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Classroom Notes

THE OSCULATING PLANE IS A NATURAL LIMITING PLANE

Jacob T. B. Beard, Jr. and Yung-Way Liu

ABSTRACT

The osculating plane at a point on a space curve is shown to be a natural limiting plane under standard smoothness hypotheses and using elementary arguments suitable for heuristic presentation in third-semester calculus courses. Given the unit tangent vector (derived in the usual way), the selection of the unit principal normal vector is then no longer a "handwave" selection among infinitely many vectors; rather, one chooses between only two vectors. Moreover, this geometric derivation of the osculating plane may enhance visualization of the moving trihedral. Possible applications range from mathematical physics to rapid prototyping manufacturing processes.

1. **INTRODUCTION.** Let a space curve C be defined by the position vector function $r(t)$ with velocity function $r'(t)$ where both of these functions are smooth; i.e., $r'(t)$ and $r''(t)$ are continuous and nonzero. Then at any point P on C, the unit tangent vector T and unit principal normal vector N both exist. It is a fundamental fact that the osculating plane to C at P, containing these unit vectors, is the "closest" plane to the curve C at the point P [3;249]. Few students populate vector analysis or differential geometry courses these days and discover this result, one not so obvious from the presentations made in current calculus texts in general.

Our own theoretical and pedagogical goals have been achieved earlier, but by *fiat*: Willmore [4;8] defined the osculating plane at P to be the limiting position as $Q \to P$ along C of that plane containing the point Q and the tangent line to C at P. Hopefully, our characterization (i) also makes the aforementioned fact "self-evident" (at least to those who "understand" the tangent line to the curve C at the point P as the limiting line of secant lines to C through P), (ii) enhances visualization of the moving trihedral at P formed by $T, N,\ B = T \times N$, and (iii) may prove to be of practical use. Our Theorem (§3), simply stated: The osculating plane for C at P is the limiting plane of "chordal" planes to C through P.

2. POSSIBLE APPLICATIONS. We are indebted to two of our colleagues at Tennessee Tech for their suggestions and conversations regarding applications of our result to obtain numerical approximations of N, the rectifying plane (containing the vectors B and T), etc.

Krishna Kumar (University Professor of Physics) suggests: The Theorem may be quite useful in minimizing/maximizing multivariate functions. One's initial guess, say for the minimum, lies on some equipotential/equifunctional surface of such a function. The tangent to a smooth curve on that surface represents the direction of (local) minimum change in functional value, while the normal to the curve yields the (locally) steepest gradient. The Theorem should help in finding the directions of such a gradient in an iterative step to the next best guess of the minimum value of the function. In particular, Kumar solved such a numerical problem in the sixties [2]. He felt then that the theoretical physics involved was already lagging behind the accuracy of his mathematical method [2;651]; now, that additional mathematical accuracy would be worthwhile.

Joseph Scardina (Professor of Mechanical Engineering, Manufacturing Center of Excellence) suggests: Another possible application is the CNC (computer numerically controlled) machining of curved surfaces. In [1], Khetawat developed a rapid prototyping CAD/CAM process which takes slices of a model along appropriate planes, determines the IGES files of the upper and lower profile of each slice, then generates the machining codes for a 5 axis CNC machine to manufacture the slices (from which the final object is assembled). Here, one wishes a mill cutter to perform as the moving rectifying plane for the model.

3. THEOREM. Let the space curve C be defined by the position vector function $r(t)$, and let Q_1, P, Q_2 be non-colinear points on C corresponding to $r(t-h_1), r(t), r(t+h_2)$ where both r, r' are smooth in a neighborhood $(t-h, t+h)$ of t with $0 < h_1, h_2 < h$. Let E_p and $E_p(h_1, h_2)$ denote the osculating plane for C at P, and the plane containing the chord $\overline{Q_1Q_2}$ and the point P, respectively. Then

$$\lim_{Q_1 \to P \leftarrow Q_2} E_p(h_1, h_2) = E_p.$$

PROOF. It suffices to show that $\lim\limits_{h_1, h_2 \to 0^+} n(t, h_1, h_2)$ is parallel to $B(t)$, where $B(t)$ is the binormal vector for C at P and $n(t, h_1, h_2)$ is a unit vector at P normal to $E_p(h_1, h_2)$. Since the vector calculus is componentwise and the position-vector function, say $r(t) = < x(t), y(t), z(t) >$, is symmetric in its components, the existence and parallelism details are argued here only for the x-direction of the vector $\lim_{h_1,h_2 \to 0^+} n(t, h_1, h_2)$. Hereafter t is arbitrary but fixed, so we suppress it (unless for clarity) and write $B = B(t)$, $n(h_1, h_2) = n(t, h_1, h_2)$, etc. Also for convenience, we say that a vector v has *x-direction number a* provided b, c exist such that a, b, c are direction numbers for v.

From their definitions:

T has x-direction number x',

N has x-direction number $y'(x''y' - x'y'') + z'(x''z' - x'z'')$,

B has x-direction number $y'z'' - y''z'$.

Let q_1, q_2 denote the vectors from the point P to the points Q_1, Q_2 respectively. From the parametrization $r(t)$ of the curve C we find that q_1, q_2 have respective x-direction numbers $x(t - h_1) - x(t), x(t + h_2) - x(t)$ which, by their Taylor expansions, yield:

$$q_1 \text{ has } x - \text{direction number} \quad -x'(t) + x''(t^*_{1x})h_1/2,$$

$$q_2 \text{ has } x - \text{direction number} \quad x'(t) + x''(t^*_{2x})h_2/2,$$

for some t^*_{1x}, t^*_{2x} satisfying $t - h_1 < t^*_{1x} < t < t^*_{2x} < t + h_2$. Since $n(h_1, h_2)$ is parallel to $q_1 \times q_2$, then

$$(*) \quad n(h_1, h_2) \text{ has } x - \text{direction number } y'(t)[z''(t^*_{1z})h_1 + z''(t^*_{2z})h_2] - z'(t)[y''(t^*_{1y})h_1 + y''(t^*_{2y})h_2].$$

We argue in detail on the first term. Since z'' is continuous on $[t - h_1, t + h_2]$, then trivially or by the Intermediate-Value Theorem there exists $t^*_z \in [t - h_1, t + h_2]$ such that $z''(t^*_z) = [z''(t^*_{1z}) + z''(t^*_{2z})]/2$, so that

$$2z''(t^*_z)(h_1 + h_2) = [z''(t^*_{1z})h_1 + z''(t^*_{2z})h_2] + [z''(t^*_{1z})h_2 + z''(t^*_{2z})h_1],$$

and

$$z''(t^*_{1z})h_1 + z''(t^*_{2z})h_2 = (h_1 + h_2)\{2z''(t^*_z) - [z''(t^*_{1z})h_2 + z''(t^*_{2z})h_1]/(h_1 + h_2)\}.$$

Thus for all $h_1, h_2 \to 0^+$, $n(t, h_1, h_2)$ has x-direction number having as a factor of its "first" summand in (*), the quantity

$$2z''(t_z^*) - [z''(t_{1z}^*)h_2 + z''(t_{2z}^*)h_1]/(h_1 + h_2).$$

Hence $\lim_{h_1, h_2 \to 0^+} n(t, h_1, h_2)$ has x-direction number involving

$$\lim_{h_2 \to 0^+} [2z''(\bar{t}_z) - z''(t)h_2/h_2] = z''(t),$$

where $\bar{t}_z = \lim_{h_1 \to 0^+} t_z^*$ (if it exists, which is immaterial since t_z^* is bounded and forced to t as $h_2 \to 0^+$). By symmetry, $\lim_{h_1, h_2 \to 0^+} n(t, h_1, h_2)$ exists and has the same direction numbers as B : $y'z'' - y''z', z'x'' - z''x', x'y'' - x''y'$.■

Authors' Comment: The straight-ahead argument on the limiting direction of $\lim_{h_1, h_2 \to 0^+} (q_1 \times q_2)||q_1 \times q_2||$ involves "big-Oh" subtleties since $\lim_{h_1, h_2 \to 0^+} (q_1 \times q_2) = \overline{0}$.

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