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ON SEMILINEAR ELLIPTIC EIGENVALUE PROBLEMS

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The existence of eigenfunctions and eigenvalues of the semilinear equation given below is reduced to the study of a constrained critical point problem. For each $r \neq 0$, the corresponding functional admits two local minima while the third critical point is obtained by a mountain pass technique.

More precisely, given $r \in \mathbb{R}$, we look for eigensolutions $(\lambda, u) \in (\mathbb{R}, H_o^1(\Omega))$ of the semilinear equation of the form

$$\begin{cases} Lu = \lambda f(x,u) & \text{in } \Omega, \\ (Lu,u) = 2r, & r \neq 0, \end{cases} \tag{P}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with a smooth boundary $\partial\Omega$. The last relation normalizes the eigensolution $u(x)$. The second order linear operator

$$(Lu)(x) = -\partial_i(a_{ij}(x)\partial_j u(x)) + c(x)u(x), \qquad \partial_i = \frac{\partial}{\partial x_i}, \qquad i,j = 1,2,\dots,N,$$

with the summation convention, is uniformly elliptic and selfadjoint ($a_{ij} = a_{ji}$ and $0 \leq c \in L^\infty(\Omega)$). Upon $f(x,t)$ we make the following hypotheses:

a) $f \in C(\bar{\Omega} \times \mathbb{R}, \mathbb{R})$, $f(x,t)t \geq 0$ and there is a neighborhood U of zero such that $f(x,t)t > 0$ for all $t \in U \setminus \{0\}$ and $x \in \Omega$;

b) $|f(x,t)| \leq k_1 + k_2|t|^p$ with $1 \leq p < \dfrac{N+2}{N-2}$ if $N \geq 3$ and p is unrestricted if $N = 1,2$.

Define $G(u) = \frac{1}{2}(Lu,u)$, $\partial G_r = \{u \in H_o^1(\Omega) \mid G(u) = r\}$ as well as

$$F(x,u) = \int_o^u f(x,t)dt \qquad \text{and} \qquad J(u) = -\int_\Omega F(x,u(x))dx.$$

One can see as in [9,Appendix B] that

1°. $J \in C^1(H_0^1(\Omega),\mathbb{R})$, so J' is locally Lipschitz continuous;

2°. J' is completely continuous, i.e., it carries weakly convergent sequences into strongly convergent ones;

3°. J is weakly continuous in $H_0^1(\Omega)$.

In our setting ∂G_r is regular, i.e. $(G'v,v) \neq 0$ for any $v \in \partial G_r$. The tangent space to ∂G_r at u coincides with the null space $N(G'u)$ and so it has the form

$$T_u = T_u(\partial G_r) = \{ v \in H_0^1(\Omega) \mid (G'u,v) = 0 \}.$$

Recall that $u \in \partial G_r$ is said to be a *critical point of* J *constrained on* ∂G_r if $(J'(u),v) = 0$ for all $v \in T_u$. It is easily seen that any local extremum of the restriction $J|_{\partial G_r}$ is a critical point [6].

Introduce on ∂G_r the mapping

$$J_r'(u) = J'(u) - \frac{(J'(u),u)}{(G'u,u)}\, G'u.$$

We can shown that $u \in \partial G_r$ is a constrained critical point of J to ∂G_r if and only if $J_r'(u) = 0$.

Indeed,define the linear projection from $H_0^1(\Omega)$ onto the tangent space T by setting

$$P_u v = v - \frac{(G'u,v)}{(G'u,u)}\, u \qquad u \in H_0^1(\Omega)$$

and the above equivalence is a result of the relations

$$(J_r(u),(I - P_u)v) = 0$$
$$(J_r'(u),P_u v) = ((J'(u),P_u v).$$

Therefore, a constrained critical point of J to ∂G_r corresponds to a weak solution of (P) with

$$\lambda = \frac{(G'u,u)}{(J'(u),u)} .$$

Therewith $c = J_r(u)$ is called *a critical value* of J_r.

REMARK 1. That $(J'(u),u) \neq 0$ follows from the hypothesis a); indeed, by [12,Lemma 1.0],if $u \neq 0$, then $\text{meas}\{ x \in \Omega \mid |u(x)| \le \varepsilon \} > 0$ for any $\varepsilon > 0$, and so $f(x,u(x))\, u(x) > 0$ on a set of positive measure.

Hence, there is a one-to-one correspondence between the critical points of J_r and the weak solutions of the problem (P). If $f(x,\circ)$ is locally Lipschitz continuous, a standard bootstrap argument shows that u is in fact a classical solution.

Several authors have studied the existence of multiple solutions of problem (P) under the additional restriction that $f(x,\circ)$ is odd [1] while a result when $f(x,\circ)$ is non-odd and ∂G_r sphere-like is given in [14]. A more general case is approached in [13].

The functional J is said to satisfy *the Palais-Smale condition at level* c, $c \in \mathbb{R}$ ((PS_c) for short), if every sequence $\{u_n\} \subset \partial G_r$ with $J_r(u_n) \to c$ and $J_r'(u_n) \to 0$ as $n \to \infty$ contains a convergent subsequence. Moreover, J_r satisfies $(PS)_-$ or $(PS)_+$ if it meets $(PS)_c$ for all $c < 0$ or $c > 0$, respectively.

Similarly as in [12], we can prove

LEMMA. *Let* $\{u_n\} \subset \partial G_r$ *be a sequence such that* $J_r(u_n) \to c$ *as* $n \to \infty$. *Then* $\{u_n\}$ *is bounded in* $H_o^1(\Omega)$.

Consider on $H_o^1(\Omega)$ a more convenient inner product given by

$$<u,v> = \int_\Omega (a_{ij}(x)\partial_i u\, \partial_j v + c(x)uv)dx$$

so that the corresponding norm $\|u\| = <u,u>^{\frac{1}{2}}$ is equivalent to the initial norm, by virtue of the Poincaré inequality.

PROPOSITION [7]. *The functional* J_r *satisfies* $(PS)_c$ *for* $c > 0$.

PROOF. If $J_r(u_n) \to c > 0$ and $J_r'(u_n) \to 0$, then $\{u_n\}$ is bounded according to the previous lemma and $u_n \rightharpoonup u_o$ in $H_o^1(\Omega)$, at least for a subsequence. Then $J_r(u_n) \to J_r(u_o) = c > 0$ by the weak continuity of J_r and $u_o \neq 0$.

Denoting

$$\mu_n = \frac{(J'(u_n),u_n)}{(G'u_n,u_n)}$$

we have $J_r'(u_n) = J'(u_n) - \mu_n G'u_n \longrightarrow 0$. If $\mu_n \to 0$ then $J'(u_n) \longrightarrow J'(u_o) = 0$ contradicting $(J'(u_o),u_o) > 0$ in the REMARK 1. If $\mu_n \to \infty$ then $G'u_n \to 0$

contradicts $(G'u_n, u_n) = 2r \neq 0$. Therefore, by passing to a subsequence, $\mu_n \to \mu_o \neq 0$ which easily implies the strong convergence of $\{G'u_n\}$. Since $(G'u_n, u_n) = \|u_n\|^2$, we infer that $\{u_n\}$ converges strongly. ∎

We obtain a positive and a negative solution by cutting the nonlinearity above or below zero,

$$f^+(x,t) = \begin{cases} f(x,t) & \text{if } t \geq 0, \\ 0 & \text{if } t < 0, \end{cases} \quad \text{and} \quad f^-(x,t) = \begin{cases} 0 & \text{if } t > 0, \\ f(x,t) & \text{if } t \leq 0, \end{cases}$$

as well as

$$F^{\pm}(x,u) = \int_o^u f^{\pm}(x,t)dt \qquad \text{and} \qquad J^{\pm}(u) = -\int_\Omega F^{\pm}(x,u(x))dx$$

and then minimizing the corresponding functionals $J^+(u)$, $J^-(u)$ over ∂G_r. Note, by a), that $J^{\pm}$ are nonpositive on ∂G_r.

Let us first consider the minimization problem

$$\inf_{u \in \partial G_r} J^+(u) = \alpha, \quad \alpha < 0,$$

and select a minimizing sequence $\{u_n\} \subset \partial G_r$ so that $u_n \rightharpoonup u^+$ in $H_o^1(\Omega)$. Due to the weak continuity we have $J^+(u^+) = \lim J^+(u_n)$ implying $-\infty < \alpha$ and $u^+ \not\equiv 0$. One can show, by contradiction as in [11], that $G(u^+) = r$ and so J^+ achieves its minimum on ∂G_r at the point u^+. By applying ultimately the Lagrange multiplier rule, there is some $\lambda \in \mathbb{R}$ such that u^+ is a weak solution in $H_o^1(\Omega)$ of the equation (P). Now, according to L^p- theory [4], $u^+ \in C^{1,\alpha}(\bar{\Omega})$, and via the maximum principle, we infer that

$$Lu^+ = \lambda f(x,u^+), \quad u^+(x) > 0 \quad \text{for} \quad x \in \Omega.$$

It is readily seen that $\lambda > 0$ and, by Hopf's boundary point lemma,

$$\frac{\partial u^+}{\partial n}(x) < 0 \qquad \text{for} \qquad x \in \partial\Omega.$$

We conclude that $J^+(u) = J(u)$ in a C^1- neighborhood of u^+ and hence u^+ is a local minimizer in the C^1- topology of J restricted to ∂G_r.

Replace now f^+ and J^+ by f^- and J^- and obtain again a nonpositive local minimizer u^- in the C^1- topology of J restricted to ∂G_r.

Related to our problem, we are ultimately interested in minimizing of J on the topology of $H_0^1(\Omega)$ and clearly an H_0^1 -neighborhood is much larger than a C^1-neighborhood. In overtaking this step, we apply now the following criterion of H.Brezis-L.Nirenberg [2] and generalized by H.T.Tehrani [11]:

Assume $u_0 \in H_0^1(\Omega)$ is a local minimizer of J in the C^1- topology restricted to ∂G_r. If $(J'(u_0),G'u_0) < 0$, then u_0 is also a minimizer of J in the H_0^1 - topology restricted to ∂G_r.

THEOREM. *Suppose that* f *satisfies hypotheses* a),b). *Then problem* (P) *has at least three solutions.*

PROOF. Pass to the new inner product and have $G(u) = \|u\|^2$. To carry out the above criterion, we need to show that $(J(u^{\pm}),u^{\pm}) < 0$. For u^+, say, one obtains

$$\lambda \int_\Omega f(x,u^+)u^+dx = \int_\Omega (Lu^+,u^+)\,dx = 2r \neq 0$$

whence $\lambda > 0$, by assumption b), and hence

$$(J'(u^+),u^+) = -\int_\Omega f(x,u^+)u^+dx = -\frac{1}{\lambda} < 0.$$

According to the Brezis-Nirenberg-Tehrani result, $u^{\pm}$ are also local minimizers of J constrained to ∂G_r with H^1- topology. Since the restriction of J on ∂G_r satisfies the $(PS)_+$ condition and J has two local minima according to a variant of the Mountain Pass Theorem (see e.g. [3,Corollary 2] or [10,Theorem 9.3]) there exists a third critical point of J on ∂G_r. ∎

This theorem is essentialy due to Li Yongqing [13] under more restrictive conditions upon the nonlinearity f:

a') $f \in C^1(\bar{\Omega} \times \mathbb{R},\mathbb{R})$, $f(x,t)t \geq 0$ and $f(x,t) = 0$ iff $t = 0$;

b') $|\partial_t f(x,t)| \leq k_1 + |t|^{p-1}$ with $1 \leq p \leq \frac{N+2}{N-2}$ if $N \geq 3$ and $1 \leq p < \infty$ if $N = 1,2$.

For the existence of the third solution,a retractable property of f is proved.

In the superlinear case, the constrained critical points under conditions a'),b') have been considered earlier (see, e.g. [5]).

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