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NONLINEAR DIRICHLET PROBLEMS IN PERFORATED DOMAINS

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Let Ω be any bounded domain in $\mathbb{R}^N$ and assume that for every $n \in \mathbb{N}$ there are a finite number of pairwise disjoint closed sets $F_i^{(n)}, i=1,\dots,I(n)$, contained in Ω. In the domain $\Omega^{(n)} = \Omega \setminus \bigcup_{i=1}^{I(n)} F_i^{(n)}$ we consider the quasilinear Dirichlet elliptic problem

$$(0.1) \qquad \sum_{i=1}^{N} \partial_i A_i(x,u,\nabla u) = A_0(x,u,\nabla u), \quad x \in \Omega^{(n)},$$

$$(0.2) \qquad u(x) = f(x), \qquad x \in \partial\Omega^{(n)},$$

where $f(x)$ is a given function on $\bar{\Omega}$. This problem is said to have a fine-grained boundary with respect to the family $\{F_i^{(n)}\}$. The domain $\Omega^{(n)}$ is obtained from a fixed domain by removing $I(n)$ "grains", representing, for example, physical impurities. The number of the sets $F_i^{(n)}$ grows unboundedly as $n \to \infty$, while their diameters approaches zero, or equivalently, the size of the grains decreases as the density increases. Below we formulate assumptions on $F_i^{(n)}$ implying, in particular, the required rate of convergence to zero of diameters of $F_i^{(n)}$ as $n \longrightarrow \infty$.

The complex structure of the domain $\Omega^{(n)}$ does not involve additional difficulties in the study of solvability of the problem (0.1),(0.2). Under certain conditions on the data of the problem (0.1),(0.2), the existence of solutions can be proved by the monotonicity methods for every n. However, for such problems with large n, it is practically impossible to achieve approximate methods of finding solutions and therefore, of great importance will be the question of the possibility of an approximate replacement of the problem (0.1),(0.2) by a simpler problem of the same type in a fixed domain, to the solution of which converge the solutions of the problem (0.1),(0.2) as $n \to \infty$.

In this setting, the following questions arise: elucidate conditions under which the solutions of the problem (0.1),(0.2) converge as $n \to \infty$ and determine the boundary value problem for the limit function.

The above mentioned problem can be related with the asymptotic methods and averagings for PDEs. It belongs to the current trend on weak convergence techniques for nonlinear equations [4]. A large number of investigations assumes additionally the periodic dispersion of the grains $F_i^{(n)}$ (see [1] or [9] and the homogenization method).

The first two sections of the paper develop I.V.Skrypnik's procedure [10, Chap.9] in terms of the capacity of sets $F_i^{(n)}$ rather than their diameters. The third section describes the G-convergence of solutions of the above problem proposed by A.A.Kovalesvskii [5],[6]. In these approaches no periodicity assumption on the dispersion of the grains is required. It is worth noticing the analogy between these methods and the technique of A-proper operators of W.V.Petryshyn [9]. This similarity becomes sharper in the case of the weak convergence of solutions to equations in divergence form involving multivalued maximal monotone mappings [2].

Finally, we mention that this kind of problems occur in real processes involving heterogeneous media or composite materials, for example, in elasticity or in the theory of filtrations, etc.

For the sake of simplicity, we restrict ourselves to the Hilbert space case.

§ 1. FORMULATION OF HYPOTHESES AND RESULTS

We assume that $f \in H^1(\Omega)$ and the functions $A_i : \bar{\Omega}\times\mathbb{R}\times\mathbb{R}^N \longmapsto \mathbb{R}$, $i=0,1,..N$, satisfy the following hypotheses:

a_1) $A_i(x,u,p)$ are Caratheodory functions, i.e. continuous in (u,p) for almost all $x \in \bar{\Omega}$, and measurable in x for all $(u,p) \in \mathbb{R}\times\mathbb{R}^N$; $A_j(x,u,0) = 0$ for $(x,u) \in \Omega\times\mathbb{R}$, $j = 1,\dots,N$;

a_2) there are positive constants ν, μ, ε such that for $2 \le p < \frac{2N}{N-2}$, $N > 2$, and for all values $x \in \bar{\Omega}$, $u,v \in \mathbb{R}$, $\bar{\xi},\xi \in \mathbb{R}^N$ the inequalities

$$\sum_{i=1}^{N} [A_i(x,u,\xi) - A_i(x,u,\xi)](\xi_i - \xi_i) \ge \nu\, |\xi - \xi| \,,$$

$$A_o(x,u,\xi)u \ge -(\nu - \varepsilon)|\xi|^2 - \varphi(x)(1 + |u|), \tag{1.1}$$

$$|A_i(x,u,\bar{\xi}) - A_i(x,v,\xi)| \le \mu(|u - v| + |\bar{\xi} - \xi|), \quad i = 1,\dots,n,$$

$$|A_o(x,u,\xi)| \le \mu\,(|u|^p + |\xi|^2)^{\frac{p-1}{p}} + \varphi(x),$$

hold for $\varphi \in L_r(\Omega)$ with $r > \frac{N}{2}$.

For the problem (0.1),(0.2) we define the generalized solution, i.e. the function $u_n \in H^1(\Omega^{(n)})$ such that $u_n - f \in H^1_o(\Omega^{(n)})$ and the integral identity

$$\int_{\Omega^{(s)}} \left\{ \sum_{i=1}^{N} A_i(x,u,\nabla u)\partial_i \varphi(x) + A_o(x,u,\nabla u)\, \varphi(x) \right\} dx = 0 \tag{1.2}$$

holds for an arbitrary function $\varphi \in H^1_o(\Omega^{(n)})$.

We can show that the finding of a function $u_n(x)$ satisfying the identity (1.2), reduces to the solving of the operator equation $Au_n = 0$ with a coercive operator A, verifying the condition $(S)_+$ (see, e.g.[7]). In addition, we can prove

THEOREM 1.1 [10]. *If the conditions* a_1),a_2) *are fulfilled, then the prob-*

lem (0.1),(0.2) has at least a solution $u_n(x)$ *for every* n. *Moreover, there is a constant* R, *independent of* s, *such that the bound*

$$\|u_n\|_{H^1(\Omega^{(n)})} \leq R$$

holds for all n.

In what follows, $u_n(x)$ stands for one of the possible solutions of the problem (0.1),(0.2), satisfying the above estimate. Therefore, the sequence $\{u_n(x)\}$ will be considered fixed. The functions $u_n(x)$, defined for $x \in \Omega^{(n)}$, are extended to Ω by setting $u_n(x) = f(x)$ for $x \in \bigcup_{i=1}^{I(n)} F_i^{(n)}$. The functions $u_n(x)$, obtained in this way, are defined for $x \in \Omega$, belong to $H^1(\Omega)$, and satisfy a bound of the form

$$\|u_n\|_{H^1(\Omega)} \leq R_1 \tag{1.3}$$

with a constant R_1, independent of n. By (1.3), the sequence $\{u_n(x)\}$ contains a weakly convergent subsequence and, consequently, passing to a subsequence if need be, we may assume that $u_n(x)$ converges weakly in $H^1(\Omega)$ to a function $u_o(x)$.

We formulate now the conditions on the sets $F_i^{(n)}$. Let $x_i^{(n)}$ be the center of a ball with radius $d_i^{(n)}$ such that $F_i^{(n)} \subset B(x_i^{(n)}, d_i^{(n)})$. From here on $B(x_o, \rho)$ is the ball with the radius ρ and center at x_o. By $r_i^{(n)}$ we denote the distance of $B(x_i^{(n)}, d_i^{(n)})$ from the set $\bigcup_{j\neq i} B(x_j^{(n)}, d_j^{(n)}) \cup \partial\Omega$.$^{j\neq i}$

Assume the following conditions:

b_1) $d_i^{(n)} \leq c_1 r_i^{(n)}$ and $\lim \max_{1\leq n\leq I(n)} r_i^{(n)} = 0$, where c_1 is a constant independent of i,n ;

b_2) for a continuous nondecreasing function $\alpha : [0,\infty) \longrightarrow [0,\infty)$, satisfying the conditions $\alpha(0) = 0, \frac{\alpha(t)}{t} \longrightarrow \infty$ as $t \to 0$, the inequality

$$(1.4) \qquad \sum_{i=1}^{I(n)} C(F_i^{(n)}) \left\{ \frac{C(F_i^{(n)}) + \alpha^N(d_i^{(n)})}{[r_i^{(n)}]^N} \right\} \le c_o$$

holds with a constant c_o independent of n. Here $C(F_i^{(n)})$ denotes the *capacity* of the set $F_i^{(n)}$, defined by

$$C(E) = \inf_{\varphi} \int_{B(x_o,1)} \left|\frac{\partial \varphi}{\partial x}\right|^2 dx$$

where the infimum is taken over functions $\varphi \in C_o^{\infty}(B(x_o,1))$ equal to one on E.

In order to formulate one more condition on $F_i^{(n)}$, guaranteeing the construction of a boundary value problem for $u_o(x)$, we need auxiliary functions $v_i^{(n)}$, which are defined below and play a crucial role in this method.

Let $\psi(x)$ be a function of class $C_o^{\infty}(B(0,1))$, equal to one in $B(0,\frac{1}{2})$. If $d^{(n)} <$ 1 ,then for any real k we denote by $v^{(n)}(x,k)$ a function vanishing on $\partial B(x_i^{(n)},1)$, $v_i^{(n)}(x,k) = k$ for $x \in F_i^{(n)}$ and satisfying the integral identity

$$(1.5) \qquad \sum_{j=1}^{N} \int_{\Omega_i^{(s)}} A_j(x,0,\nabla v_i^{(n)})\, \partial_j \varphi(x)\, dx = 0 \quad \forall\, \varphi \in H_o^1(\tilde{\Omega}_i^{(n)})$$

where $\tilde{\Omega}_i^{(n)} = B(x_i^{(n)},1) \setminus F_i^{(n)}$.

The existence and uniqueness of the function $v_i^{(n)}$ follow from the monotonicity methods. We take $v_i^{(n)}(x,\sigma) = \sigma\psi(x - x_i^{(n)})$ outside $\Omega_i^{(n)}$.

Assume also the following condition:

c) there is a continuous function $c(x,\sigma)$ such that for an arbitrary ball $B \subset \Omega$ the equality

$$(1.6) \quad \lim_{n \to \infty} \sum_{j \in I_n^{(B)}} \sum_{i=1}^{N} \frac{1}{\sigma} \int_{\Omega} A_j(x,0,\nabla v_j^{(n)}(x,\sigma))\, \partial_i v_j^{(n)}(x,\sigma)\, dx = \int_B c(x,\sigma)\, dx$$

holds, where the convergence to the limit in (1.6) is uniform with respect to k on any bounded interval. In (1.6) $I_n^{(B)}$ stands for the set of those indices for which $1 \le j \le I(n)$, $x_j^{(n)} \in B$.

THEOREM 1.2. *Assume that the conditions* $a_1), a_2), b_1), b_2), c)$ *are satisfied. Let* $f \in W_q^1(\Omega)$, $q > n$, *and let* $\{u_n(x)\}$ *be a sequence of solutions of the problem* (0.1),(0.2), *weakly convergent to* $u_0(x)$. *Then the sequence* $\{u_n(x)\}$ *converges strongly in* $W_p^1(\Omega)$ *for every* $p < 2$ *and the function* $u_0(x)$ *is a generalized solution of the problem*

$$\sum_{i=1}^{N} \partial_i A_i(x,u,\nabla u) - A_0(x,u,\nabla u) + c(x,f(x) - u(x)) = 0 \quad x \in \Omega,$$

$$(1.7) \qquad u(x) = f(x), \qquad x \in \partial\Omega.$$

In the proof of Theorem 1.2, I.V.Skrypnik [10] first establishes a priori estimates, used basically for the investigation of the convergence of the sequence $\{u_n(x)\}$ as well as the derivation of the limit problem. After these, an asymptotic expansion of $u_n(x)$, based on the separation of the principal term by means of the functions $v_i^{(n)}(x,k)$ is introduced. The strong convergence in $H^1(\Omega)$ of the remainder of the expansion is proved as well. The limiting process in the differential equation, based on the asymptotic expansion will be carried out, where the problem (1.7) will be established for the limit function $u_0(x)$.

We mention that an analysis of the asymptotic expansion makes it possible to clarify the effect of both of the sets $F_i^{(n)}$ and the terms contained in the original equation on the correction term $c(x,f - u)$ in (1.7).

§ 2. AVERAGING THE DIRICHLET PROBLEM FOR EQUATIONS OF HIGHER ORDER

The above results remain valid also for higher order divergence elliptic equations.

We deal with the sequence of semilinear Dirichlet problems

$$(2.1)\qquad \sum_{|\alpha|\le m} (-1)^{|\alpha|} D^{\alpha} A_{\alpha}(x,\nabla u,\ldots,\nabla^{m} u) = 0, \quad x \in \Omega^{(s)}$$

$$(2.2)\qquad D^{\alpha}\{u(x) - f(x)\} = 0, \quad x \in \partial\Omega^{(n)}, \ |\alpha| \le m-1,$$

where $\Omega^{(n)} = \Omega \setminus \bigcup_{i=1}^{I(n)} F_i^{(n)}$ are perforated domains, $F_i^{(n)}$, $i = 1,\ldots,I(n)$, are pairwise disjoint closed sets,contained in a bounded domain $\Omega \subset R^N$. Let M be the number of multi-indices $\alpha = (\alpha_1,\ldots,\alpha_N)$ of length $|\alpha| = \alpha_1+\ldots+\alpha_N \le m$.

We suppose the hypotheses:

A_1) the functions $A_{\alpha}(x,\xi)$, $|\alpha| \le m$ for $x \in \bar{\Omega}$, $\xi \in R^M$ satisfy the Caratheodory conditions;

A_2) the representation

$$(2.3)\qquad A_{\alpha}(x,\xi) = \sum_{|\beta| = m} A_{\alpha\beta}(x)\xi_{\beta} + B_{\alpha}(x,\xi) \qquad \text{for} \quad |\alpha| = m$$

hold for $x \in \bar{\Omega}$, $\xi \in R^M$ and the estimates

$$\sum_{|\alpha|\le m} A_{\alpha}(x,\xi)\xi_{\alpha} \ge \nu \sum_{|\alpha|=m} |\xi_{\alpha}|^2 - \varphi(x),$$

$$(2.4)\qquad |B_{\alpha}(x,\xi)| \le \mu \sum_{|\beta|=m} |\xi_{\beta}|^r + \mu \sum_{|\beta|<m} |\xi_{\beta}| + \varphi(x), \quad |\alpha| = m,$$

$$|A_{\alpha}(x,\xi)| \le \mu \sum_{|\beta|\le m} |\xi_{\beta}| + \varphi(x), \qquad |\alpha| < m$$

are satisfied where $A_{\alpha\beta} \in C^{0,\lambda}(\bar{\Omega})$, $\lambda \in (0,1)$, $0 < r < 1$, ν, μ are positive constants and $\varphi(x) \in L_2(\Omega)$.

We state further hypotheses, preserving the notations $d_i^{(n)}$, $r_i^{(n)}$ in § 1. Regarding the sets $F_i^{(n)}$ we suppose that there are constants c',c'', independent of i,s, such that:

B') $\lim \max_{1\le i\le I(n)} r_i^{(n)} = 0$, $d_i^{(n)} \le c' r_i^{(n)}$;

$$B'') \sum_{i=1}^{I(n)} \frac{[d_i^{(n)}]^{2(N-2m)}}{[r_i^{(n)}]^N} \le c'' \qquad \text{for } N > 2m,$$

$$\sum_{i=1}^{I(n)} \frac{[\ln d_i^{(n)}]^{-2}}{[r_i^{(n)}]^N} \le c'' \qquad \text{for } N < 2m,$$

To formulate one more hypothesis on $F_i^{(n)}$ we make use of the auxiliary functions $v_i^{(n)}(x) \in H_o^m(B(x_i^{(n)},1))$. The function $v_i^{(n)}(x)$ for $d_i^{(n)} < 1/2$ is equal to 1 for $x \in F_i^{(n)}$ and for $x \in B(x_i^{(n)},1) \setminus F_i^{(n)}$ is defined as a generalized solution of the problem

$$(2.5) \qquad \sum_{|\alpha|,|\beta|=m} A_{\alpha\beta}(x_i^{(n)})\, D^{\alpha+\beta} v(x) = 0, \qquad x \in B(x_i^{(n)},1) \setminus F_i^{(n)}$$

$$(2.6) \qquad v - g \in H_o^m(B(x_i^{(n)},1))$$

where $g \in C_o^\infty(B(x_i^{(n)},1))$ with $g(x) \equiv 1$ for $x \in B(x_i^{(n)},\frac{1}{2})$.

We suppose an additional hypothesis:

C) there is a continuous function $c(x)$ in Ω such that for any ball $B \subset \Omega$ the equality

$$\lim_{n\to\infty} \sum_{j\in I(n,B)} \sum_{|\alpha|,|\beta|=m} \int_{B(x_i^{(n)},1)} A_{\alpha\beta}(x_i^{(n)})\, D^\alpha v_i^{(n)}(x)\, D^\beta v_i^{(n)}(x) = \int_B c(x)dx$$

holds, where $I(n,B)$ denotes the set of indices j, for which $x_j^{(n)} \in B$, $1 \le j \le I(n)$.

THEOREM 2.1. *Assume the hypotheses* A_1), A_2), B'), B''), C'), $f(x) \in H^m(\Omega)$ *and that* $u_n(x)$ *is a generalized solution of the problem* (2.1),(2.2). *Let* $\tilde u_n(x)$ *be equal to* $u_n(x)$ *for* $x \in \Omega^{(n)}$ *and to* $f(x)$ *for* $x \in \Omega \setminus \Omega^{(n)}$. *Suppose that the sequence* $\tilde u_n(x)$ *converges weakly in* $H^m(\Omega)$. *Then* $\tilde u_n(x)$ *converges strongly in* $W_p^m(\Omega)$ *for any* $p < 2$, *while the function* $u_o(x)$ *is a genera-*

lized solution of the problem

$$\sum_{|\alpha|\le m} (-1)^{|\alpha|} D^{\alpha} A_{\alpha}(x,u,\dots,\nabla^m u) + c(x)u(x) = c(x)f(x), \quad x \in \Omega,$$

$$D^{\alpha}\{u(x) - f(x)\} = 0, \quad x \in \partial\Omega, \quad |\alpha| \le m-1.$$

Here $c(x)$ *is the function in the hypothesis* C).

§ 3. G-CONVERGENCE OF OPERATORS WITH DIFFERENT DOMAINS

For the problem described above, we consider an G-convergence of sequences of elliptic operators $\{A_n\}$ with various domains of definition. By definition, the G-convergence of a sequence of operators $\{A_n\}$ means the weak convergence of their inverses.

Let Ω be a bounded in $\mathbb{R}^N, N \ge 2$, with a Lipschitz boundary. Let $x_i^{(n)}$, $d_i^{(n)}$ and $r_i^{(n)}$ determined above, satisfying b_1). Then $\Omega \setminus \Omega^{(n)} = \bigcup_{i\in I(n)} B(x_i^{(n)}, d_i^{(n)})$, $B(x_i^{(n)}, d_i^{(n)}) \subset \Omega$ for any $n \in \mathbb{N}$. Moreover, setting $B_i^{(n)} = B(x_i^{(n)}, d_i^{(n)} + \frac{1}{2}r_i^{(n)})$ have $B_i^{(n)} \subset \Omega$ and $\operatorname{int} B_i^{(n)} \cap \operatorname{int} B_j^{(n)} \ne 0$ for all $i,j \in I(n)$, $i \ne j$, $n \in \mathbb{N}$.

Let $\frac{1}{p} + \frac{1}{p'} = 1$ with $p > 1$. Assume that every function $u^{(n)} \in W_p^1(\Omega^{(n)})$ admits an extension $\bar{u} \in W_p^1(\Omega)$ and that $\{\Omega^{(n)}\}$ are strongly connected domains, i.e. there is a constant k does not depend on n such that

$$\|\bar{u}\|_{W_p^1(\Omega)} \le k\,\|u\|_{W_p^1(\Omega^{(n)})}$$

and denote $X_n = W_p^1(\Omega^{(n)})$, $X = W_p^1(\Omega)$ and by X_n^*, X^* their dual spaces.

For any $n \in \mathbb{N}$, let X_n be a real separable reflexive Banach space with norm $\|\cdot\|_n$ and let X_n^* be its dual space with norm $\|\cdot\|_{*,n}$; X is a real separable Banach space with norm $\|\cdot\|$ and X^* its dual with norm $\|\cdot\|_*$. Consider a family of linear extensions $\{P_n\}$ where $P_n : X_n \longmapsto X$ for any $n \in \mathbb{N}$ and assume that there exist constants $k, \gamma \geq 1$ such that the following conditions hold:

(a) $k^{-1}\|u\|_n \leq \|P_n u\| \leq k\,\|u\|_n$ for all $n \in \mathbb{N}$ and $u \in X_n$;

(b) For each $u \in X$ there is a sequence $\{u_n\}$, $u_n \in X_n$ such that

$$\limsup \|u\|_n \leq \gamma\,\|u\| \quad \text{and} \quad P_n u_n \longrightarrow u \text{ in } X.$$

Here and subsequently, "$\longrightarrow$" and "$\rightharpoonup$" denote strong and weak convergence, respectively, and $(\cdot,\cdot)$ stands for the duality either between X and X^* or between X_n and X_n^*. We can easily prove that $\|f \circ P_n\|_{*,n} \leq \chi\,\|f\|_*$ for all $f \in X^*$ and $n \in \mathbb{N}$ with a constant $\chi > 0$ depending on k.

DEFINITION 3.1. Assume that for any $n \in \mathbb{N}$ the operators $A_n : X_n \longmapsto X_n^*$ and $A : X \longmapsto X^*$ are invertible. The sequence $\{A_n\}$ is said to be *G-convergent* to A and write $A_n \xrightarrow{G} A$ if $P_n A_n^{-1}(f \circ P_n) \longrightarrow A^{-1}f$ whenever $f \in X^*$.

This G-convergence definition is a generalization of that for operators with the same domain ([3]).

We detail now the G-convergence for nonlinear elliptic operators of divergence form in the case of Hilbert spaces X_n, X.

Let $\lambda \geq 1$ $q \geq 2$ and let $A_n : X_n \longmapsto X_n^*$ be sequence of injective operators satisfying

$$\|A_s u - A_s v\|_{*,s} \leq \lambda\,\|u - v\|_s \tag{3.1}$$

$$(A_n u - A_n v, u - v) \geq \frac{1}{\lambda}\,(1 + \|u\|_n + \|v\|_n)^{2-q}\|u - v\|^q \tag{3.2}$$

for all $u, v \in X_n$ and $n \in \mathbb{N}$. From the above conditions it follows that

$$\|A_n u\|_{*,n} \leq \mu\,(\|u\|_n + 1) \quad \text{and} \quad (A_n u, u) \geq \frac{1}{\mu}\,(\|u\|_n - 1) \quad \text{for some } \mu > 0$$

and so the operators $A_n : X_n \longmapsto X_n^*$ are invertible [7]. Moreover, we infer that $\|A_n^{-1}f\|_n \leq \mu(\|f\|_{*,n} + 1)$ and $\|A_n^{-1}f - A_n^{-1}g\| \leq 2\mu^2(1 + \|f\|_{*,n} + \|g\|_{*,n})\|f - g\|_{*,n}$ holds, for all $n \in \mathbb{N}$ and $f, g \in X_n^*$.

THEOREM 3.1 [5]. *Under assumptions* (3.1),(3.2) *on the sequence of operators* $\{A_n\}$, *there exist an increasing sequence* $\{n'\} \subseteq \{n\}$ *and an injective operator* $A:X \longmapsto X^*$ *such that* $\{A_{n'}\}$ *is* G-*convergent to* A *and the following inequalities*

$$\|Au - Av\|_{*,} \leq \lambda_o(1 + \|u\| + \|v\|)^{2-p'} \|u - v\|^{p'}$$

$$(Au - Av, u - v) \geq \frac{1}{\lambda_o}(1 + \|u\| + \|v\|)^{2-q}\|u - v\|^q$$

hold for $u,v \in X$ *with* $p' = \frac{2}{q-1}$ *and some constant* $\lambda_o > 1$, *depending only on* k, γ, χ, λ.

On the other hand, let $f_n \in X_n^*$ for all $n \in \mathbb{N}$ and $f \in X^*$. Then the sequence $\{f_n\}$ converges to f means $\lim_{n \to \infty} \|f_n - f \circ P_n\|_{*,n} = 0$. The relationship between the G-convergence of operators $\{A_n\}$ and the convergence of solutions of the corresponding operator equations is established by

THEOREM 3.2 [5]. *Let* $u_n = A_n^{-1} f_n$ *for all* $n \in \mathbb{N}$ *and* $u = A^{-1}f$. *Assume that assumptions* (3.1),(3.2) *are satisfied and the sequence* $\{A_n\}$ *is* G-*convergent to* A. *If* $\{f_n\}$ *converges to* f *then* $P_n u_n \rightharpoonup u$ *in* X.

Let us consider now for any $s \in \mathbb{N}$ the perturbation $T_s : X_s \longmapsto X_s^*$ and $T:X \longmapsto X^*$ satisfying the following condition:

(D) For any $v_n \in X_n$, $n \in \mathbb{N}$ and $v \in X$ with $P_n v_n \rightharpoonup v$ in X it follows that

$$\lim_{n \to \infty} \|T_n v_n - (Tv) \circ P_n\|_{*,n} = 0.$$

(E) There are $u \in X$ and $f \in X^*$, such that $P_n u_n \rightharpoonup u$ in X, where $u_n \in X_n$ is a solution of the perturbed equation

$$A_n u_n + T_n u_n = f_n \quad \text{with } f_n \in X_n^* \text{ for any } n \in \mathbb{N}$$

and the sequence $\{f_n\}$ converges to $f \in X^*$.

THEOREM 3.3 [6]. *Let* $A:X \longmapsto X^*$ *be an invertible operator and* $\{A_n\}$ *a* G-*convergent sequence to* A. *Assume that perturbations* T_n *for any* $n \in \mathbb{N}$ *and* T *satisfy conditions* (D),(E). *Then* $Au + Tu = f$.

Assume further that for any $n \in \mathbb{N}$ there are sets V_n in X_n and W in X with the properties:

(F) Let $P_n v_n \longrightarrow v$ in X, with $v \in X$ and $v_n \in X_n$ for all $n \in \mathbb{N}$. If $v_{n'} \in V_{n'}$ for a subsequence $\{n'\} \subseteq \{n\}$, then $v \in V$;

(G) For any $v \in V$ there is a sequence $v_n \in V_n$ such that

$$\lim_{n \to \infty} \| v_n - A_n^{-1}((Av) \circ P_n\|_n = 0.$$

A simple example of nonempty sets satisfying assumption (F) are

$$V_n = \{ u \in H_o^m(\Omega^{(n)}) \mid \|u\|_{m-1} \le 1\} \quad \text{and} \quad V = \{ u \in H_o^m(\Omega) \mid \|u\|_{m-1} \le 1\}.$$

THEOREM 3.4 [6]. *Let* $A: X \longmapsto X^*$ *be an invertible operator and* $\{A_n\}$ *a G-convergent sequence to* A. *Assume there are sets* $\{V_n\}$, V *satisfying conditions* (F),(G) *and the sequence* $\{f_s\}$, $f_n \in X_n^*$ *convergent to* $f \in X^*$ *and there is* $u_n \in V_n$ *for any* $n \in \mathbb{N}$ *such that the inequality*

$$(A_n u_n - f_n, v - u_n) \ge 0 \quad \forall \ v \in V_n, \quad n \in \mathbb{N}$$

holds. Then, there exists an element $u \in V$ *such that*

(a) $(Au - f, v - u) \ge 0 \quad \forall \ v \in V$;

(b) $P_n u_n \longrightarrow u$ in X;

(c) $\lim_{n \to \infty} \|A_n u_n - (Au) \circ P_n\| = 0$.

In the case when A satisfies inequalities of the form (3.1),(3.2), it is an invertible operator and the sequence $\{A_n\}$, $A_n: X_n \longmapsto X_n^*$ can be defined by

$$A_n u = (A(P_n u) \circ P_n \quad \forall \ n \in \mathbb{N}.$$

Moreover, if for any $u \in X$ there is a sequence $\{u_n\}$, $u_n \in X_n$ so that $P_n u_n \longrightarrow u$ the sequence $\{A_n\}$ is G-convergent to A.

Finally, we will characterize the corrector for divergence equation (0.1), (0.2) in the simpler case $A_i \in C(\mathbb{R}^N)$, $A_i(0) = 0$ and $\lambda > 0$ such that

$$\sum_{i=1}^{N} (A_i(\bar{\xi}) - A_i(\xi)) \le \lambda \ |\bar{\xi} - \xi|,$$

$$\sum_{i=1}^{N} (A_i(\bar{\xi}) - A_i(\xi))(\bar{\xi}_i - \xi_i) \geq \frac{1}{\lambda} |\bar{\xi} - \xi|^2$$

for $\bar{\xi}, \xi \in \mathbb{R}^N$. The corresponding operator $A:\overset{\circ}{H}{}^1(\Omega) \longmapsto H^{-1}(\Omega)$ is defined by

$$(Au,v) = \int_\Omega \left\{ \sum_{i=1}^{N} A_i(\nabla u)\, \partial_i v \right\} dx \quad \forall\ u,v \in \overset{\circ}{H}{}^1(\Omega)$$

and similarly, $A_n:\overset{\circ}{H}{}^1(\Omega_n) \longmapsto H^{-1}(\Omega_n)$ replacing Ω by Ω_n for any $n \in \mathbb{N}$. Under the above hypotheses, the operators A_n are invertible.

Let $\delta = \operatorname{diam} \Omega$ and for $\sigma \in \mathbb{R}$, $n \in \mathbb{N}$ and $j \in I(n)$ denote by $u(\sigma,n,j)$ a generalized solution of the problem

$$\sum_{i=1}^{N} \partial_i A_i(\nabla u) = 0, \qquad x \in B(x_n^j,\delta) \setminus B(x_n^j,r_n^j)$$

$$u(x) = 0 \qquad \text{for } |x - x_n^j| = \delta,$$

$$u(x) = \sigma \qquad \text{for } |x - x_s^j| = r_s^j,$$

which belongs to $H^1(B(x_n^j,\delta) \setminus B(x_n^j,r_n^j))$.

Now, assume that hypothesis $B_2)$ is satisfied and there is a $h \in C(\bar{\Omega} \times \mathbb{R})$ and $\mu > 0$ such that

$$|h(x,\sigma)| \leq \mu|\sigma| \quad \text{and} \quad (h(x,\bar{\sigma}) - h(x,\sigma))(\bar{\sigma} - \sigma) \geq 0 \quad \forall\ x \in \Omega,\ \bar{\sigma},\sigma \in \mathbb{R}.$$

For any open cube Q, such that $\bar{Q} \subset \Omega$ sppose that

$$\lim_{n \to \infty} \sum_{j \in I(n)} \sum_{i=1}^{N} \frac{1}{\sigma} \int_\Omega A_i(\nabla u(\sigma,n,j))\partial_i u(\sigma,n,j)\, dx = \int_Q h(x,\sigma)\, dx$$

and the limit is uniform with respect to σ on any bounded interval. Set

$$(Tu,v) = - \int_\Omega h(x,-u)\, v\, dx \qquad \forall\ u,v \in \overset{\circ}{H}{}^1(\Omega).$$

The operator $T:\overset{\circ}{H}{}^1(\Omega) \longmapsto H^{-1}(\Omega)$ so defined meets hypotheses (D),(E).Moreover, we can prove ([6]) that the sequence $\{A_n\}$ is G-convergent to the operator $A + T$. This is the operator form of Theorem 1.2.

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