

On Some Partial Differential Subordinations For Holomorphic Mappings In C^n

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Abstract

In this paper the author obtains some results concerning partial differential subordinations for holomorphic mappings defined on some domains in C^n .

1. INTRODUCTION Very recently G. Kohr and P. Liczberski [3] considered the analytic functions defined on the unit ball from C^n with an arbitrarily fixed norm, and deduced some properties for these mappings, concerning inclusion relations or subordinations. These results are based on a generalization of Jack's, Miller's and Mocanu's Lemma [5] and also, recently, G. Kohr and C. Pinteau [4] gave an extension of this lemma for holomorphic mappings defined on some domains in C^n .

In this paper using that result we shall obtain some applications concerning inclusion relations or subordinations.

2. PRELIMINARIES Let C^n be denote the space of n complex variables

$$z = (z_1, \dots, z_n)'$$

with the Euclidean inner product

$$\langle z, w \rangle = \sum_{j=1}^n z_j \bar{w}_j$$

and the Euclidean norm

$$\|z\| = \langle z, z \rangle^{\frac{1}{2}}$$

The open Euclidean ball $\{z \in C^n : \|z\| < r\}$ is denoted by B_r and the open unit Euclidean ball is abbreviated by $B_1 = B$. As usual by $L(C^n, C^m)$ we denote the space of all continuous linear operators from C^n into C^m with the standard operator norm. The letter I will always represent the identity operator in $L(C^n, C^n)$.

The origin $(0, \dots, 0)'$ is always denoted by 0 .

The class of holomorphic mappings from a domain $G \subseteq C^n$ into C^n is denoted by $H(G)$.

A mapping $f \in H(G)$ is said to be locally biholomorphic in G if its Frechet derivative

$$Df(z) = \left[\frac{\partial f_j}{\partial z_k}(z) \right]_{1 \leq j, k \leq n}$$

as an element of $L(C^n, C^n)$ is nonsingular at each point $z \in G$ (or, equivalently, if it has a locally inverse at any point $z \in G$).

A mapping $f \in H(G)$ is called biholomorphic if the inverse mapping f^{-1} does exist, is holomorphic on a domain Ω and $f^{-1}(\Omega) = G$.

If $D^2f(z)$ means the Frechet derivative of the second order for $f \in H(G)$ at the point z , then of course $D^2f(z)$ is a continuous bilinear operator from $C^n \times C^n$ into C^n and its restriction $D^2f(z)(u, \cdot)$ to $u \times C^n$ belongs to $L(C^n, C^n)$.

For a set $G \subseteq C^n$ the closure and the topological bord will be denoted by \bar{G} and ∂G .

The symbol $'$ means the transpose of elements and matrix defined on C^n .

If g is a C^2 real valued function, defined on a domain $\Omega \subseteq C^n$, we denote by

$$\frac{\partial g}{\partial z}(z) = \left(\frac{\partial g}{\partial z_1}(z), \dots, \frac{\partial g}{\partial z_n}(z) \right)',$$

$$\frac{\partial^2 g}{\partial z^2}(z) = \left[\frac{\partial^2 g}{\partial z_i \partial z_j}(z) \right]_{1 \leq i, j \leq n}$$

and

$$\frac{\partial^2 g}{\partial \bar{z} \partial z}(z) = \left[\frac{\partial^2 g}{\partial \bar{z}_i \partial z_j}(z) \right]_{1 \leq i, j \leq n},$$

respectively, where z is an arbitrary point from Ω .
We say that z_0 is a critical point for g if

$$\frac{\partial g}{\partial z}(z_0) = 0$$

Let denote by $\underline{C}(g)$ the set of critical points of the function g . We also, call a real number t regular value for g , if either $g^{-1}(t) = \emptyset$ or $g^{-1}(t) \cap \underline{C}(g) = \emptyset$.

Let D be a domain in C^n whose border ∂D is a C^2 real hypersurface which can be represented under the form

$$\partial D = \varphi^{-1}(t),$$

where t is a regular value of a C^2 differentiable function φ defined on a neighborhood of ∂D .

The main results of this paper are based on the following lemma.

LEMMA 2.1. [4] Let D be a domain in C^n satisfying the above conditions with $0 \in D$, $z_0 \in \partial D$ and $f \in H(D \cup \{z_0\}) \cap C(\bar{D})$ such that $f(0) = 0$ and f locally biholomorphic at z_0 .

If $\phi \in C^2(V, R)$, where V is a neighborhood of $f(\partial D)$, with $\frac{\partial \phi}{\partial w} \neq 0$ in V and suppose that

$$(2.1) \quad z_0 \in \underline{C}(\phi \circ f|_{\partial D}),$$

then there exists a real nonzero number s such that the following condition is satisfied

$$(2.2) \quad ([Df(z_0)]^{-1})' \frac{\partial \varphi}{\partial z}(z_0) = s \frac{\partial \phi}{\partial w}(w_0),$$

where $w_0 = f(z_0)$.

REMARK 2.1. In [4] the authors showed that under the conditions of Lemma 2.1, if z_0 is particularly a point of local maximum of the function $\phi \circ f|_{\partial D}$, then the condition (2.2) is obviously satisfied, and furthermore the following condition holds

$$(2.3) \quad \operatorname{Re} \left\{ (Df(z_0)v)' \frac{\partial^2 \phi}{\partial w^2}(w_0) Df(z_0)v \right\} + \\ + \overline{(Df(z_0)v)'} \frac{\partial^2 \phi}{\partial \bar{w} \partial w}(w_0) Df(z_0)v + \operatorname{Re} \left(\frac{\partial \phi}{\partial w}(w_0) \right)' D^2 f(z_0)(v, v) \leq \\ \leq \frac{1}{s} \left[\operatorname{Re} v' \frac{\partial^2 \varphi}{\partial z^2}(z_0)v + \bar{v}' \frac{\partial^2 \varphi}{\partial \bar{z} \partial z}(z_0)v \right],$$

for all $v \in C^n \setminus \{0\}$ with $\operatorname{Re} \left\langle \frac{\partial \varphi}{\partial z}(z_0), \bar{v} \right\rangle = 0$.

Furthermore we shall consider bounded schlicht domains D in C^n for which the kernel function becomes infinite everywhere on the boundary ∂D , i.e. it is union of an increasing sequence of strictly pseudo-convex domains

$$(2.4) \quad D_t = \{z \in D : K_D(z, \bar{z}) - t < 0\},$$

for some number $t > 0$, where $z = (z_1, \dots, z_n)'$ (see for example [1], [2]).

Suppose that $\frac{\partial K_D}{\partial z}(z, \bar{z}) \neq 0$, $z \neq 0$ in D , and $K_D(0, 0) = \min_{z \in D} K_D(z, \bar{z})$ at only $z = 0$.

Let denote the function $K_D(z, \bar{z})$ by $\varphi(z)$ and $\varphi_t(z) = \varphi(z) - t$, for all $t > K_D(0, 0)$ and $z \in D$.

In the next we shall consider a positive fixed number $t > K_D(0, 0)$.

DEFINITION 2.1. Let f and g be holomorphic mappings on the domain $D_t \subset D$, where $t > K_D(0, 0)$. We say that f is subordinate to g (written $f \prec g$ or $f(z) \prec g(z)$) if there exists a holomorphic mapping $\omega : D_t \rightarrow C^n$ with the following properties: $\omega(0) = 0$, $\omega(D_t) \subseteq D_t$ and $f = g \circ \omega$ in D_t .

It is easy to observe that if f is subordinate to g , then $f(0) = g(0)$ and $f(D_t) \subseteq g(D_t)$. But, if g is biholomorphic on D_t , then f is subordinate to g iff $f(0) = g(0)$ and $f(D_t) \subseteq g(D_t)$.

DEFINITION 2.2. Let g be a holomorphic mapping on a neighborhood V of \bar{D}_t . We say that g is biholomorphic on \bar{D}_t if there exists a positive ϵ such that g is biholomorphic on $D_{t+\epsilon} \subset V$.

3.MAIN RESULTS

THEOREM 3.1. Let t be a positive number, with $t > K_D(0, 0)$ and let g be a biholomorphic mapping on \bar{D}_t . Let f be a holomorphic mapping on D_t such that $f(0) = g(0)$ and suppose there exist points $z_0 \in D_t$, $\zeta_0 \in \partial D_t$, with $\varphi(z_0) = r_0$ where $K_D(0, 0) < r_0 < t$, such that f is locally biholomorphic at z_0 and

$$(3.1) \quad f(z_0) = g(\zeta_0) \quad , \quad f(D_{r_0}) \subseteq g(D_t).$$

Then there exist a real nonzero number s with

$$(3.2) \quad ([Df(z_0)]^{-1})' \frac{\partial \varphi}{\partial z}(z_0) = s ([Dg(\zeta_0)]^{-1})' \frac{\partial \varphi}{\partial z}(\zeta_0)$$

and

$$(3.3) \quad \operatorname{Re} \left\{ w' \frac{\partial^2 \varphi}{\partial z^2}(\zeta_0) w \right\} + \bar{w}' \frac{\partial^2 \varphi}{\partial \bar{z} \partial z}(\zeta_0) w \leq$$

$$\begin{aligned} &\leq \frac{1}{s} \left[\operatorname{Re} v' \frac{\partial^2 \varphi}{\partial z^2} (z_0) v + \bar{v}' \frac{\partial^2 \varphi}{\partial \bar{z} \partial z} (z_0) v \right] - \\ &-\frac{1}{s} \left\{ \operatorname{Re} \left(\frac{\partial \varphi}{\partial z} (z_0) \right)' [Df(z_0)]^{-1} D^2 f(z_0) (v, v) - \right. \\ &\left. - \operatorname{Re} \left(\frac{\partial \varphi}{\partial z} (z_0) \right)' [Df(z_0)]^{-1} D^2 g(\zeta_0) (w, w) \right\}, \end{aligned}$$

for all $v \in C^n \setminus \{0\}$ with $\operatorname{Re} \left\langle \frac{\partial \varphi}{\partial z} (z_0), \bar{v} \right\rangle = 0$, where $w = [Dg(\zeta_0)]^{-1} Df(z_0) v$.

Furthermore,

$$(3.4) \quad s \operatorname{Re} \left\langle \frac{\partial \varphi}{\partial z} (\zeta_0), \bar{\zeta}_0 \right\rangle \geq 0.$$

Proof. Let consider the mapping

$$h: \bar{D}_{r_0} \rightarrow \bar{D}_t, \quad h(z) = (g^{-1} \circ f)(z), \quad z \in \bar{D}_{r_0},$$

then h is holomorphic on \bar{D}_{r_0} , $h(0) = 0$ and

$$t = \varphi(h(z_0)) = \max \{ \varphi(h(z)) : \varphi(z) \leq \varphi(z_0) \}$$

Thus, h satisfy the assumptions of Lemma 2.1, hence there exists a real nonzero s , such that

$$\left([Dh(z_0)]^{-1} \right)' \frac{\partial \varphi}{\partial z} (z_0) = s \frac{\partial \varphi}{\partial z} (\zeta_0).$$

A straightforward calculation yields

$$Dh(z_0) = [Dg(\zeta_0)]^{-1} Df(z_0),$$

which implies the relation (3.2).

On the other hand, it is easy to see that

$$D^2f(z_0)(v, v) = D^2g(h(z_0))(Dh(z_0)v, Dh(z_0)v) + \\ Dg(z_0)D^2h(z_0)(v, v), \text{ for all } v \in C^n.$$

Multiplying in the both sides of the above equality with $\left(\frac{\partial\varphi}{\partial z}(z_0)\right)' [Df(z_0)]^{-1}$ and using (3.2.), we obtain

$$\left(\frac{\partial\varphi}{\partial z}(z_0)\right)' [Df(z_0)]^{-1} D^2f(z_0)(v, v) = \\ = \left(\frac{\partial\varphi}{\partial z}(z_0)\right)' [Df(z_0)]^{-1} D^2g(\zeta_0)(w, w) + \\ + s \left(\frac{\partial\varphi}{\partial z}(\zeta_0)\right)' D^2h(z_0)(v, v),$$

where $w = Dh(z_0)v$.

Now it is enough to use the relation (2.3) with $\Phi(z) \equiv \varphi(z)$, $z \in D$, and also the relation (3.2) to obtain (3.3).

It remains only to prove (3.4).

Since h is locally biholomorphic at z_0 , then there exists an open neighborhood W of z_0 such that for $\epsilon > 0$ sufficiently small the following relations hold

$$(1 + \epsilon)h(z_0) \in h(W)$$

and

$$(1 + \epsilon)h(z_0) \notin h(D_{r_0}).$$

Hence, if we denote by $\Psi(w) = (\varphi \circ h^{-1})(w)$, $w \in h(W)$,

then we deduce

$$0 \leq \lim_{\epsilon \searrow 0} \frac{\Psi((1 + \epsilon)h(z_0)) - \Psi(h(z_0))}{\epsilon} =$$

$$= 2\operatorname{Re} \left(\frac{\partial \varphi}{\partial z}(z_0) \right)' [Dh(z_0)]^{-1} h(z_0).$$

Using the relation (3.2), we get (3.4).

THEOREM 3.2. Let f be a locally biholomorphic mapping on D_t , where $t > K_D(0,0)$, let g be a biholomorphic mapping on \overline{D}_t such that $f(0) = g(0)$ and suppose that f is not subordinate to g . Then there exist a real number r_0 with $K_D(0,0) < r_0 < t$, the points $z_0 \in \partial D_{r_0}$, and $\zeta_0 \in \partial D_t$, such that the following relations hold

$$(3.5.) \quad f(z_0) = g(\zeta_0), \quad f(D_{r_0}) \subseteq g(D_t),$$

$$(3.6.) \quad ([Df(z_0)]^{-1})' \frac{\partial \varphi}{\partial z}(z_0) = s ([Dg(\zeta_0)]^{-1})' \frac{\partial \varphi}{\partial z}(\zeta_0),$$

where s is a nonzero real number, with

$$(3.7.) \quad s \operatorname{Re} \left\langle \frac{\partial \varphi}{\partial z}(\zeta_0), \overline{\zeta_0} \right\rangle \geq 0$$

Proof. Since f is not subordinate to g and $f(0) = g(0)$, we deduce that $f(D_t) \not\subseteq g(D_t)$. It is easy to see that there is a positive number R such that $B_R \subset D_t$ and $f(B_R) \subseteq g(D_t)$. Also, there exist a positive small r such that $r > k_D(0,0)$ and $D_r \subseteq B_R$, so $f(D_r) \subseteq g(D_t)$.

On the other hand, if we denote by

$$r_0 = \sup \{r : f(D_r) \subseteq g(D_t)\},$$

then $K_D(0,0) < r_0 < t$ and $f(\overline{D}_{r_0}) \subseteq g(\overline{D}_t) \setminus g(D_t)$, hence we can find the points $z_0 \in \partial D_{r_0}$ and $\zeta_0 \in \partial D_t$ such that $f(z_0) = g(\zeta_0)$.

Now it is enough to apply the result of Theorem 3.1.

REMARK 3.1. If D is the unit ball of C^n , then it is well known that

$$K_D(z, \bar{z}) = \frac{n!}{\pi^n (1 - \|z\|^2)^{n+1}}.$$

Using the result of Theorem 3.2., we obtain

COROLLARY 3.1. Let f be a locally holomorphic mapping on B_r , where $0 < r < 1$.

Let g be a biholomorphic mapping on $\overline{B_r}$ such that $f(0) = g(0)$. If f is not subordinate to g then there exist the real numbers $0 < r_0 < r, s > 0$ and points $z_0 \in \partial B_{r_0}, \zeta_0 \in \partial B_r$, such that

$$f(z_0) = g(\zeta_0), f(B_{r_0}) \subseteq g(B_r)$$

and

$$([Df(z_0)]^{-1})' \bar{z}_0 = s ([Dg(\zeta_0)]^{-1})' \bar{\zeta}_0$$

Proof. It is sufficient to apply the result of Theorem 3.2., for the domain D_t , where

$$D_t = \{z \in B : K_B(z, \bar{z}) < t\},$$

and

$$1 - r^2 = \sqrt[n+1]{\frac{n!}{\pi^n t}}.$$

Also, in this case, it is clear that

$$\operatorname{Re} \left\langle \frac{\partial \varphi}{\partial z}(\zeta_0), \bar{\zeta}_0 \right\rangle > 0$$

so using (3.7.) we conclude that $s > 0$.

The concept of admissible class is given in the following definition.

DEFINITION 3.1. Let $\Omega \subseteq C^{2n}$ and $\Delta \subseteq C^n$ be two domains and let g be a biholomorphic mapping on \overline{D}_t , with $t > K_D(0, 0)$.

Suppose that $(g(0), 0) \in \Omega$ and $Q(g) \subseteq \Omega$, where

$$Q(g) = \bigcup_{\substack{s \in \mathbb{R} \setminus \{0\} \\ \zeta \in \partial D_t}} Q_s(g, \zeta),$$

and

$$Q_s(g, \zeta) = \left\{ (u, v) \in C^{2n} : u = g(\zeta), v = s \left([Dg(\zeta)]^{-1} \right)' \frac{\partial \varphi}{\partial z}(\zeta) \right\}.$$

The "admissible class" $X(\Omega, \Delta, g)$ consist of those mappings

$$\Psi : \Omega \times D_t \rightarrow C^n$$

which are continuous and satisfy the following conditions:

$$\Psi(g(0), 0; 0) \in \Delta$$

and

$$(3.8.) \quad \Psi(u, v; z) \notin \Delta,$$

for all $(u, v) \in Q(g)$ and $z \in D_t$.

Following this definition, we obtain

THEOREM 3.3. Let f be a locally biholomorphic mapping on D_t and let g be a biholomorphic mapping on \overline{D}_t , such that $f(0) = g(0)$.

Suppose there exists a mapping $\Psi \in X(\Omega, \Delta, g)$ such that

$$\left(f(z), \left([Df(z)]^{-1} \right)' \frac{\partial \varphi}{\partial z}(z) \right) \in \Omega$$

and

$$(3.9.) \quad \Psi \left(f(z), ([Df(z)]^{-1})' \frac{\partial \varphi}{\partial z}(z); z \right) \in \Delta,$$

for all $z \in D_t$, then $f \prec g$.

Proof. If the subordination $f \prec g$ does not hold, then in view of Theorem 3.1, there exist points $z_0 \in \partial D_{r_0}$ with $K_D(0, 0) < r_0 < t$, $\zeta_0 \in \partial D_t$ and a real nonzero number s such that the relations (3.5.) – (3.7.) hold.

Let $u = f(z_0)$ and $v = ([Df(z_0)]^{-1})' \frac{\partial \varphi}{\partial z}(z_0)$, then according to (3.8.), we conclude that $\Psi(u, v; z_0) \notin \Delta$, which is a contradiction with (3.9.). Hence, $f \prec g$.

Let consider the mapping $g : D_t \rightarrow C^n, g(z) = z, z \in D_t$, then g is biholomorphic on $\overline{D_t}$ and by simple calculations, we obtain

$$Q_s(z, \zeta) = \left\{ (u, v) \in C^{2n} : u = \zeta, v = s \frac{\partial K_D}{\partial z}(\zeta, \bar{\zeta}) \right\},$$

where s is a real nonzero number and $\zeta \in \partial D_t$.

Also, let $Q(1) = \bigcup_{\substack{s \in \mathbb{R} \setminus \{0\} \\ \zeta \in \partial D_t}} Q_s(z, \zeta)$

Let $\Omega \subseteq C^{2n}$ and $\Delta \subseteq C^n$ be two domains, with $(0, 0) \in \Omega$.

Let $X(1)$ the set of those continuous mappings $\Psi : \Omega \times D_t \rightarrow C^n$, which satisfy

$$\Psi(0, 0; 0) \in \Delta$$

and

$$\Psi(u, v; z) \notin \Delta,$$

for all $(u, v) \in Q(1)$ and $z \in D_t$.

An immediate consequence of Theorem 3.3. is given in the following result

COROLLARY 3.2. Let t be a real number, $t > K_D(0,0)$, let f be a locally biholomorphic mapping on D_t , with $f(0) = 0$.

Suppose there exist $\Psi \in X(1)$ such that

$$\left(f(z), ([Df(z)]^{-1})' \frac{\partial K_D}{\partial z}(z, \bar{z}) \right) \in \Omega$$

and

$$\Psi \left(f(z), ([Df(z)]^{-1})' \frac{\partial K_D}{\partial z}(z, \bar{z}); z \right) \in \Delta$$

for all $z \in D_t$, then $K_D(f(z), \overline{f(z)}) < t$ in D_t .

Proof. It is sufficient to show that $f(z) \prec z, z \in D_t$. If this assertion does not hold, then in view of Theorem 3.2., there exist $z_0 \in D_t, \zeta_0 \in \partial D_t$ and a real nonzero number s such that

$$f(z_0) = \zeta_0$$

and

$$([Df(z_0)]^{-1})' \frac{\partial K_D}{\partial z}(z_0, \bar{z}_0) = s \frac{\partial K_D}{\partial z}(\zeta_0, \bar{\zeta}_0)$$

Let $u = f(z_0)$ and $v = ([Df(z_0)]^{-1})' \frac{\partial K_D}{\partial z}(z_0, \bar{z}_0)$ then $(u, v) \in Q(1)$ and according the definition of class $X(1)$, we conclude that $\Psi(u, v; z_0) \notin \Delta$, but this is a contradiction with the hypothesis. So $f(z) \prec z, z \in D_t$, hence $K_D(f(z), \overline{f(z)}) < t$, on D_t , too.

Another applications of Theorem 3.2. are given in the following results

THEOREM 3.4. Let $A(z)$ be a function defined on the domain D_t , with $t > K_D(0,0)$. Let f be a locally biholomorphic mapping on D_t with $f(0) = 0$ and g be a biholomorphic mapping on $\overline{D_t}$ such that $g(0) = 0$ and $g(D_t)$ is a convex domain on C^n .

Suppose that

$$(3.10.) \quad \operatorname{Re} A(z) \geq 0, z \in D_t,$$

and

$$(3.11.) \quad \operatorname{Re} \left\langle \frac{\partial \varphi}{\partial z}(z), \bar{z} \right\rangle > 0, z \in D \setminus \{0\}.$$

If

$$(3.12.) \quad f(z) + A(z) \overline{([D(f(z))])^{-1} \frac{\partial \varphi}{\partial z}(z)} \in g(D_t),$$

for all $z \in D_t$, then $f \prec g$.

Proof. If the subordination $f \prec g$ does not hold, then in view of Theorem 3.2., there exist a real nonzero number s and points $z_0 \in D_t, \zeta_0 \in \partial D_t$ such that

$$f(z_0) = g(\zeta_0)$$

and

$$([Df(z_0)]^{-1})' \frac{\partial \varphi}{\partial z}(z_0) = s ([Dg(\zeta_0)]^{-1})' \frac{\partial \varphi}{\partial z}(\zeta_0)$$

respectively.

Let us $\Psi(z) = f(z) + A(z) \overline{([Df(z))])^{-1} \frac{\partial \varphi}{\partial z}(z)}$, then at $z = z_0$ we have

$$\Psi(z_0) = g(\zeta_0) + sA(z_0) \overline{([Dg(\zeta_0)]^{-1})' \frac{\partial \varphi}{\partial z}(\zeta_0)}.$$

On the other hand, using the relation (3.11.) and the result of Theorem 3.2., we conclude that $s > 0$, hence following (3.10.), we obtain

$$\operatorname{Re} \left\langle \Psi(z_0) - g(\zeta_0), \overline{([Dg(\zeta_0)]^{-1})' \frac{\partial \varphi}{\partial z}(\zeta_0)} \right\rangle \geq 0.$$

Since $g(D_t)$ is a convex domain the above condition is a contradiction with (3.12.). Hence $f \prec g$.

Let $g(z) \equiv z$ in Theorem 3.4. and D be the Euclidean ball of C^n , then using the above result we obtain

COROLLARY 3.3. Let r be a positive number with $r < 1$, let f be a locally biholomorphic mapping on B_r , such that $f(0) = 0$.

Let $A(z)$ be a function defined on B_r such that $\operatorname{Re} A(z) \geq 0, z \in B_r$.

$$\text{If } \left\| f(z) + A(z) \overline{([Df(z)]^{-1})' z} \right\| < r, z \in B_r,$$

then

$$\|f(z)\| < r \text{ on } B_r$$

REFERENCES

1. R. Gunning, H. Rossi, *Analytic functions of several complex variables*, Prentice Hall, Englewood Cliffs, N. J. (1965).
2. K. Kikucki, Starlike and convex mappings in several complex variables, *Pacif. J. of Mathematics*, vol 44, no 2 (1973), 569 – 580.
3. G. Kohr, P. Liczberski, General partial differential subordinations for holomorphic mappings in C^n (to appear).
4. G. Kohr, C. Pinteá, An extension of Jack's-Miller's-Mocanu's Lemma for holomorphic mappings defined on some domains in C^n (to appear).
5. S. S. Miller, P. T. Mocanu, Second order differential inequalities in the complex plane, *J. of Math. Anal. Appl.*, 69 (1978), 289 – 305.
6. S. S. Miller, P. T. Mocanu, Differential subordinations and inequalities in the complex plane, *J. of Diff. Equations*, vol 67, no 2 (1987), 199–211.
7. T. J. Suffridge, Starlike and convex maps in Banach spaces, *Pacif. J. of Math.* 46 (1973), 575 – 589.
8. T. J. Suffridge, Starlikeness, convexity and other geometric properties of holomorphic maps in higher dimensions, *Lecture Notes in Mathematics*, vol 599 (1976), 146 – 159