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A DISTRIBUTIONAL EQUATION RELATED TO RAO'S QUADRATIC ENTROPY

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Abstract

In this paper a method will be developed to define and solve the functional equation $f\left(\frac{x+y}{2}\right)+f\left(\frac{x-y}{2}\right)=2f\left(\frac{x}{2}\right)+2f\left(\frac{y}{2}\right)+\lambda f(x)f(y)$ in distributions. This equation was used by Lau to characterize Rao's quadratic entropies. It will be shown that the distributional equation reduces to the classical functional equation when the solutions are regular distributions.

1 Introduction

In connection with a characterization of Rao's quadratic entropies ([9] and [10]), Lau [7] solved the following functional equation

$$f\left(\frac{x+y}{2}\right)+f\left(\frac{x-y}{2}\right)=2f\left(\frac{x}{2}\right)+2f\left(\frac{y}{2}\right)+\lambda f(x)f(y), \tag{1.1}$$

where $f\colon \mathbb{R} \to \mathbb{R}$ ($\mathbb{R} = (-\infty,\infty)$) is an unknown function, and $\lambda \geq 0$. In [7], the solution was obtained by assuming that the function f is even, continuous, nonnegative on $[-1,1]$, and infinitely differentiable on $(-1,1)$. These conditions later were shown to be unnecessary, and the solution of equation (1.1) was obtained in a more abstract setting by Chung, Ebanks, Ng and Sahoo in [1].

A method of solving a functional equation is to assume differentiability of the underlying function. If in Equation (1.1) we assume, for example, that f is a thrice differentiable real function, then by applying $\frac{\partial^2}{\partial x^2}$ and $\frac{\partial^2}{\partial y^2}$ and comparing the two resulting equations we observe that

$$\frac{1}{2}f''(x) + \lambda f''(x)f(y) = \frac{1}{2}f''(y) + \lambda f(x)f''(y).$$

If $\lambda = 0$ this implies that f'' is a constant. If $\lambda \neq 0$, apply $\frac{\partial^2}{\partial x \partial y}$ to the last equation to deduce that

$$\lambda f'''(x)f'(y) = \lambda f'(x)f'''(y),$$

from which it follows that $f''' = cf$ for some constant c. Solving this equation and substituting back into (1.1) we find all the thrice differentiable solutions. The assumption of differentiability is typically unnatural and can be waived if we study Equation (1.1) in distributions.

In this paper we shall reformulate and solve Equation (1.1) in the domain of distributions. Since the introduction by Schwartz [11] and Sobolev [12], the theory of distributions has found numerous applications in differential equations and quantum physics. The advantage of using distributions becomes more conspicuous especially when the unknown function f has possible deficiency. The study of functional equations in the domain of distributions has probably been initiated by Fenyő [4], and followed by others [8], [6], [2] and [3].

After introducing some notation in the next section, we shall define a few distributional operators in Section 3 that mirror the functional operations such as addition, multiplication, and linear combination of the variables. In Sections 4 and 5, equation (1.1) will be reformulated and solved in the domain of distributions. It turns out that this distributional equation reduces to the classical functional equation when the solutions are regular distributions.

2 Notation

Let $\mathbb{R}$ be the field of all real numbers $\mathbb{R} = (-\infty, \infty)$, and $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$. $\mathcal{E}(\mathbb{R})$ and $\mathcal{E}(\mathbb{R})$ represent the spaces of infinitely differentiable functions on

$\mathbb{R}$ and $\mathbb{R}^2$, whereas $\mathcal{E}'(\mathbb{R})$ and $\mathcal{E}'(\mathbb{R}^2)$ are the duals of $\mathcal{E}(\mathbb{R})$ and $\mathcal{E}(\mathbb{R}^2)$, respectively. Similarily, $\mathcal{D}(\mathbb{R})$ and $\mathcal{D}(\mathbb{R}^2)$ designate the spaces of infinitely differentiable functions on $\mathbb{R}$ and $\mathbb{R}^2$ with compact support, and $\mathcal{D}'(\mathbb{R})$ and $\mathcal{D}'(\mathbb{R}^2)$ are their duals. There is an important relation among these spaces: $\mathcal{D}(\mathbb{R}) \subset \mathcal{E}(\mathbb{R}) \subset \mathcal{E}'(\mathbb{R}) \subset \mathcal{D}'(\mathbb{R})$, [11].

Next we define $L_{Loc}(\mathbb{R})$ and $L_{Loc}(\mathbb{R}^2)$ to be the spaces of equivalence classes of locally integrable functions on $\mathbb{R}$ and $\mathbb{R}^2$, respectively. The regular distribution corresponding to a locally integrable function $f \in L_{Loc}(\mathbb{R})$ will be denoted by λ_f. We have

$$< \lambda_f, \phi >= \int_{\mathbb{R}} f(x)\phi(x)dx \tag{2.1}$$

for any $\phi \in \mathcal{D}(\mathbb{R})$.

D is the differentiation operator defined on $\mathcal{D}'(\mathbb{R})$. On the space $\mathcal{D}'(\mathbb{R}^2)$ we need two differentiation operators D_1 and D_2 to specify the partial differentiation with respect to the first and second variable from $\mathbb{R}^2$. These symbols will also be used to denote the differentiation operators on $\mathcal{D}(\mathbb{R})$ and $\mathcal{D}(\mathbb{R}^2)$, the subspaces of $\mathcal{D}'(\mathbb{R})$ and $\mathcal{D}'(\mathbb{R}^2)$.

3 Some Linear Operators on $\mathcal{D}$

We define two integration operators H_1 and H_2 from $\mathcal{D}(\mathbb{R}^2)$ to $\mathcal{D}(\mathbb{R})$ by

$$H_1[\phi](x) = 2\int_{\mathbb{R}} \phi(2x, y)dy, \tag{3.1}$$

and

$$H_2[\phi](y) = 2\int_{\mathbb{R}} \phi(x, 2y)dx, \tag{3.2}$$

respectively, for any $\phi \in \mathcal{D}(\mathbb{R}^2)$.

It is obvious that H_1 and H_2 are continuous linear operators on $\mathcal{D}(\mathbb{R}^2)$ and we shall denote this by membership in $L[\mathcal{D}(\mathbb{R}^2); \mathcal{D}(\mathbb{R})]$. Their adjoints H_1^* and H_2^* again are continuous linear operators from $\mathcal{D}'(\mathbb{R})$ to $\mathcal{D}'(\mathbb{R}^2)$ and are given by

$$< H_1^*[T], \phi >=< T, H_1[\phi] >=< T(x), 2\int_{\mathbb{R}} \phi(2x, y)dy > \tag{3.3}$$

and

$$< H_2^*[T], \phi > = < T, H_2[\phi] > = < T(y), 2\int_{\mathbb{R}} \phi(x, 2y)dx > \qquad (3.4)$$

for any $\phi \in \mathcal{D}(\mathbb{R}^2)$, and $T \in \mathcal{D}'(\mathbb{R})$.

Proposition 3.1 *The following is a summary of the properties of the operators H_1^* and H_2^*.*

1. *If $T \in \mathcal{D}'(\mathbb{R})$, then*

$$D_1 H_1^*[T] = \frac{1}{2} H_1^*[DT] \quad \text{and} \quad D_2 H_1^*[T] = 0,$$
$$D_2 H_2^*[T] = \frac{1}{2} H_2^*[DT] \quad \text{and} \quad D_1 H_2^*[T] = 0,$$

2. *If $f \in L_{Loc}(\mathbb{R})$ then $H_1^*[\lambda_f] \in L_{Loc}(\mathbb{R}^2)$, and $H_1^*[\lambda_f] = f\left(\frac{x}{2}\right)$.*

3. *If $f \in L_{Loc}(\mathbb{R})$ then $H_2^*[\lambda_f] \in L_{Loc}(\mathbb{R}^2)$, and $H_2^*[\lambda_f] = f\left(\frac{y}{2}\right)$.*

Proof of the first line of Part 1: For any $\phi \in \mathcal{D}(\mathbb{R}^2)$,

$$\begin{aligned}
< D_1 H_1^*[T], \phi > &= < H_1^*[T], -D_1[\phi] > \\
&= < T, 2\int_{\mathbb{R}} -D_1\phi(2x, y)dy > \\
&= < T, -2\frac{\partial}{\partial x}\int_{\mathbb{R}} \phi(2x, y)\frac{1}{2}dy > \quad \text{by chain rule} \\
&= < DT, \frac{1}{2}H_1[\phi] > \\
&= < \frac{1}{2}H_1^*[DT], \phi > .
\end{aligned}$$

$$\begin{aligned} < D_2H_1^*[T], \phi > &= < H_1^*[T], -D_2\phi > \\ &= < T, H_1[-D_2\phi] > \\ &= < T(x), 2\int_{\mathbb{R}} -D_2\phi(2x,y)dy > \\ &= < T(x), -2\int_{\mathbb{R}} \frac{\partial}{\partial y}\phi(2x,y)dy > \\ &= < T(x), -2\phi(2x,y) \mid_{y=-\infty}^{y=\infty} > \\ &= < T(x), 0 >= 0, \end{aligned}$$

since ϕ has compact support.

The second line of Part 1 is proved analogously.

Proof of Part 2: For any $\phi \in \mathcal{D}(\mathbb{R}^2)$,

$$\begin{aligned} < H_1^*[\lambda_f], \phi > &= < \lambda_f, H_1[\phi] > \\ &= 2\int_{\mathbb{R}^2} f(x)\phi(2x,y)dxdy.. \end{aligned}$$

Set $u = 2x$ and $v = y$. Then $2dxdy = dudv$. Hence

$$\begin{aligned} < H_1^*[\lambda_f], \phi > &= \int_{\mathbb{R}^2} f\left(\frac{u}{2}\right)\phi(u,v)dudv \\ &= < f\left(\frac{x}{2}\right), \phi(x,y) > \end{aligned}$$

Thus, for $f \in L_{Loc}(\mathbb{R})$, $H_1^*(\lambda_f) = f\left(\frac{x}{2}\right)$.

Proof of Part 3: The proof is analogues to that of Part 2 and hence we shall omit it.

In order to write equation (1.1) in distributions we need to define the operators M_+, M_-, and their adjoints, as well as the tensor product operator P.

Let M_+ be an operator from $\mathcal{D}(\mathbb{R}^2)$ to $\mathcal{D}(\mathbb{R})$ given by

$$M_+[\phi](x) = 2\int_{\mathbb{R}} \phi(2x - y, y)dy. \tag{3.5}$$

for any $\phi \in \mathcal{D}(\mathbb{R}^2)$. We note that $M_+ \in L[\mathcal{D}(\mathbb{R}^2); \mathcal{D}(\mathbb{R})]$. The adjoint of M_+ is designated by M_+^*, which is in $L[\mathcal{D}'(\mathbb{R}); \mathcal{D}'(\mathbb{R}^2)]$ and is defined by

$$< M_+^*[T], \phi > = < T, M_+[\phi] > \tag{3.6}$$

for any $\phi \in \mathcal{D}(\mathbb{R}^2)$, and $T \in \mathcal{D}'(\mathbb{R})$.

We also define the operator M_- from $\mathcal{D}(\mathbb{R}^2)$ to $\mathcal{D}(\mathbb{R})$ by

$$M_-[\phi](x) = 2\int_{\mathbb{R}} \phi(2x+y, y)dy. \tag{3.7}$$

for any $\phi \in \mathcal{D}(\mathbb{R}^2)$. We note that M_- is also in $L[\mathcal{D}(\mathbb{R}^2); \mathcal{D}(\mathbb{R})]$. The adjoint M_-^*: $\mathcal{D}'(\mathbb{R}) \to \mathcal{D}'(\mathbb{R}^2)$ is defined by

$$< M_-^*[T], \phi > = < T, M_-[\phi] > \tag{3.8}$$

for any $\phi \in \mathcal{D}(\mathbb{R}^2)$, and $T \in \mathcal{D}'(\mathbb{R})$.

Proposition 3.2 *The operators M_+^* and M_-^* have the following properties.*

1. *If $f \in L_{Loc}(\mathbb{R})$ then $M_+^*[\lambda_f] \in L_{Loc}(\mathbb{R}^2)$, and $M_+^*[\lambda_f] = f\left(\frac{x+y}{2}\right)$.*

2. *If $f \in L_{Loc}(\mathbb{R})$ then $M_-^*[\lambda_f] \in L_{Loc}(\mathbb{R}^2)$, and $M_-^*[\lambda_f] = f\left(\frac{x-y}{2}\right)$.*

3. $D_1 M_+^*[T] = D_2 M_+^*[T] = \frac{1}{2}M_+^*[DT]$.

4. $D_1 M_-^*[T] = \frac{1}{2}M_-^*[DT]$ *and* $D_2 M_-^*[T] = -\frac{1}{2}M_-^*[DT]$.

Proof of Part 1: For any $\phi \in \mathcal{D}(\mathbb{R}^2)$,

$$\begin{aligned} < M_+^*[\lambda_f], \phi > &= < \lambda_f, M_+[\phi] > \\ &= 2\int_{\mathbb{R}^2} f(x)\phi(2x-y, y)dxdy. \end{aligned}$$

Set $u = 2x - y$ and $v = y$. Then $2dxdy = dudv$. Hence

$$\begin{aligned} < M_+^*[\lambda_f], \phi > &= \int_{\mathbb{R}^2} f\left(\frac{u+v}{2}\right)\phi(u,v)dudv \\ &= < f\left(\frac{x+y}{2}\right), \phi(x,y) > \end{aligned}$$

Thus, for $f \in L_{Loc}(\mathbb{R})$, $M_+^*(\lambda_f) = f\left(\frac{x+y}{2}\right)$.

Proof of Part 2: For any $\phi \in \mathcal{D}(\mathbb{R}^2)$,

$$\begin{aligned} < M_-^*[\lambda_f], \phi > &= < \lambda_f, M_-[\phi] > \\ &= 2\int_{\mathbb{R}^2} f(x)\phi(2x+y,y)dxdy. \end{aligned}$$

Set $u = 2x + y$ and $v = y$. Then $2dxdy = dudv$. Hence

$$\begin{aligned} < M_-^*[\lambda_f], \phi > &= \int_{\mathbb{R}^2} f\left(\frac{u-v}{2}\right)\phi(u,v)dudv \\ &= < f\left(\frac{x-y}{2}\right), \phi(x,y) > \end{aligned}$$

Thus, for $f \in L_{Loc}(\mathbb{R})$, $M_-^*(\lambda_f) = f\left(\frac{x-y}{2}\right)$.

Proof of Part 3: For any $\phi \in \mathcal{D}(\mathbb{R}^2)$,

$$\begin{aligned} < D_1 M_+^*[T], \phi > &= < M_+^*[T], -D_1\phi > \\ &= < T, M_+[-D_1\phi] > \\ &= < T, 2\int_{\mathbb{R}} -D_1\phi(2x-y,y)dy > \\ &= < T, 2\int_{\mathbb{R}} -\frac{\partial}{\partial x}\phi(2x-y,y)\frac{1}{2}dy > \quad \text{by chain rule} \\ &= < T, -2\frac{\partial}{\partial x}\int_{\mathbb{R}} \phi(2x-y,y)\frac{1}{2}dy > \\ &= < DT, \frac{1}{2}M_+[\phi] > \\ &= < \frac{1}{2}M_+^*[DT], \phi > . \end{aligned}$$

and

$$\begin{aligned} < D_2 M_+^*[T], \phi > &= < M_+^*[T], -D_2\phi > \\ &= < T, M_+[-D_2\phi] > \\ &= < T, 2\int_{\mathbb{R}} -D_2\phi(2x-y,y)dy > \end{aligned}$$

Let $u = 2x - y$, then $y = 2x - u$ and $dy = -du$.

$$\begin{aligned} < D_2 M_+^*[T], \phi > &= < T, 2\int_{\mathbb{R}} D_2\phi(u, 2x-u)du > \\ &= < T, 2\int_{\mathbb{R}} \frac{\partial}{\partial x}\phi(u, 2x-u)\frac{1}{2}du > \quad \text{by chain rule} \\ &= < -DT, 2\int_{\mathbb{R}} \phi(u, 2x-u)\frac{1}{2}du > \\ &= < -DT, 2\int_{\mathbb{R}} \phi(2x-y, y)\frac{1}{2}(-dy) > \\ &= < -DT, -\frac{1}{2}M_+[\phi] > \\ &= < \frac{1}{2}M_+^*[DT], \phi > . \end{aligned}$$

The proof of Part 4 is similar, and hence we omit it.

In order to find a distributional version of equation (1.1) a tensor product operator needs to be defined on $\mathcal{D}'(\mathbb{R}) \times \mathcal{D}'(\mathbb{R})$ (see [5]). We define the linear operator P: $\mathcal{D}'(\mathbb{R}) \times \mathcal{D}'(\mathbb{R}) \to \mathcal{D}'(\mathbb{R}^2)$ by

$$< P[T,S], \phi > = < T(x), < S(y), \phi(x,y) >> = < S(y), < T(x), \phi(x,y) >> \tag{3.9}$$

for any T, S in $\mathcal{D}'(\mathbb{R})$ and any $\phi \in \mathcal{D}(\mathbb{R}^2)$.

Proposition 3.3 *If T and S are regular distributions, that is, there are functions f and g in $L_{Loc}(\mathbb{R})$ such that $\lambda_f = T$ and $\lambda_g = S$, then*

$$P[\lambda_f, \lambda_g] = f(x)g(y).$$

Proof: The definition of the tensor product implies that

$$\begin{aligned} < P[\lambda_f, \lambda_g], \phi > &= < \lambda_f, < \lambda_g, \phi(x,y) >> \\ &= \int_{\mathbb{R}^2} f(x)g(y)\phi(x,y)dydx \\ &= < f(x)g(y), \phi(x,y) >, \end{aligned}$$

for any $\phi \in \mathcal{D}(\mathbb{R}^2)$.

Therefore, the product operator P will be used to represent the product $f(x)g(y)$ when the equation is interpreted in distributions. The following proposition describes some important properties of the operator P. We refer the reader to [2] for a proof.

Proposition 3.4 *For any T, S, $U \in \mathcal{D}'(\mathbb{R})$,*

1. *$D_1P[S;T] = P[DS;T]$ and $D_2P[S;T] = P[S;DT]$.*
2. *Suppose S, T, U, and V are nonzero distributions in $\mathcal{D}'(\mathbb{R})$. Then $P[S;T] = P[U;V]$ if and only if there exist nonzero real numbers c_1 and c_2 such that $S = c_1U$ and $T = c_2V$.*

Remark: The operators M_+^*, M_-^*, H_1^* and H_2^* are explicit and constructive "pullback" operators of Hörmander [5], and that our product operator P is indeed a tensor product operator akin to Hörmander's operator $\otimes$.

4 The Functional Equation in Distributions

In this section we shall employ the operators H_1^*, H_2^*, $M_\pm^*$ and the tensor product P we introduced in the last section to reformulate equation (1.1) in the domain of distributions.

Let F be in $\mathcal{D}'(\mathbb{R}^2)$. The following equation is the distributional analog of equation (1.1)

$$M_+^*[F] + M_-^*[F] = 2H_1^*[F] + 2H_2^*[F] + \lambda P[F;F]. \tag{4.1}$$

The next proposition shows that (4.1) reduces to (1.1) in case F is a regular distribution.

Proposition 4.1 *If F is a regular distribution, that is, there is a function $f \in L_{Loc}(\mathbb{R})$ such that $\lambda_f = F$, then equation (4.1) reduces to (1.1).*

Proof: If equation (4.1) holds true for a regular distribution $F = \lambda_f$, that is,

$$M_+^*[\lambda_f] + M_-^*[\lambda_f] = 2H_1^*[\lambda_f] + 2H_2^*[\lambda_f] + \lambda P[\lambda_f;\lambda_f]$$

then for any $\phi \in \mathcal{D}(I^2)$,

$$< M_+^*[\lambda_f] + M_-^*[\lambda_f] - 2H_1^*[\lambda_f] - 2H_2^*[\lambda_f] - \lambda P[\lambda_f;\lambda_f], \phi >= 0. \tag{4.2}$$

It follows from Propositions 3.1, 3.2 and 3.3, and equation (4.2) that

$$\int_{\mathbb{R}^2}\left[f\left(\frac{x+y}{2}\right)+f\left(\frac{x-y)}{2}\right)-2f\left(\frac{x}{2}\right)-2f\left(\frac{y}{2}\right)-\lambda f(x)f(y)\right]\phi(x,y)dxdy=0.$$

We thus conclude that

$$f\left(\frac{x+y}{2}\right)+f\left(\frac{x-y)}{2}\right)-2f\left(\frac{x}{2}\right)-2f\left(\frac{y}{2}\right)-\lambda f(x)f(y)=0$$

almost everywhere.

5 Solution to the Distributional Equation

In this section we shall use the operators and the properties which we discussed above to solve the distributional equation. We shall also point out that this distributional equation reduces to the classical equation (1.1) when the solutions are regular distributions.

Depending on whether $\lambda = 0$ or not, equation (4.1) leads to two different solutions.

Case 1: $\lambda \neq 0$.

Apply D_1^2 and D_2^2 to equation (4.1) and use the properties developed in Propositions 3.1, 3.2, and 3.4. We obtain the system

$$\begin{aligned}\frac{1}{4}M_+^*[D^2F]+\frac{1}{4}M_-^*[D^2F] &= \frac{1}{2}H_1^*[D^2F]+\lambda P[D^2F;F],\\ \frac{1}{4}M_+^*[D^2F]+\frac{1}{4}M_-^*[D^2F] &= \frac{1}{2}H_2^*[D^2F]+\lambda P[F;D^2F].\end{aligned}$$

This implies that

$$\frac{1}{2}H_1^*[D^2F]+\lambda P[D^2F;F]=\frac{1}{2}H_2^*[D^2F]+\lambda P[F;D^2F]. \tag{5.1}$$

Applying D_1D_2 to equation (5.1) and using Proposition 3.1 and 3.3, we obtain $\lambda P[D^3F;DF]=\lambda P[DF;D^3F]$, or

$$P[D^3F;DF]=P[DF;D^3F], \tag{5.2}$$

since $\lambda \neq 0$. By virtue of Proposition 3.4, there is a nonzero real number c such that $D^3F = cDF$. This linear homogeneous differential equation yields

$$F = k_0 + k_1 e^{\sqrt{c}t} + k_2 e^{-\sqrt{c}t}, \tag{5.3}$$

where k_0, k_1 and k_2 are to be determined.

We have thus shown

Proposition 5.1 *If $F \in \mathcal{D}'(\mathbb{R})$ satisfies equation (4.1) with $\lambda \neq 0$, then there exists a nonzero real number c such that $F = \lambda_f$, where f satisfies the differential equation $f^{(3)} = cf'$.*

To determine k_0, k_1 and k_2, we apply the operators $M_\pm^*$, H_i^* $(i = 1, 2)$, and P to $F = k_0 + k_1 e^{\sqrt{c}t} + k_2 e^{-\sqrt{c}t}$ in (5.3). For $\phi \in \mathcal{D}(\mathbb{R}^2)$,

$$\begin{aligned} & < M_+^*[F], \phi > \\ = & < k_0 + k_1 e^{\sqrt{c}x} + k_2 e^{-\sqrt{c}x}, M_+[\phi] > \\ = & < k_0 + k_1 e^{\sqrt{c}x} + k_2 e^{-\sqrt{c}x}, 2\int_{\mathbb{R}} \phi(2x - y, y) dy > \\ = & 2\int_{\mathbb{R}^2} (k_0 + k_1 e^{\sqrt{c}x} + k_2 e^{-\sqrt{c}x}) \phi(2x - y, y) dy dx \\ = & \int_{\mathbb{R}^2} (k_0 + k_1 e^{\sqrt{c}(u+y)/2} + k_2 e^{-\sqrt{c}(u+y)/2}) \phi(u, y) dy du \\ = & < k_0 + k_1 e^{\sqrt{c}(x+y)/2} + k_2 e^{-\sqrt{c}(x+y)/2}, \phi > . \end{aligned} \tag{5.4}$$

Similarly,

$$< M_-^*[F], \phi > = < k_0 + k_1 e^{\sqrt{c}(x-y)/2} + k_2 e^{-\sqrt{c}(x-y)/2}, \phi >, \tag{5.5}$$

$$< H_1^*[F], \phi > = < k_0 + k_1 e^{\sqrt{c}x/2} + k_2 e^{-\sqrt{c}x/2}, \phi >, \tag{5.6}$$

$$< H_2^*[F], \phi > = < k_0 + k_1 e^{\sqrt{c}y/2} + k_2 e^{-\sqrt{c}y/2}, \phi >, \tag{5.7}$$

and

$$\begin{aligned} & < P[F; F], \phi > \\ = & < k_0 + k_1 e^{\sqrt{c}x} + k_2 e^{-\sqrt{c}x}, < k_0 + k_1 e^{\sqrt{c}y} + k_2 e^{-\sqrt{c}y}, \phi(x, y) > \\ = & < k_0^2 + k_0 k_1 (e^{\sqrt{c}x} + e^{\sqrt{c}y}) + k_0 k_2 (e^{-\sqrt{c}x} + e^{-\sqrt{c}y}) + k_1^2 e^{\sqrt{c}(x+y)} \\ & + k_2^2 e^{-\sqrt{c}(x+y)} + k_1 k_2 (e^{\sqrt{c}(x-y)} + e^{-\sqrt{c}(x-y)}), \phi(x, y) > . \end{aligned} \tag{5.8}$$

Substituting (5.4), (5.5), (5.6), (5.7) and (5.8) into equation (4.1) and comparing the coefficients, we see that

$$2k_0 = 4k_0 + \lambda k_0^2, \quad k_1 = k_2 = 0.$$

The first equation leads to $k_0 = 0$ or $k_0 = -2\lambda^{-1}$. Therefore, we provide the following proposition as a conclusion of Case 1: $\lambda \neq 0$.

Proposition 5.2 *If $F \in \mathcal{D}'(\mathbb{R})$ satisfies equation (4.1) with $\lambda \neq 0$, then $F = \lambda_f$ where $f(x) = 0$ or $f(x) = -2\lambda^{-1}$, for all $x \in \mathbb{R}$.*

Case 2: $\lambda = 0$.

When $\lambda = 0$ equations (4.1) and (5.1) reduce to

$$M_+^*[F] + M_-^*[F] = 2H_1^*[F] + 2H_2^*[F], \tag{5.9}$$

$$\text{and} \quad H_1^*[D^2F] = H_2^*[D^2F], \tag{5.10}$$

respectively.

Apply D_1 to equation (5.10) and use the properties in Proposition 3.1 we obtain $H_1^*[D^3F] = 0$. This equation implies that $D^3F = 0$.

Indeed, since H_1^* is a surjective map, for any $\phi \in \mathcal{D}(\mathbb{R}^2)$,

$$0 =< H_1^*[D^3F], \phi >=< D^3F, H_1[\phi] >=< D^3F, \psi >,$$

for every $\psi \in \mathcal{D}(\mathbb{R}^2)$.

The solution of the equation $D^3F = 0$ is

$$F = l_2x^2 + l_1x + l_0, \tag{5.11}$$

with the coefficients l_i $(i = 0, 1, 2)$ to be determined.

To determine the coefficients l_i $(i = 0, 1, 2)$ we substitute (5.11) into equation (5.10). For $\phi \in \mathcal{D}(\mathbb{R}^2)$,

$$\begin{aligned} & < M_+^*[F] + M_-^*[F], \phi > \\ = & < F, M_+[\phi] > + < F, M_-[\phi] > \\ = & \ 2\int_{\mathbb{R}^2} (l_2x^2 + l_1x + l_0)\phi(2x - y, y)dxdy \end{aligned} \tag{5.12}$$

$$
\begin{aligned}
&+2\int_{\mathbb{R}^2}(l_2x^2+l_1x+l_0)\phi(2x+y,y)dxdy \\
=\;& \int_{\mathbb{R}^2}\left(l_2\left(\frac{x+y}{2}\right)^2+l_1\left(\frac{x+y}{2}\right)+l_0\right)\phi(x,y)dxdy \\
&+\int_{\mathbb{R}^2}\left(l_2\left(\frac{x-y}{2}\right)^2+l_1\left(\frac{x-y}{2}\right)+l_0\right)\phi(x,y)dxdy \\
=\;& < l_2\left(\frac{x^2+y^2}{2}\right)+l_1x+2l_0,\phi > .
\end{aligned} \tag{5.13}
$$

On the other hand,

$$
\begin{aligned}
& < 2H_1^*[F]+2H_2^*[DF],\phi > \\
=\;& < 2F(x),H_1[\phi] > + < 2F(y),H_2[\phi] > \\
=\;& 2\int_{\mathbb{R}^2}(l_2x^2+l_1x+l_0)\phi(2x,y)dxdy \\
&+2\int_{\mathbb{R}^2}(l_2y^2+l_1y+l_0)\phi(x,2y)dxdy \\
=\;& \int_{\mathbb{R}^2}2\left(l_2\left(\frac{u}{2}\right)^2+l_1\left(\frac{u}{2}\right)+l_0\right)\phi(u,y)dudy \\
&+\int_{\mathbb{R}^2}2\left(l_2\left(\frac{v}{2}\right)^2+l_1\left(\frac{v}{2}\right)+l_0\right)\phi(x,v)dxdv \\
=\;& < l_2\left(\frac{x^2+y^2}{2}\right)+l_1(x+y)+4l_0,\phi >
\end{aligned} \tag{5.14}
$$

Equating (5.13) and (5.14), we conclude that $l_0 = l_1 = 0$, and l_2 can be an arbitrary constant c. Thus the solution of equation (5.9) is $F = cx^2$.

We thus conclude the case of $\lambda = 0$ with the following proposition.

Proposition 5.3 *If $F \in \mathcal{D}'(\mathbb{R})$ satisfies equation (4.1) with $\lambda = 0$, then $F = \lambda_f$ where $f(x) = cx^2$ for all $x \in \mathbb{R}$ with arbitrary constant c.*

Finally we summarize our main result in the following theorem.

Theorem 5.1 *If $F \in \mathcal{D}'(\mathbb{R})$ satisfies equation (4.1), then it is a regular distribution generated by the function $f(x) = cx^2$, if $\lambda = 0$; or, if $\lambda \neq 0$, by $f(x) = 0$ or $f(x) = -2\lambda^{-1}$, for all $x \in \mathbb{R}$.*

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