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ON UNIFORM L^p - STABILITY IN VARIATION OF INTEGRO - DIFFERENTIAL EQUATIONS

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1. Introduction

The study of stability and asymptotic behavior of solutions of ordinary differential equations has made considerable use of the variation of constants formula, both the classical linear version and the nonlinear version of Alekseev [9; Theorem 2.6.3].

The nonlinear variation of constants formula of Alekseev provides a most useful technique for investigating the effect of a perturbation on the solutions of nonlinear systems of differential equations.

In contrast to the usual Lyapunov function method, the use of this formula does not require that the unperturbed system satisfy very stringent hypotheses, and qualitative estimates are easily obtained.

Many authors have obtained results in qualitative behavior of solutions of perturbed nonlinear systems, using the nonlinear variation of constants formula of Alekseev. Since 1970 when Brauer and Strauss [2] defined a new type of stability, so-called uniform stability, considerable attention has been paid to the development of the theory of this type of stability, especially, in connection with the theory of Lyapunov function and using the formula of Alekseev.

I quote here the contributions of Athanassov [1], Dannan and Elaydi [5], Choi, Koo and Lee [3], Voskresenskij [14] and Morchalo [10]. Strauss [12] defined the concept of L^p-stability and Morchalo [11] extended it to so-called L^p-stability in variation.

In this paper, a different approach, based on the Alekseev formula and the theory of integral inequalities, is used to obtain some results on the asymptotic behavior and growth properties of solutions of perturbed nonlinear integro-differential systems.

Actually I continue the study of the relationship between different kinds of behavior for integro-differential equations begun in my earlier paper [13].

Our approach is inspired by paper of Hara, Yoneyama and Itoh [7]. The different types of stability and growth properties of perturbed nonlinear integro-differential systems are discussed and several theorems are studied. In the final of section 3, a simple example to illustrate one of our results is given.

2. Notation and general setting

We are interested in the relations between the solutions of the unperturbed system

$$(P_0)\quad x' = f(t,x)$$

and the solutions of the perturbed system

$$(P_1)\quad y' = f(t,x) + \int_0^t g(t,s,y(s))\,ds$$

respectively,

$$(P_2)\quad y' = f(t,y) + h(t,y) + \int_0^t g(t,s,y(s))\,ds.$$

Here x,y,f,g and h are elements of $\mathbb{R}^n$, an n-dimensional real Euclidian space. Let $\mathbb{R}^+$ denote the interval $[0,\infty]$ and $C(X,Y)$ denote the space of continuous functions from X to Y, where X and Y are any convenient spaces.

We shall always assume that $f,h \in \mathbf{C}(\mathbb{R}^+ \times \mathbb{R}^n, \mathbb{R}^n)$, and that f has partial derivatives $\partial f/\partial x$, on $\mathbb{R}^+ \times \mathbb{R}^n$ for all $t \in \mathbb{R}^+$. With respect to the function $g(t,s,y)$ we shall assume that it is a continuous n-vector on $0 \le s \le t < \infty$ and $|x| < H \le \infty$.

The symbol $|\cdot|$ will be used to denote arbitrary vector norm in $\mathbb{R}^n$. Throught this work $x(t;t_0,x_0)$ will denote the unique solution of (P_0), satisfying the initial condition $x(t_0;t_0,x_0) = x_0$. We shall denote by $\Phi(t,t_0,x_0)$ the fundamental matrix solution of the variational system

$$(V)\quad z' = f_x'(t, x(t;t_0,x_0))\,z$$

of (P_0) with respect to the solution $x(t;t_0,x_0)$ of (P_0) with $\Phi(t_0,t_0,x_0) = I$ (identity matrix).

It is known [9; Theorem 2.5.3], [8; Theorem 2.1.3] that

$$(2.1)\quad \phi(t,t_0,x_0) = \frac{\partial}{\partial x_0} x(t;t_0,x_0)$$

and, in addition,

$$(2.2)\quad \frac{\partial x(t;t_0,x_0)}{\partial t_0} + \phi(t,t_0,x_0)\,f(t_0,x_0) = 0.$$

We note that in case when $f(t,x) \equiv A(t)x$, where $A(t)$ is a continuous n×n matrix on $\mathbb{R}^+$, we have $\Phi(t,t_0,x_0) = X(t)X^{-1}(t_0)$, $X(t)$

being a fundamental matrix for the corresponding system

$$(L)\qquad x'=A(t)x\ .$$

We regard (P_1) and (P_2) as perturbed systems of (P_0) to obtain some new results on the asymptotic behavior of solutions of (P_1) and (P_2) using the matrix $\Phi(t,t_0,x_0)$ and the various stability and growth properties for (P_0) characterized by those of $\Phi(t,t_0,x_0)$.

The solution of (P_1) or (P_2) with initial values (t_0,φ) will be denoted by $y(t;t_0,\varphi)$, where $t_0\geq 0$ and $\varphi\in\mathbf{C}([0,t_0],\mathbf{R}^n)$.

For $\varphi\in\mathbf{C}(\mathbf{R}^+,\mathbf{R}^n)$ and $t\in\mathbf{R}^+$, define $\|\varphi\|_t=\max\{|\varphi(s)|;0\leq s\leq t\}$. Writing $y(t):=y(t;t_0,\varphi)$, in view of (2.2), we see that

$$(2.3)\qquad \frac{d}{ds}[x(t;s,y(s)]=\frac{\partial x(t;s,y(s))}{\partial s}+\frac{\partial x(t;s,y(s))}{\partial x_0}y'(s)=$$
$$=\phi(t,s,y(s)[y'(s)-f(s,y(s)]\ .$$

Noting that $x(t;t,y(t))=y(t)$ and $y'(s)-f(s,y(s))=\int_0^s g(t,\tau,y(\tau))\,d\tau$,

by integrating (2.3) from t_0 to $t\geq t_0$, we obtain

$$(2.4)\qquad y(t;t_0,\varphi)=x(t;t_0,\varphi(t_0))+\int_{t_0}^{t}\phi(t,s,y(s;t_0,\varphi))\cdot$$
$$\cdot\int_0^s g(s,\tau,y(\tau;t_0,\varphi))\,d\tau ds.$$

Also, by using the same type of reasoning as above, for $y(t):=y(t;t_0,\varphi)$ solution of (P_2), we have

$$(2.5)\qquad y(t;t_0,\varphi)=x(t;t_0,\varphi(t_0))+\int_{t_0}^{t}\phi(t,s,y(s;t_0,\varphi)\cdot$$
$$\cdot[h(s,y(s;t_0,\varphi))+\int_0^s g(s,\tau,y(\tau;t_0,\varphi))\,d\tau]\,ds.$$

Formula (2.4) (or (2.5)) will be the main tool in our analysis and it may be regarded as a variation of constants formula and represents an extension of the corresponding formula from [9; Theorem 2.6.3].

We note that under the above hypotheses made on $f(t,x)$, if $x(t;t_0,x_1)$ and $x(t;t_0,x_2)$ are two solutions of (P_0) existing for $t\geq t_0$, such that x_1,x_2 belong to a convex subset of $\mathbf{R}^n$, then [9; Theorem 2.6.4], for $t\geq t_0$

$$(2.6)\qquad x(t;t_0,x_2)-x(t;t_0,x_1)=[\int_0^1\phi(t,t_0,x_1+s(x_2-x_1))\,ds]\cdot(x_2-x_1)\ .$$

A noteworthy particular case of (2.6) is obtained for $f(t,0)\equiv 0$, namely

$$(2.7)\quad x(t;t_0,x_0)=[\int_0^1 \Phi(t,t_0,s\cdot x_0)\,ds]\,x_0\ ,\ \text{for } t\geq t_0.$$

We now give the definitions of different kinds of stability and of the concept of slowly-growing, in terms of the behavior of solutions of (P_0) and of the variational system (V).

__Definition 2.1__ The solution x=0 of (P_0) is said to be :

a) ***Uniformly stable in variation (USV)*** if there exists constant $M\geq 1$ such that $|\Phi(t,t_0,x_0)|\leq M$ for all $t\geq t_0\geq 0$, whenever $|x_0|<H$;

Here and in the sequel $H\leq\infty$.

b) ***Exponentially asymptotically stable in variation (EASV)*** if there exist constants $L\geq 1$ and $\lambda>0$ such that $|\Phi(t,t_0,x_0)|\leq Le^{-\lambda(t-t_0)}$ for all $t\geq t_0\geq 0$, whenever $|x_0| < H$;

c) ***Uniformly L^p-stable in variation ($p\geq 1$) (UL^p-SV)*** if it is (USV) and $\int_0^\infty |x(t;t_0,x_0)|^p dt<\infty$ for all $|x_0|<H$;

d) ***Generalized slowly-growing in variation (GSGV)*** if

$$|\Phi(t,t_0,x_0)|\leq L(t_0)\exp(\int_{t_0}^{t}\alpha(\tau)\,d\tau)\ ,\ \text{for all } t\geq t_0\geq 0,\ \text{whenever } |x_0|<H,$$

where $L\in C(\mathbf{R}^+,\mathbf{R}^+)$, $\alpha\in C(\mathbf{R}^+,\mathbf{R})$;

e) When $L(t_0)\equiv L\equiv \text{const.}\geq 1$, the zero solution x=0 is said to be ***generalized uniformly slowly-growing in variation (GUSGV)***.

To be in agreement with the most part of the authors concerning the terminology, we should have used, almost everywhere, the expression "globally". We have omitted it for brevity.

With regard to the definitions of different kinds of stability of the zero solution of (P_1) or (P_2) we shall use the classical ones with only slight change due to the form of the initial condition [6]. We shall assume that $g(t,s,0)\equiv 0$.

__Definition 2.2__ The zero solution y=0 of (P_1) or (P_2) is said to be:

a) ***Stable (S)***, if for every $\epsilon>0$ and any $t_0\geq 0$ there exists $\delta>0$ such that $\|\varphi\|_{t_0}<\delta$ and $t\geq t_0$ imply $|y(t;t_0,\varphi)|<\epsilon$;

b) ***Uniformly stable (US)***, if it is (S) and the above δ is independent of t_0;

c) ***Exponentially asymptotically stable (EAS)***, if there exists $\lambda>0$ and for every $\epsilon>0$ there exists $\delta>0$ such that $t_0\geq 0$, $\|\varphi\|_{t_0}<\delta$ and $t\geq t_0$ imply $|y(t;t_0,\varphi)|<\epsilon\exp(-\lambda(t-t_0))$. It is ***globally exponentially asymptotically stable (GEAS)***, if there exists $\lambda>0$ and for any $\alpha>0$ there exists $L(\alpha)>0$ such that $\|\varphi\|_{t_0}<\alpha$ and $t\geq t_0$ imply $|y(t;t_0,\varphi)|\leq L(\alpha)\exp(-\lambda(t-t_0))\|\varphi\|_{t_0}$;

d) ***Uniformly L^p-stable (UL^p-S), $p\geq 1$***, if it is (US) and $\int_{t_0}^{\infty}|y(t;t_0,\varphi)|^p dt<\infty$ for all $\|\varphi\|_{t_0}<\alpha$.

Finally we give

Definition 2.3 The solutions of (P_1) or (P_2) are ***uniformly bounded (UB)***, if for every $\alpha>0$ there exists $\beta(\alpha)>0$ such that $t_0\geq 0$, $\|\varphi\|_{t_0}<\alpha$ and $t\geq t_0$ imply $|y(t;t_0,\varphi)|\leq\beta(\alpha)$.

Remark 2.1 The above definitions may be also formulated for the zero solution $x=0$ of (P_0). In this case $\|\varphi\|_{t_0}$ will be replaced by $|x_0|$.

As regarding the functions $h(t,y)$ and $g(t,y)$, throughout this paper we shall make growth hypotheses of the following kind:

Assume that there exist continuous positive functions $a(t)$ and $b(t,s)$ for $t\geq s\geq 0$ such that for $|y|<H$, for some $H>0$

(H1) : $|g(t,s,y)|\leq b(t,s)|y|$;

(H2) : $|h(t,y)|\leq a(t)$, $|g(t,s,y)|\leq b(t,s)|y|$;

(H3) : $|g(t,s,y)|\leq b(t,s)|y|^m$, $0<m<1$;

(H4) : $|h(t,y)|\leq a(t)|y|^m$, $|g(t,s,y)|\leq b(t,s)|y|^m$, $0<m<1$;

(H5) : $|g(t,s,y)| \le a(t)+b(t,s)|y|^m$, $0<m<1$.

In view to develop different stability or growth relationships between solutions of systems (P_0) and (P_1) or (P_2) we need some integral inequality which are generalizations of Bihari's inequality.

Lema 2.1 **Let there exist continuous nonnegative functions u(t), v(t), h(t,s) such that for any $t \ge t_0$ we have**

$$u(t) \le c+\int_{t_0}^{t}[v(s)\omega(u(s))+\int_{0}^{s}h(s,\tau)\omega(u(\tau))d\tau]ds \ , \tag{2.8}$$

where $c \ge u(t_0)$ is a positive constant and $\omega(\gamma)$ is a positive and nondecreasing function for $\gamma>0$, $\omega(0)\ge 0$. Then, for any $t \ge t_0$ we have

$$\Omega(u(t)) \le \Omega(c)+\int_{t_0}^{t}[v(s)+\int_{0}^{s}h(s,\tau)d\tau]ds \ , \tag{2.9}$$

where $\Omega(z)=\int \frac{dz}{\omega(z)}$.

Proof. Denote the right hand side of (2.8) by U(t) so that $U(t_0)=c$ and $U'(t) \le v(t)\omega(u(t))+\int_{0}^{t}h(t,\tau)\omega(u(\tau))d\tau$. Since ω is nondecreasing and $u(t) \le U(t)$, we get

$$U'(t) \le \omega(U(t))\cdot[v(t)+\int_{0}^{t}h(t,\tau)d\tau] \ .$$

Integrating from t_0 to t, we obtain the desired result.

Remark 2.3 When $\omega(r)=r$, (2.9) becomes

$$u(t) \le c\cdot\exp\{\int_{t_0}^{t}[v(s)+\int_{0}^{s}h(s,\tau)d\tau]ds\} \ . \tag{2.10}$$

For $h(t,s)=b(t)c(s)$ one obtains Lemma 2.5 [3]; for $v(t)\equiv 0$, (2.9) reduces to the generalized Gronwall's inequality; if $\omega(r)=r$ and $h(t,s)\equiv 0$, one obtains from (2.9) the Gronwall classical inequality.

Using the properties of ω it follows that there exists Ω^{-1} and (2.9) may be written as

(2.11) $$u(t)\le\Omega^{-1}\left(\Omega(c)+\int_{t_0}^{t}[v(s)+\int_0^s h(s,\tau)\,d\tau]\,ds\right),\ t_0\le t\le T\ ,$$

where $T=\sup\{\ t\ge t_0;\Omega(c)+\int_{t_0}^{t}[v(s)+\int_0^s h(s,\tau)\,d\tau]\,ds\in \mathrm{dom}\ \Omega^{-1}\ \}$.

In particular case when $\omega(r)=r^m$, $0<m<1$, we obtains the estimate

(2.12) $$u(t)\le\left(c^{1-m}+(1-m)\int_{t_0}^{t}[v(s)+\int_0^s h(s,\tau)\,d\tau]\,ds\right)^{1/(1-m)},\ t\ge t_0\ .$$

Lemma 2.2 **Let there exist continuous nonnegative functions u(t), v(t), h(t,s) such that for any $t\ge t_0$ we have**

(2.13) $$u(t)\le v(t)+\int_{t_0}^{t}\int_0^s h(s,\tau)\cdot u^m(\tau)\,d\tau,\ 0<m\le 1\ .$$

Then, for any $t\ge t_0$ we have

(2.14) $$u(t)\le v(t_0)\exp\left[\int_{t_0}^{t}\int_0^s h(s,\tau)\,d\tau ds\right]+\int_{t_0}^{t}v'(s)\exp\left[\int_s^t\int_0^\tau h(\tau,\nu)\,d\nu d\tau\right]ds\ ,$$

if m=1 and v is differentiable, respectively

(2.15) $$u(t)\le\frac{v(t)}{1-m}+\left[\int_{t_0}^{t}\int_0^s h(s,\tau)\,d\tau ds\right]^{1/(1-m)}$$ **, if 0<m<1.**

Proof. Denote by $\gamma(t)=\sup\{u(s);0\le s\le t\}$ for a fixed $t\ge t_0$. Then from (2.13) we get

$$\gamma(t)\le v(t)+\int_{t_0}^{t}\left(\int_0^s h(s,\tau)\,d\tau\right)\gamma(s)\,ds.$$

Now the estimate (2.14) follows from Theorem 1.1.2 [8]. When v(t) is assumed to be nondecreasing and positive, then (2.14) reduces to ([8;p.6])

(2.14)′ $$u(t)\le v(t)\cdot\exp\left[\int_{t_0}^{t}\int_0^s h(s,\tau)\,d\tau ds\right],\ t\ge t_0.$$

To prove (2.15) (0<m<1) we first remark that from (2.13) we have

$$\gamma(t) \le v(t) + \int_{t_0}^{t} \left(\int_0^s h(s,\tau)\,d\tau\right)\gamma^m(s)\,ds.$$

From here, by using the same type of reasoning as in [11] it follows the estimate (2.15).

As immediate consequences of the two previous lemmas we have the following results:

Corollary 2.1 **Assume that (H5) holds with $0<m\le 1$ and that $\int_{t_0}^{\infty} sa(s)\,ds<\infty$ and $\int_{t_0}^{\infty}\int_0^s b(s,\tau)\,d\tau ds<\infty$. Then the solutions of (P_1) are (UB).**

Indeed, this time we have

$$|y(t;t_0,\varphi)| \le M[\|\varphi\|_{t_0} + \int_{t_0}^{t} sa(s)\,ds + \int_{t_0}^{t}\int_0^s b(s,\tau)\,|y(s;t_0),\varphi)|^m d\tau ds],\ t\ge t_0 \ .$$

By using Lemma 2.2, formula (2.15), we get

$$|y(t;t_0,\varphi)| \le M[\|\varphi\|_{t_0} + \int_{t_0}^{t} sa(s)\,ds]\,(1-m)^{-1} + [M\int_{t_0}^{t}\int_0^s b(s,\tau)\,d\tau]^{1/(1-m)},$$

from where the desired result.

To prove our assertion if m=1, we shall use Lemma 2.2, formula (2.14), since in this case $v(t)=M[\|\varphi\|_{t_0}+\int_{t_0}^{t} sa(s)\,ds]$ is nondecreasing and positive.

Corollary 2.2 **Assume that the solution x=0 of (P_0) is (USV) and that (H_4) holds with 0<m<1. Moreover, we assume that there exists a constant K>0 such that**

$$\text{(2.16)} \quad \int_0^{\infty} [a(s) + \int_0^s b(s,\tau)\,d\tau]\,ds \le K < \infty .$$

Then the solutions of (P_2) are (UB).

To arrive at the formulated result we shell use Lemma 2.1, formula (2.10) if m=1, and formula (2.12) if 0<m<1.

3. Main results

This section is devoted to the study of the relationship between the uniform L^p-stability in variation of the zero solution of (P_o) and the uniform L^p-stability of the zero solution of (P_1) or (P_2). The hypotheses which will be used are related to those from [13].

In the following we shall assume that there exists a constant K>0 such that

$$(3.1)\qquad \int_0^\infty\int_0^s b(s,\tau)\,d\tau\,ds \le K\infty .$$

Theorem 6.1 **Assume that the solution x=0 of (P_0) is (UL^p-SV) and that (H1) and (3.1) hold. Furthermore we assume that**

$$(3.2)\qquad v(t)\equiv\Big(\int_0^t\Big[\int_0^s b(s,\tau)\,d\tau\Big]^q ds\Big)^{p/q}\in L^1([t_0,\infty))\ ,\ p^{-1}+q^{-1}=1.$$

Then the zero solution of (P_1) is (UL^p-S).

Proof. According to the definition of the (UL^p-SV), the solution x=0 is (USV) and since (3.1) holds, by [13; Theorem 3.1[it follows that the zero solution of (P_1) is (US).

To show that $\int_{t_0}^\infty |y(t;t_0,\varphi)|^p dt<\infty$ whenever $\|\varphi\|_{t_0}<H$, we shall use the variation of constants formula (2.4).

So, we have

$$|y(t;t_0,\varphi)|^p\le 2^{p-1}\Big\{|x(t;t_0,\varphi(t_0))|^p+M^p\Big[\int_{t_0}^t\int_0^s b(s,\tau)\,|y(\tau;t_0,\varphi)|\,d\tau\,ds\Big]^p\Big\}\le$$

$$\le 2^{p-1}|x(t;t_0,\varphi(t_0)|^p+2^{p-1}M^p\Big[\int_{t_0}^t\int_0^s b(s,\tau)\,d\tau)\,\gamma(s)\,ds\Big]^p,$$

where, as in [13; section 4], $\gamma(t)=\sup\{|y(s;t_0,\varphi)|;0\le s\le t\}$.
Hence,

$$|y(t;t_0,\varphi)|^p\le\gamma(t)^p\le 2^{p-1}|x(t;t_0,\varphi(t_0)|^p+2^{p-1}M^p\Big(\int_{t_0}^t\Big[\int_0^s b(s,\tau)\,d\tau\Big]^q ds\Big)^{p/q}\cdot$$

$$\cdot\int_{t_0}^t\gamma^p(s)\,ds=2^{p-1}|x(t;t_0,\varphi(t_0)|^p+2^{p-1}M^p v(t)\int_{t_0}^t\gamma^p(s)\,ds.$$

For arbitrary $u\ge t_0\ge 0$, we obtain

$$\int_{t_0}^{u}\gamma(t)^p dt \le 2^{p-1}\int_{t_0}^{u}|x(t;t_0,\varphi(t_0))|^p dt+2^{p-1}M^p\int_{t_0}^{u}v(t)\int_{t_0}^{t}\gamma(s)^p ds dt \le$$

$$\le K_1+K_2\int_{t_0}^{u}v(t)\int_{t_0}^{t}\gamma(s)^p ds dt,$$

where $\qquad K_1=2^{p-1}\int_{t_0}^{\infty}|x(t;t_0,\varphi(t_0))|^{pdt}\ ,\ K_2=2^{p-1}M^p.$

From here, by Bihari's inequality, it follows

$$\int_{t_0}^{\tau}\gamma(t)^p dt \le K_1\exp[K_2\int_{t_0}^{\tau}v(t)\,dt] \le K_1\exp K_2\int_{t_0}^{\infty}v(t)\,dt,\ \textit{for any}\ \ u\ge t_0\ge 0$$

and therefore $\qquad \int_{t_0}^{\infty}|y(t;t_0,\varphi)|^p dt \le \int_{t_0}^{\infty}\gamma(t)^p dt<\infty.$

This inequality yields the desired conclusion, and Theorem 3.1 is proved.

Remark 3.1 An analogous result can be obtained for the zero solution of (P_2). This time, we shall assume that (H2) and (2.16) hold and, in addition,

$$(3.3) \qquad v(t)\equiv[\int_{0}^{t}(a(s)+\int_{0}^{s}b(s,\tau)\,d\tau)^q ds]^{p/q}\in L^1([t_0,\infty)).$$

Corollary 3.1 **Assume that the solution x=0 of (P_1) is (UL^p-SV) and that (H3) holds. Furthermore, we assume that**

$$\gamma(t)\equiv[\int_{0}^{t}(\int_{0}^{s}b(s,\tau)\,d\tau)^r ds]^{1/r}\in L^1([t_0,\infty))\ ,\ \textit{where}\ r=\frac{p}{p-m};$$

Then all solutions of (P_1) belong to $L^p([t_0,\infty))$.

Proof. By using hypothesis (H3) and the Minkowski-Hölder inequality, we have

$$|y(t;t_0,\varphi)|^p \le 2^{p-1}\{|x(t;t_0,\varphi(t_0))|^p+M^p[\int_{t_0}^{t}(\int_{0}^{s}b(s,\tau)\,d\tau)^r ds]^{p/r}[\int_{0}^{t}\gamma^p(s)\,ds]^m\}.$$

From here, for arbitrary $u\ge t_0\ge 0$, we obtain

$$\int_{t_0}^{u}\gamma^p(t)\,dt\le 2^{p-1}\int_{t_0}^{u}|x(t;t_0,\varphi(t_0))|^p dt+$$

$$+2^{p-1}M^p\int_{t_0}^{u}v(t)\,[\int_{t_0}^{t}\gamma^p(s)\,ds]^m dt\le K_1+K_2\int_{t_0}^{t}v(t)\,[\int_{t_0}^{t}\gamma^p(s)\,ds]^m dt\ ,$$

$$\text{where}\quad K_1=2^{p-1}\int_{t_0}^{\infty}|x(t;t_0,\varphi(t_0)|^p dt\ ,\ K_2=2^{p-1}M^p.$$

From here by the inequality (2.12) it follows

$$\int_{t_0}^{u}\gamma^p(t)\,dt\le [K_1^{1-m}+(1-m)\,K_2\int_{t_0}^{u}v(t)\,dt]^{\frac{1}{1-m}},$$

for any $u\ge t_0\ge 0$, and therefore

$$\int_{t_0}^{\infty}|y(t;t_0,\varphi)|^p dt\le\int_{t_0}^{\infty}\gamma^p(t)\,dt<\infty.$$

__Remark 3.2__ A similar result can be formulated for the system (P_2) if we assume that (H4) holds and, in addition,

$$v(t)\equiv[\int_0^t(a(s)+\int_0^s b(s,\tau)\,d\tau)^r ds]^{1/r}\in L^1([t_0,\infty)).$$

We can not formulate some similar results to preceding ones when p=1 by just changing p=1.

__Theorem 3.2__ **Assume that the solution x=0 of (P_0) is uniformly L^1-stable in variation, that (H1) holds and (3.1) is satisfied with $K<M^{-1}$. Then the zero solution of (P_1) is uniformly L^1-stable.**

__Proof.__ Like in the first part of the proof of Theorem 3.1 it follows that y=0 is (US). Further, we have

$$|y(t;t_0,\varphi)|\le|x(t;t_0,\varphi(t_0))|+M\int_{t_0}^{t}\int_0^{s}b(s,\tau)\,|y(\tau,t_0,\varphi)|\,d\tau ds\le$$

$$\le|x(t;t_0,\varphi(t_0))|+M\int_{t_0}^{t}[\int_0^{s}b(s,\tau)\,d\tau]\,\gamma(s)\,ds,$$

from where

$$\gamma(t)\le|x(t;t_0,\varphi(t_0))|+MK\gamma(t)\quad\text{for all } t\ge t_0\ge 0.$$

Therefore, $$\int_{t_0}^{\infty}|y(t;t_0,\varphi)|\,dt\le\int_{t_0}^{\infty}\gamma(t)\,dt\le(1-MK)^{-1}\int_{t_0}^{\infty}|x(t;t_0,\varphi(t_0))|\,dt<\infty.$$

Remark 3.3 If we assume that (H2) holds and (2.16) is satisfied with $K<M^{-1}$ then, as above, it follows that the zero solution of (P_2) is uniformly L^1-stable.

Corollary 3.2 **Assume that the solution $x=0$ of (P_0) is $(UL^1\text{-}SV)$ and that (H3) holds. Furthermore, we assume that**

$$v(t)\equiv\Big[\int_0^t\Big(\int_0^s b(s,\tau)\,d\tau\Big)^r ds\Big]^{1/r}\in L^1([t_0,\infty))\ ,\ \text{where } r=\frac{1}{1-m}.$$

Then all solutions of (P_1) belong to $L^1([t_0,\infty))$.

Indeed, this time, we have

$$|y(t;t_0,\varphi)|\le|x(t;t_0,\varphi(t_0))|+M\Big[\int_{t_0}^t\Big(\int_0^s b(s,\tau)\,d\tau\Big)^r d\tau\Big]^{1/r}\cdot\Big[\int_{t_0}^t\gamma(s)\,ds\Big]^{1/m}.$$

Further, by using the same argument as in the proof of Corollary 3.1, one obtains the desired assertion.

Remark 3.4 If we assume that (H4) holds and, in addition, $v(t)$ given in Remark 6.2 (with $r=1/(1-m)$) belongs to $L^1([t_0,\infty))$, then all solutions of (P_2) belong to $L^1([t_0,\infty))$.

In the sequel we wish to study the relationship between the (GUSGV) of the zero solution of (P_0) and the uniform L^p-stability of the zero solution of (P_1) and (P_2).

To this end, with respect to the function $\alpha(t)$ from (GUSGV) we make the assumption that there exists a positive number K such that if $t\in R^+$, then

$$\int_0^t\exp\Big(\int_s^t\alpha(\tau)\,d\tau\Big)\,ds\le K. \tag{3.4}$$

We remark that (3.4) and Lemma 1 [4;p.68] yield to

$$\lim_{t\to\infty}\exp\Big(\int_0^t\alpha(\tau)\,d\tau\Big)=0\,. \tag{3.5}$$

Furthemore, we shall assume that

$$(3.6)\qquad \sup_{t>0}\int_0^t b(s,\tau)\,d\tau < \frac{1}{KL} \quad \text{for all } t\geq 0.$$

Theorem 3.3 **Assume that the solution $x=0$ of (P_0) is (GUSGV) and that (H1), (3.4) and (3.6) hold. Furthermore we assume that**

$$(3.7)\qquad \exp\Big(p\int_{t_0}^t \alpha(\tau)\,d\tau\Big)\in L^1([t_0,\infty)),$$

$$(3.8)\qquad \int_{t_0}^t \exp\Big(-q\int_{t_0}^s \alpha(\tau)\,d\tau\Big)\Big(\int_0^s b(s,\tau)\,d\tau\Big)^q ds\leq K_1\ ,\ \textit{for all } t\geq t_0\ ,\ p^{-1}+q^{-1}=1.$$

Then the zero solution of (P_1) is (UL^p-S). Moreover, for every $\epsilon>0$ there exists $\delta(\epsilon,t_0)>0$ such that for a solution $y(t;t_0,\varphi)$ of (P_1) for which $\|\varphi\|_{t_0}<\delta$ the relation $\int_{t_0}^\infty |y(t;t_0,\varphi)|^p dt<\epsilon$ holds.

Proof. By the definition of (GUSGV) and hypotheses (H1), (3.4) and (3.6), in accordance with proof of the first part of the Theorem 5.1 [13], it follows that the zero solution of (P_1) is (US).

Then, by using the same type of reasoning as in the proof of the Theorem 3.1, we have

$$|y(t;t_0,\varphi)|^p\leq 2^{p-1}\Big\{L^p\exp\Big(p\int_{t_0}^t\alpha(\tau)\,d\tau\Big)\|\varphi\|_{t_0}^{\ p}+L^p\exp\Big(p\int_{t_0}^t\alpha(\tau)\,d\tau\Big)\cdot$$
$$\cdot\Big[\int_{t_0}^t\exp\Big(-q\int_{t_0}^s\alpha(\tau)\,d\tau\Big)\Big(\int_0^s b(s,\tau)\,d\tau\Big)^q ds\Big]^{p/q}\int_{t_0}^t\gamma^p(s)\,ds\Big\}.$$

From here, for arbitrary $u\geq t_0\geq 0$, one obtains

$$\int_{t_0}^u\gamma^p(t)\,ds\leq K_2\|\varphi\|_{t_0}^{\ p}+K_3\int_{t_0}^u\exp\Big(p\int_{t_0}^t\alpha(\tau)\,d\tau\Big)\cdot\int_{t_0}^t\gamma^p(s)\,ds\,dt\ ,$$

where $K_2=2^{p-1}L^p\int_{t_0}^\infty\exp\Big(p\int_{t_0}^t\alpha(\tau)\,d\tau\Big)dt\ ,\ K_3=2^{p-1}L^pK_1^{\ p/q}.$

Therefore,

$$\int_{t_0}^\infty|y(t;t_0,\varphi)|^p dt\leq\int_{t_0}^\infty\gamma^p(t)\,dt\leq K_2\|\varphi\|_{t_0}^{\ p}\exp K_3\int_{t_0}^\infty\exp\Big(p\int_{t_0}^t\alpha(\tau)\,d\tau\Big)dt<\infty.$$

The last assertion of the theorem immediately follows.

Remark 3.4 By using (2.7), it is a simple matter to show that (GUSGV) and (3.7) imply the (UL^p-SV) of the zero solution of (P_0).

A similar result can be formulated for (P_2) if we accept (H2),

$$(3.9)\qquad \sup_{t\geq 0}\left[a(t)+\int_0^t b(t,s)\,ds\right] < 1/3\,KL$$

and

$$(3.10)\qquad \int_{t_0}^{t}\exp\left(-q\int_0^s \alpha(\tau)\,d\tau\right)\left[a(s)+\int_0^s b(t,\tau)\,d\tau\right]^q ds \leq K_1 \quad \text{for all} \quad t\geq t_0.$$

Theorem 3.3 takes a particularly simple form when $\alpha(t)=-\lambda$, where λ is a positive constant and $b(t,\tau)=e^{-(\lambda+\sigma)t}\cdot e^{\sigma\tau}\cdot c(\tau)$, $\sigma>0$. We note that in this case the (GUSGV) of the zero solution of (P_0) reduces to the its (EASV).

Corollary 3.3 **Assume that the solution $x=0$ of (P_0) is (EASV) and $c(t)\in L^1(\mathbb{R}^+)$. Then the conclusions of Theorem 6.3 occur.**

Proof. Clearly (3.4) and (2.16) are satisfied. Then, we have

$$\int_0^t b(t,\tau)\,d\tau = e^{-(\lambda+\sigma)t}\int_0^t e^{\sigma\tau}c(\tau)\,d\tau \leq e^{-\sigma t}\int_0^t e^{\sigma\tau}c(\tau)\,d\tau \to 0\ , \ \text{as } t\to\infty$$

[9; Theorem 2.14.6], so that (3.6) holds.

Since $c(t)\in L^1(\mathbb{R}^+)$, $c(t)>0$, it follows [7; Lemma 1] that

$\int_0^t e^{-\sigma(t-\tau)}\cdot c(\tau)\,d\tau \in L^q(R^+)$, $q\geq 1$, and therefore (3.8) is satisfied. All hypotheses of Theorem 3.3 being accomplished, the conclusion follows from above.

We remark that under the mentioned conditions the hypotheses of Corollary 4.1 [13] occur and because in this case the zero solution of (P_1) is (EAS), the conclusion of Corollary 3.1 also follows and by this way.

Again in the special case $p=1$ the corresponding result takes the following form.

Theorem 3.4 **Assume that the solution $x=0$ of (P_0) is (GUSGV) and that (H1), (3.4) and (3.6) hold. Furthermore, we assume that**

$$(3.11)\qquad \exp\left(\int_{t_0}^{t}\alpha(\tau)\,d\tau\right)\in L^1([t_0,\infty)),$$

$$(3.12)\qquad \exp\left(-\int_{t_0}^{t}\alpha(\tau)\,d\tau\right)\int_{0}^{t}b(t,\tau)\,d\tau\le K_1<\infty.$$

Then the zero solution of (P_1) is $(UL^1\text{-}S)$. Moreover, for every $\epsilon>0$ there exists $\delta(\epsilon,t_0)>0$ such that for a solution $y(t;t_0,\varphi)$ of (P_1) for which $\|\varphi\|_{t_0}<\delta$ the relation $\int_{t_0}^{\infty}|y(t;t_0,\varphi)|\,dt<\epsilon$ holds.

Proof. The uniform stability of the zero solution of (P_1) follows by the same arguments made in the proof of the Theorem 3.3. Then we have

$$|y(t;t_0,\varphi)|\le L\exp\left(\int_{t_0}^{t}\alpha(\tau)\,d\tau\right)\|\varphi\|_{t_0}+L\exp\left(\int_{t_0}^{t}\alpha(\tau)\,d\tau\right)\cdot$$

$$\cdot\left[\int_{t_0}^{t}\exp\left(-\int_{t_0}^{s}\alpha(\tau)\,d\tau\right)\left(\int_{0}^{s}b(s,\tau)\,d\tau\right)\gamma(s)\,ds.\right.$$

From here, for an arbitrary $u\ge t_0\ge 0$, one obtains

$$\int_{t_0}^{u}\gamma(t)\,dt\le K_2\|\varphi\|_{t_0}+LK_1\int_{t_0}^{u}\left[\exp\left(\int_{t_0}^{t}\alpha(\tau)\,d\tau\right)\int_{t_0}^{t}\gamma(s)\,ds\right]dt,$$

$$\text{where}\quad K_2=\int_{t_0}^{\infty}\exp\left(\int_{t_0}^{t}\alpha(\tau)\,d\tau\right)dt.$$

By Bihari's inequality one obtains

$$\int_{t_0}^{\infty}|y(t;t_0,\varphi)|\,dt\le\int_{t_0}^{\infty}\gamma(t)\,dt\le K_2\|\varphi\|_{t_0}\exp\left(LK_1\int_{t_0}^{\infty}\exp\left(\int_{t_0}^{t}\alpha(\tau)\,d\tau\right)dt\right)\le$$

$$\le K_2\|\varphi\|_{t_0}\exp(LK_1K_2)<\infty.$$

From here it follows also the last part of the theorem.

Example 3.1 Let us consider the scalar differential equation

$$(3.13)\qquad x'=-\left(2t+\frac{1}{t}\right)x-t^2x^3\quad,\quad x(t_0)=x_0\quad,\quad t_0\ge\frac{1}{2}$$

and its corresponding perturbed integro-differential equation

$$(3.14)\qquad y'=-\left(2t+\frac{1}{2}\right)y-t^2y^3+\int_{0}^{t}e^{-2t^2}\cdot\frac{y(s)\sin y(s)}{1+(t+s)^2}\,ds\ .$$

The solution of (3.13) is given by

$$x(t;t_0,x_0)=e^{-\int_{t_0}^{t}(2\tau+\frac{1}{\tau})d\tau}x_0[1+2x_0^2\int_{t_0}^{t}e^{-\int_{t_0}^{s}(2\tau+\frac{1}{\tau})d\tau}s^2ds]^{-1/2}$$

and

$$\phi(t;t_0,x_0)=e^{-\int_{t_0}^{t}(2\tau+\frac{1}{\tau})d\tau}\cdot[1+2x_0^2\int_{t_0}^{t}e^{-\int_{t_0}^{s}(2\tau+\frac{1}{\tau})d\tau}\cdot s^2ds]^{-3/2}\le e^{-\int_{t_0}^{t}(2\tau+\frac{1}{\tau})d\tau},$$

for all $t\geq t_0>0$ and all $x_0\in R$.
We note that all assumptions of Theorem 3.4 hold with $L=1$; (3.4) with $K=1$; (3.6) becomes

$$\sup_{t\geq0}\int_0^t b(t,s)\,ds=\sup_{t\geq0}\int_0^t e^{-2t^2}\frac{ds}{1+(t+s^2)}ds\le\sup_{t\geq0}\int_0^t\frac{ds}{1+(t+s^2)}\le0.34<\frac{1}{KL}=1;$$

(3.11) becomes $\exp(\int_{t_0}^{t}\alpha(\tau)\,d\tau)=t_0e^{t_0}e^{-t^2}t^{-1}\in L_1([t_0,\infty)]$,

and (3.12) holds since

$$te^{t^2}\int_0^t e^{-2t^2}\frac{d\tau}{1+(t+\tau)^2}=te^{-t^2}[\mathrm{arctg}\;2t-\mathrm{arctg}\;t]<0.34\cdot(2e)^{-1/2}=K_1.$$

Therefore the zero solution of (3.14) is (UL^1-S) and, in addition, the estimate from the last part of Theorem 3.4 holds.

Theorem 3.5 **Assume that the solution $x=0$ of (P_0) is (GUSGV) and that (H3) and (3.7) hold. Furthermore we assume that**

$$(3.15)\qquad\int_{t_0}^{t}\exp(-r\int_{t_0}^{s}\alpha(\tau)\,d\tau)\,(\int_0^s b(s,\tau)\,d\tau)^r ds\le K_1\;,\;\textit{for all }t\geq t_0,$$

where $r=p/(p-m)$. Then all solutions of (P_1) belong to $L^p([t_0,\infty))$.

Proof. By using hypotheses (H3) and the Hölder inequality, we have

$$|y(t;t_0,\varphi)|^p\le2^{p-1}\{L^p\exp(p\int_{t_0}^{t}\alpha(\tau)\,d\tau)\|\varphi\|_{t_0}^p+L^p\exp(p\int_{t_0}^{t}\alpha(\tau)\,d\tau)\cdot$$
$$\cdot[\int_{t_0}^{t}\exp(-r\int_{t_0}^{s}\alpha(\tau)\,d\tau)\,(\int_0^s b(s,\tau)\,d\tau)^r ds]^{p/r}[\int_0^t|\gamma(s)^p ds]^{m/p}\}.$$

From here, one obtains

$$\int_{t_0}^{u}\gamma(t)^p dt \le K_2\|\varphi\|_{t_0}^p + K_3\int_{t_0}^{u}\exp\left(p\int_{t_0}^{t}\alpha(\tau)\,d\tau\right)\left(\int_{t_0}^{t}|\gamma(s)|^p ds\right)^m, \quad u\ge t_0\ge 0,$$

where $K_2=2^{p-1}L^p\int_0^{\infty}\exp\left(p\int_{t_0}^{t}\alpha(\tau)\,d\tau\right)dt$, $K_3=2^{p-1}L^pK_1^{p/r}=2^{p-1}L^pK_1^{p-m}$.

An application of the Bihari inequality (2.12) (with $v(t)\equiv 0$) [8; Theorem 1.3.4], yields to

$$\int_{t_0}^{u}\gamma(t)^p \le \left[(K_2\|\varphi\|_{t_0}^p)^{1-m} + (1-m)K_3\int_{t_0}^{u}\exp\left(p\int_{t_0}^{t}\alpha(\tau)\,d\tau\right)dt\right]^{\frac{1}{1-m}},$$

from where one obtains the desired result.

Remark 3.5 Analogous results can be formulated as regard the solutions of (P_1) when $p=1$, respectively, for the solutions of (P_2) under hypothesis (H4), $p\ge 1$.

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