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A DIFFERENTIATION RULE FOR MULTIPLE SINGULAR INTEGRALS WITH ISOLATED SINGULARITIES

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Summary.

The concepts of ultrasingular integral[2],integral with strong isolated singularity [8,11], and the derivation rule for integrals with a week isolated singularity [8,11], are reminded. A derivation rule for integrals with strong isolated singularities is established in terms of an ultra-singular integral. The conditions for the existence of such an integral are verified.

1. Introduction.

We shall consider singular integrals of the form

$$v(x)=\int_{E_m}\frac{f(x,\theta)}{r^{m+p}}u(y)\,dy \qquad (1)$$

where $x=(x_1,x_2,\ldots,x_m)$ and $y=(y_1,y_2,\ldots,y_m)$ are points in the m-dimensional euclidean space E_m (or in a domain $D\subset E_m$), $r=|y-x|$ is the euclidean distance between y and x , $\theta=\frac{y-x}{r}$ is the central projection of the point y on the unit sphere S having the centre x .

The function $f(x,\theta)$ is called characteristics of the integral (1), $u(y)$ is the density and x is the pole(singularity) of the integral. The number p is an integer which we shall precise.

If $p=-1$, (1) becomes

$$v(x) = \int_{E_m} \frac{f(x,\theta)}{r^{m-1}} u(y)\,dy \tag{2}$$

that is an integral with a week isolated(i.e. integrable) singularity.

If $p=0$ we get

$$v(x) = \int_{E_m} \frac{f(x,\theta)}{r^{m}} u(y)\,dy \tag{3}$$

that means an integral with strong isolated singularity(non integrable,in the usual sense). That is why we have to consider the last integral defined in principal value, namely:

$$\lim_{\varepsilon \to +0} \int_{r>\varepsilon} \frac{f(x,\theta)}{r^{m}} u(y)\,dy$$

If $p>0$,than the isolated singularity is accentuating, the corresponding integral might be called <u>ultrasingular</u>(this is the term used by the author in[2]; compare his definition with the coresponding definition of a <u>hypersingular</u> integral) and be understood in a sense we shall precise. But let us recall first the following theorem of Tricomi [13] and Michlin [9] .

Theorem.If:

1) in any bounded domain of E_m the density $u(x)$ is a Lipschitz function with exponent α , $0<\alpha\leq 1$;

2) at great distances from the origin $u(x)=O(|x|^{-k})$, $k>0$;

3) the characteristic is a bounded function and for fixed x is a continuous function of θ , than

for the existence(in principal value) of the singular integral (3) it is necessary and sufficient that

$$\int_S F(x,\theta)\,dS=0 \tag{4}$$

where S is the unit sphere from E_m which is swept by the point θ and dS -the surface element of this sphere.

The proof of this theorem may be found in [9]. The above mentioned authors have established the differentiation rule for integrals with week singularities :

$$w(x)=\int_D \frac{\phi(x,\theta)}{r^{m-1}}u(y)\,dy$$

This rule is the following:

$$\frac{\partial w(x)}{\partial x_k}=\int_D u(y)\frac{\partial}{\partial x_k}\left[\frac{\phi(x,\theta)}{r^{m-1}}\right]dy-u(x)\int_S \phi(x,\theta)\cos(r,x_k)\,d\sigma \tag{5}$$

k = 1,2,...,m

This rule is valid under the assumptions that $\phi(x,\theta)$ is continuous and bounded together with its first partial derivatives with respect to the cartesian coordinates $x_1,x_2,\dots,x_m$ of the point $x\in D$ and $\theta_1,\theta_2,\dots,\theta_m$ of the point $\theta\in S$.

2.Ultrasingular integrals.

Consider again the integral (1) and let $p\geq 0$. The one dimensional case $m=1, p\geq 0$ was studied by Charles Fox [5] who considered singular integrals of the form

$$\int_a^b \frac{u(y)}{(y-x)^{1+p}}dy, \qquad x\in(a,b) \tag{6}$$

Taking from J.Hadamard [7] the idea of finite part of a divergent integral, Charles Fox defines the principal value of (1) as

$$\lim_{\epsilon\to 0}\left[\int_a^{x-\epsilon}\frac{u(y)}{(y-x)^{1+p}}dy+\int_{x+\epsilon}^{b}\frac{u(y)}{(y-x)^{1+p}}dy-H_{1p}(x,\epsilon)\right] \tag{7}$$

where

$$H_1(x,\epsilon)=\begin{cases} 0 & \text{for } p=0 \\ \sum_{i=0}^{p-1}\frac{u^{(i)}(x)}{i!}\left[\frac{1-(-1)^{p-i}}{(p-i)\epsilon^{p-i}}\right] & \text{for } p\neq 0 \end{cases} \tag{8}$$

For $p=0$ the principal value of the integral (1) in Fox's sense reduces at the principal value of the integral

$$\int_a^b\frac{u(y)}{y-x}dy,$$

considered in the usual sense of Cauchy.

In the same article Charles Fox proved that the principal value, in his sense, exists if the function $u(y)$ verifies the following conditions:

i) $u(y)$ is p times derivable in (a,b) ;

ii) $u^{(p)}(y)\in Lip\alpha$,that is, $u(y)$ is a Lipschitz function in (a,b) with exponent $\alpha, 0<\alpha\leq 1$. .

Charles Fox's definition is extended in a natural way to the case $a=-\infty, b=+\infty$, under the supplementary hypothesis that $u(x)=O(|x|^{-l})$, with $0<l<1$, for $|x|\to\infty$.

In the multidimensional case $m>1$ on the functions $u(x)$ and $f(x,\theta)$ we make the following hypothesis:

1) in any bounded domain from E_m the density $u(x)$ admits partial derivatives up to order p included;

2) the partial derivatives of order p , $D^p u(x)$,are Lipschitz functions in any bounded domain from E_m that means that: for any x and y in such a domain there exists a constant $A>0$ and a number $\alpha, 0<\alpha\leq 1$ such that

$$|D^p u(y)-D^p u(x)|\leq A r^{\alpha} \tag{9}$$

3) at infinity , $u(x)=O(|x|^{\beta})$ with $\beta<p)$;

4) the characteristics $f(x,\theta)$ is a bounded function and for fixed x a continuous function of θ .

Definition. One calls <u>ultrasingular</u> <u>integral</u> the following limit:

$$\lim_{\epsilon\to 0}\left[\int_{\epsilon<r<R}\frac{f(x,\theta)}{r^{m+p}}u(y)\,dy-H_{mp}(x,\epsilon)\right], \tag{10}$$

under the hypothesis that this limit exists and is finite , and

$$H_{mp}(x,\epsilon)=\begin{cases} 0 & \text{for } p=0 \\ \sum_{i=0}^{p-1}\frac{1}{i!}\frac{1}{(p-i)\epsilon^{p-i}}\int_S f(x,\theta)\left[\frac{\partial u(x+r\theta)}{\partial r}\right]^i dS & \text{for } p>0 \end{cases} \tag{11}$$

In (10) we have designated by $\epsilon<r<R$ the set of points $y\in E_m$ satisfying the property that $\epsilon<|y-x|<R, x\in E_m$. In (11), $\left[\frac{\partial u(x+r\theta)}{\partial r}\right]^i$ is the derivative of the function $u(x+r\theta)$ along the radius of the sphere S , at the symbolic power i. The limit (10), if it exists and is finite, is denoted by

$$\overset{*}{\int}_{E_m}\frac{f(x,\theta)}{r^{m+p}}u(y)\,dy \tag{12}$$

and because the above definition represents a generalisation for definition (7) of the principal value of the integral (6) we could call (10) the principal value, in the sense of Charles Fox, of the integral (1).

We shall remind the following

Theorem: if the conditions 1)-4) are satisfied, than for the existence of the ultrasingular integral (10) it is necessary and sufficient that

$$H_{mp}^{(1)}(x)=0 \tag{13}$$

where

$$H_{m,p}^{(1)}(x)=\begin{cases}\int_S f(x,\theta)\,dS & \text{for } p=0\\ \frac{1}{p!}\int_S f(x,\theta)\,[\frac{\partial u(x+r\theta)}{\partial r}]^i ds & \text{for } p\neq 0\end{cases} \tag{14}$$

The proof of this theorem may be found in [2].

3.The derivation of integrals with strong singularities.

Consider the following integral with strong isolated singularity:

$$v(x)=\int_D \frac{\phi(x,\theta)}{r^m}u(y)\,dy, \qquad x\in D \tag{15}$$

Suppose that:

1) the function $\phi(x,\theta)$ is continuous and bounded for any $x\in D,\theta\in S$;

2) $\phi(x,\theta)$ has first partial derivatives with respect to the cartesian coordinates of the point θ , being also continuous and bounded when $x\in D, \theta\in S$;

3) $u(y)$ is derivable and its first partial derivatives are Lipschitz functions with exponent $\alpha, 0<\alpha\leq 1$;

4) at infinity $u(x)=O(|x|^{-1}), l>0$; ;

5) $\int_S \phi(x,\theta)\,dS=0$

hence the necessary and sufficient condition for the existence of the integral (15) is verified.

Let us isolate the point x by a ball $r<\epsilon$, let $D_\epsilon=D-(r<\epsilon)$ and calculate the derivative

$$\frac{\partial}{\partial x_1}\int_{D_\epsilon}\frac{\phi(x,\theta)}{r^m}u(y)\,dy=\int_{D_\epsilon}D_1''\left[\frac{\phi(x,\theta)}{r^m}\right]u(y)\,dy+ \tag{16}$$

$$+\int_{D_\epsilon}\frac{D_1'[\phi(x,\theta)]}{r^m}u(y)\,dy-\int_{S_\epsilon}u(y)\,\frac{\phi(x,\theta)}{\epsilon^m}\cos(r,x_1)\,dS$$

In this formula we have designed by:

D_1 -the derivative with respect to x_1 , calculated under the assumption that r and θ do not depend on x_1 ;

D_1 -the derivative with respect to x_1 calculated under the assumption that only r and x_1 depend on θ ;

S_ϵ -the ball of radius ϵ of centre x ;

$dS_\epsilon=\epsilon^{m-1}dS$ -the surface element of S_ϵ ;

$\alpha_1=(r,x_1)$ -the angle between the vector $y-x$ and the unit vector of the Ox_1 axis.

We have:

$$D_1''\left[\frac{\phi(x,\theta)}{r^m}\right]=\frac{\partial\phi}{\partial\alpha_1}\frac{\partial\alpha_1}{\partial x_1}\frac{1}{r^m}-\frac{m}{r^{m+1}}\phi\frac{\partial r}{\partial x_1}=$$

$$=\frac{1}{r^{m+1}}\left[m\phi\cos\alpha_1+\frac{\partial\phi}{\partial\alpha_1}\sin\alpha_1\right] \tag{17}$$

and putting

$$f(x,\theta)=m\phi\cos\alpha_1+\frac{\partial\phi}{\partial\alpha_1}\sin\alpha_1$$

we have

$$D_1''\left[\frac{\phi(x,\theta)}{r^m}\right]=\frac{f(x,\theta)}{r^{m+1}} \tag{18}$$

Now let us prove that the function $f(x,\theta)$ satisfies the condition

$$\int_S f(x,\theta)\left[\frac{\partial u(x)}{\partial y_1}\theta_1+\frac{\partial u(x)}{\partial y_2}\theta_2+\ldots+\frac{\partial u(x)}{\partial y_m}\theta_m\right]dS=0 \tag{19}$$

We shall write (19) so:

$$\sum_{i=1}^{m} I_i\frac{\partial u(x)}{\partial y_i}=0 \tag{20}$$

where

$$I_i=\int_S f(x,\theta)\,\theta_i dS \tag{21}$$

We shall prove next that $I_i=0\,(i=1,2,\ldots,m)$ so that the condition (19) will be satisfied.

According to (21) and (17) we have:

$$I_1=\int_S f(x,\theta)\cos\alpha_1 dS=\int_{-\pi}^{\pi} d\alpha_{m-1}\int_0^{\pi}\sin\alpha_{m-2}d\alpha_{m-2}\dots\int_0^{\pi}\sin\alpha^{m-3}\alpha_2 d\alpha_2\times$$

$$\times\int_0^{\pi}[m\phi\cos\alpha_1+\frac{\partial\phi}{\partial\alpha_1}\sin\alpha_1]\cos\alpha_1\sin^{m-2}\alpha_1 d\alpha_1$$

but

$$[m\phi\cos^2\alpha_1+\frac{\partial\phi}{\partial\alpha_1}\sin\alpha_1\cos\alpha_1]\sin^{m-2}\alpha_1=$$

$$=\phi\sin^{m-2}\alpha_1-[\phi\cos 2\alpha_1+(m-2)\phi\cos^2\alpha_1+\frac{1}{2}\frac{\partial\phi}{\partial\alpha_1}\sin 2\alpha_1]\sin^{m-2}\alpha_1=$$

$$=\phi\sin^{m-2}\alpha_1+\frac{\partial}{\partial\alpha_1}[\phi\sin 2\alpha_1\sin^{m-2}\alpha_1]$$

and

$$I_1=\int_S\phi(x,\theta)\,dS+\frac{1}{2}\int_{-\pi}^{\pi}d\alpha_{m-1}\int_0^{\pi}\sin\alpha_{m-2}d\alpha_{m-2}\dots\int_0^{\pi}\sin\alpha^{m-3}\alpha_2 d\alpha_2\times$$

$$\times\int_0^{\pi}\frac{\partial}{\partial\alpha_1}[\phi\sin 2\alpha_1\sin^{m-2}\alpha_1]\,d\alpha_1$$

but

$$\int_0^{\pi}\frac{\partial}{\partial\alpha_1}[\phi\sin 2\alpha_1\sin^{m-2}\alpha_1]\,d\alpha_1=0$$

so $I_1=\int_S\phi(x,\theta)\,dS$, and taking into account 5), $I_1=0$. Generally, for $i=2,3,\dots m-1,$, we have:

$$I_1=\int_{-\pi}^{\pi}d\alpha_{m-1}\int_0^{\pi}\sin\alpha_{m-2}d\alpha_{m-2}\dots\int_0^{\pi}\sin^{m-2}\alpha_i\cos\alpha_i d\alpha_i\times$$

$$\times\int_0^\pi \sin^{m-i+1}\alpha_{i-1}\,d\alpha_{i-1}\ldots\int_0^\pi \sin^{m-2}\alpha_2\,d\alpha_2\times$$

$$\times\int_0^\pi [m\phi\cos\alpha_1+\frac{\partial\phi}{\partial\alpha_1}\sin\alpha_1]\sin^{m-1}\alpha_1\,d\alpha_1$$

because $\theta_i=\frac{y_i-x_i}{r}|_{r=1}=\sin\alpha_1\sin\alpha_2\ldots\sin\alpha_{i-1}\cos\alpha_i$

but

$$m\phi\sin^{m-1}\alpha_1\cos\alpha_1+\frac{\partial\phi}{\partial\alpha_1}\sin^m\alpha_1=\frac{\partial}{\partial\alpha_1}[\phi\sin^m\alpha_1]$$

so

$$\int_0^\pi [m\phi\cos\alpha_1+\frac{\partial\phi}{\partial\alpha_1}\sin\alpha_1]\sin^{m-1}\alpha_1\,d\alpha_1=0$$

hence $I_i=0 \quad (i=2,3,\ldots,m-1)$.

For $i=m$, we have

$\theta_m=\frac{y_m-x_m}{r}|_{r=1}=\sin\alpha_1\sin\alpha_2\ldots\sin\alpha_{m-2}\sin\alpha_{m-1}$

and

$$I_m=\int_{-\pi}^{\pi}\sin\alpha_{m-1}\,d\alpha_{m-1}\int_0^\pi\sin^2\alpha_{m-2}\,d\alpha_{m-2}\ldots\int_0^\pi\sin\alpha^{m-2}\alpha_2\,d\alpha_2\int_0^\pi\frac{\partial}{\partial\alpha_1}[\phi\sin^m\alpha_1]\,d\alpha_1=0$$

hence the condition (19) is satisfied.

Developing the function according to Taylor's formula we can write

$$u(y)=u(x)+[\frac{\partial u(x)}{\partial y_1}(y_1-x_1)+\frac{\partial u(x)}{\partial y_2}(y_2-x_2)+\ldots+\frac{\partial u(x)}{\partial y_m}(y_m-x_m)]+R_1(y)$$

where $R_1(y)$ is the Taylor's formulae rest,

$$R_1(y)=w(y)\,r$$

The function $w(y)$ is continuous in the point x and equal to 0 in this point and, as we have shown, from the condition 3), it results that there exist a constant $B>0$ such that

$$|w(y)|\le Br^{\alpha}$$

Under these circumstances the last integral from the right part of the relation (16) may be written:

$$\int_{S_\varepsilon} u(y)\,\frac{\phi(x,\theta)}{\varepsilon^{m}}\cos(r,x_1)\,dS_\varepsilon=\frac{1}{\varepsilon}u(x)\int_S \phi(x,\theta)\cos\alpha_1\,dS+$$

$$+\int_S\left[\frac{\partial u(x)}{\partial y_1}\theta_1+\frac{\partial u(x)}{\partial y_2}\theta_2+\ldots+\frac{\partial u(x)}{\partial y_m}\theta_m\right]\phi(x,\theta)\cos\alpha_1\,dS+$$

$$+\frac{1}{\varepsilon^{m-1}}\int_{S_\varepsilon} w(y)\,\phi(x,\theta)\cos\alpha_1\,dS_\varepsilon$$

Now we shall prove that for $\varepsilon\to0$, the integral

$$I_\varepsilon=\frac{1}{\varepsilon^{m-1}}\int_{S_\varepsilon} w(y)\,\phi(x,\theta)\cos\alpha_1\,dS_\varepsilon$$

uniformly converges towards 0 in any compact domain of E_m ; indeed

$$|I_\varepsilon|\le\frac{1}{\varepsilon^{m-1}}\int_{S_\varepsilon}|w(y)|\,|\phi(x,\theta)|\,dS_\varepsilon\le B\varepsilon^{\alpha}\int_S|\phi(x,\theta)|\,dS$$

and taking into account 1), it follows that

$$|I_\varepsilon|\le C\varepsilon^{\alpha} \tag{22}$$

where

$$C=B\int_S|\phi(x,\theta)|\,dS$$

From (22) follows that $\lim I_{e_{e\to 0}}=0$. Now let us calculate

$$J_1=\int_S \phi(x,\theta)\cos\alpha_1 dS$$

According to (17) we have:

$$J_1=\int_S f(x,\theta)\,dS-\int_0^{\pi}[(m-1)\phi\cos\alpha_1+\frac{\partial\phi}{\partial\alpha_1}\sin\alpha_1]\,dS=$$

$$=\int_S f(x,\theta)\,dS-\int_{-\pi}^{\pi}d\alpha_{m-1}\int_0^{\pi}\sin\alpha_{m-2}d\alpha_{m-2}\ldots\int_0^{\pi}\sin\alpha^{m-3}\alpha_2 d\alpha_2\times$$

$$\times\int_0^{\pi}\frac{\partial}{\partial x_1}[\phi\sin^{m-1}\alpha_1]\,d\alpha_1=\int_S f(x,\theta)\,dS$$

because the last integral is certainly 0 . Under these conditions, the relation (16) may be written:

$$\frac{\partial}{\partial x_1}\int_{D_e}\frac{\phi(x,\theta)}{r^m}u(y)\,dy=[\int_{D_e}\frac{f(x,\theta)}{r^{m+1}}u(y)\,dy-\frac{1}{e}u(x)\int_S f(x,\theta)\,dS]+$$

$$+\int_{D_e}\frac{D_1'[\phi(x,\theta)]}{r^m}u(y)\,dy-\int_S\phi(x,\theta)\,\frac{\partial u(x+r\theta)}{\partial r}\cos\alpha_1 dS+I_e \qquad (23)$$

Passing to the limit for $e\to 0$ in (23), we get:

$$\frac{\partial}{\partial x_1}\int_D\frac{\phi(x,\theta)}{r^m}u(y)\,dy=\int_D^{*}\frac{f(x,\theta)}{r^{m+1}}u(y)\,dy+$$

$$+\int_D\frac{D_1'[\phi(x,\theta)]}{r^m}u(y)\,dy-\int_S\phi(x,\theta)\,\frac{\partial u(x+r\theta)}{\partial r}\cos\alpha_1 dS \qquad (24)$$

where

$$\int_D^{*} \frac{f(x,\theta)}{r^{m=1}} u(y)\, dy = \lim_{\varepsilon - 0} [\int_{D_\varepsilon} \frac{f(x,\theta)}{r^{m+1}} u(y)\, dy - \frac{1}{\varepsilon} u(x) \int_S f(x,\theta)\, dS] \qquad (25)$$

is the ultrasingular integral(or the principal value in Charles Fox's sense) over the domain D . For the existence of the limit (25) it is necessary and sufficient that (19) will be verified. The integral with strong singularity from (24) exists because

$$\int_S D_1'[\phi(x,\theta)]\, dS = 0$$

which follows from 2) and 5). Finally, in the last integral from (24), $\frac{\partial u(x+r\theta)}{\partial r}$ means the derivative of the function $u(x+r\theta)$ in the direction of the vector $y-x$, that means in the direction of the radius of the unit ball S . This result may immediately be generalized for the derivatives $\frac{\partial v}{\partial x_k}, (k=1,2,\ldots,m)$ . .

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