sum_{n=1}^L \alpha_3^p(m, n)$$

where

$$(33) \quad \alpha_3(m, n) = \frac{1}{n} \sum_{y=1}^n \sum_{x=1}^m b(x, y).$$

From (33) and using the inequality (5) we observe that

$$(34) \quad \alpha_3^p(m, n) - \left(\frac{p}{p-1}\right) \left\{ \sum_{x=1}^m b(x, n) \right\} \alpha_3^{p-1}(m, n) \leq \left(\frac{1}{p-1}\right) \left\{ (n-1) \alpha_3^p(m, n-1) - n \alpha_3^p(m, n) \right\}.$$

Now by following the same procedure below (23) up to (26) we get

$$(35) \quad \sum_{n=1}^L \alpha_3^p(m, n) \leq \left(\frac{p}{p-1}\right)^p \sum_{n=1}^L \left\{ \sum_{x=1}^m b(x, y) \right\}^p$$

From (32) and (35) we observe that

$$(36) \quad S_{ML} \leq \left(\frac{p}{p-1}\right)^{3p} \sum_{n=1}^L \sum_{m=1}^M \alpha_4^p(m, n)$$

where

$$(37) \quad \alpha_4(m, n) = \frac{1}{m} \sum_{x=1}^m b(x, n).$$

From (37) and using the inequality (5) we observe that

$$(38) \quad \alpha_4^p(m, n) - \left(\frac{p}{p-1}\right) b(m, n) \alpha_4^{p-1}(m, n) \leq \left(\frac{1}{p-1}\right) \left\{ (m-1) \alpha_4^p(m-1, n) - m \alpha_4^p(m, n) \right\}.$$

Now following the same procedure below (29) up to (31) we get

$$(39) \quad \sum_{m=1}^M \alpha_4^p(m, n) \leq \left(\frac{p}{p-1}\right)^p \sum_{m=1}^M b(m, n).$$

From (36) and (39) we observe that

$$(40) \quad S_{ML} \leq \left(\frac{p}{p-1}\right)^{4p} \sum_{m=1}^M \sum_{n=1}^L b(m, n)$$

By letting  $M, L$  tend to infinity in (40), we get the desired inequality in (19). The proof is complete.

**Remark 2.** If we define  $B(m, n)$  in (18) by

$$(41) \quad B(m, n) = \frac{1}{mn} \sum_{s_1=1}^m \sum_{t_1=1}^n \frac{1}{s_1 t_1} \sum_{s_2=1}^{s_1} \sum_{t_2=1}^{t_1} \dots \sum_{s_{r-1}=1}^{s_{r-2}} \sum_{t_{r-1}=1}^{t_{r-2}} \frac{1}{s_{r-1} t_{r-1}} \sum_{s_r=1}^{s_{r-1}} \sum_{t_r=1}^{t_{r-1}} b(s_r, t_r)$$

for  $m, n \in N$ , then in place of inequality (19) we get

$$(42) \quad \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} B^p(m, n) \leq \left(\frac{p}{p-1}\right)^{2rp} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} b^p(m, n)$$

The proof of inequality (42) is a natural extension of the proof of Theorem

2 given above. We note that the inequality obtained in (42) is a two independent variable version of the inequality established in Theorem 1. Furthermore, we also note that the inequality obtained in (42) can very easily be extended to the functions of several independent variables. The precise formulation of such a result is very close to that given in Theorem 2 with suitable modifications and hence we do not discuss it here.

#### References

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