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NOTES ON TOPOLOGICAL APPLICATIONS OF REGULAR OR σ-SMOOTH MEASURES TO WALLMAN TYPE SPACES

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1. INTRODUCTION. Let X be an arbitrary set, and $\mathcal{L}$ a lattice of subsets of X. It is assumed throughout the paper that $\emptyset, X \in \mathcal{L}$.

We adhere to the customary lattice - topological definitions which can be found for example in [1],[2],[4],[7] and [10]. Here, we just note some of the measure theoretic equivalents. For this purpose we introduce the following notations: $\mathcal{A}(\mathcal{L})$ denotes the algebra generated by $\mathcal{L}$, and $I(\mathcal{L})$ the set of non-trivial zero-one valued finitely additive measures on $\mathcal{A}(\mathcal{L})$. $I_R(\mathcal{L})$ the set of $\mathcal{L}$-regular measures of $I(\mathcal{L})$, where $\mu \in I(\mathcal{L})$ is $\mathcal{L}$-regular if for any $A \in \mathcal{A}(\mathcal{L})$ $\mu(A) = \sup\{\mu(L) / L \subset A, L \in \mathcal{L}\}$. $I_\sigma(\mathcal{L})$ the set of σ-smooth measures of $I(\mathcal{L})$ on $\mathcal{L}$, where $\mu \in I(\mathcal{L})$ is σ-smooth on $\mathcal{L}$ if for all sequences $\{L_n\}$ of sets of $\mathcal{L}$ with $L_n \downarrow \emptyset$, $\mu(L_n) \longrightarrow 0$. $I^\sigma(\mathcal{L})$ the set of σ-smooth measures on $\mathcal{A}(\mathcal{L})$ of $I(\mathcal{L})$. $I_R^\sigma(\mathcal{L})$ the set of $\mathcal{L}$-regular measures of $I^\sigma(\mathcal{L})$. $\Pi(\mathcal{L}) = \{\pi$, defined on $\mathcal{L}$, non-trivial, monotone, and $\pi(A \cap B) = \pi(A)\pi(B)$, $A, B \in \mathcal{L}\}$ the set of all premeasures on $\mathcal{L}$. $\Pi_\sigma(\mathcal{L})$ is the set of all premeasures on $\mathcal{L}$ which are σ-smooth on $\mathcal{L}$.

Note that there exists a one-to-one correspondence between:

$\mathcal{L}$-filters $\mathcal{F}$ and elements of $\Pi(\mathcal{L})$ given by $\pi(L)=1$ iff $L \in \mathcal{F}$.

$\mathcal{L}$-filters with countable intersection property and $\Pi_\sigma(\mathcal{L})$.

All elements of $I(\mathcal{L})$ and all prime $\mathcal{L}$-filters, given by: for any $\mu \in I(\mathcal{L})$ we associate the prime $\mathcal{L}$-filter given by :

$\mathcal{F} = \{ A \in \mathcal{L} / \mu(A)=1 \}$.

All elements of $I_R(\mathcal{L})$ and all $\mathcal{L}$-ultrafilters, given by the following rule: with each $\mathcal{L}$-ultrafilter $\mathcal{F}$ we associate the zero-one valued measure defined on $\mathcal{A}(\mathcal{L})$ by:

$$\mu(E) = \begin{cases} 1 \text{ if there exists } A \in \mathcal{F},\ A \subset E \\ 0 \text{ if there exists } A \in \mathcal{F},\ A \subset E' \end{cases} .$$

The support of $\mu \in I(\mathcal{L})$ is $S(\mu) = \bigcap \{L \in \mathcal{L} / \mu(L)=1\}$.
With this notation, we now note: $\mathcal{L}$ is compact iff $S(\mu) \neq \emptyset$ for every $\mu \in I_R(\mathcal{L})$. $\mathcal{L}$ is countably compact iff $I_R(\mathcal{L}) = I_R^{\sigma}(\mathcal{L})$. $\mathcal{L}$ is normal iff for each $\mu \in I(\mathcal{L})$, there exists a unique $\nu \in I_R(\mathcal{L})$ such that $\mu \leq \nu\ (\mathcal{L})$ i.e. $\mu(L) \leq \nu(L)$ for all $L \in \mathcal{L}$. $\mathcal{L}$ is regular iff whenever $\mu_1, \mu_2 \in I(\mathcal{L})$ and $\mu_1 \leq \mu_2(\mathcal{L})$, then $S(\mu_1) = S(\mu_2)$. $\mathcal{L}$ is replete iff for any $\mu \in I_R^{\sigma}(\mathcal{L})$, $S(\mu) \neq \emptyset$. $\mathcal{L}$ is prime-complete iff for any $\mu \in I_{\sigma}(\mathcal{L})$, $S(\mu) \neq \emptyset$. $\mathcal{L}$ is Lindelöf iff for any $\pi \in \Pi_{\sigma}(\mathcal{L})$, $S(\pi) \neq \emptyset$.

2. THE SPACES $I_R^{\sigma}(\mathcal{L})$, $I_{\sigma}(\mathcal{L})$ AND THE LATTICES $\mathcal{W}_{\sigma}(\mathcal{L})$, $\mathcal{V}_{\sigma}(\mathcal{L})$

We consider in this section the important space $I_R^{\sigma}(\mathcal{L})$; for $A \in \mathcal{A}(\mathcal{L})$ define $W_{\sigma}(A) = \{ \mu \in I_R^{\sigma}(\mathcal{L}) \mid \mu(A)=1 \}$
Then, assuming $\mathcal{L}$ is disjunctive, $\mathcal{W}_{\sigma}(\mathcal{L}) = \{W_{\sigma}(L) / L \in \mathcal{L}\}$ is a lattice in $I_R^{\sigma}(\mathcal{L})$ isomorphic to $\mathcal{L}$, under the map $L \to W_{\sigma}(L)$, $L \in \mathcal{L}$, and $\mathcal{A}(\mathcal{W}_{\sigma}(\mathcal{L})) = \mathcal{W}_{\sigma}(\mathcal{A}(\mathcal{L}))$. Also the map $\mu \to \mu'$, where $\mu'(W_{\sigma}(A)) = \mu(A)$, $A \in \mathcal{A}(\mathcal{L})$ is a bijection between $I_R^{\sigma}(\mathcal{L})$ and $I_R^{\sigma}(\mathcal{W}_{\sigma}(\mathcal{L}))$.It is well known that $\mathcal{W}_{\sigma}(\mathcal{L})$ is replete and is a basis for the closed sets $\tau\mathcal{W}_{\sigma}(\mathcal{L})$, all arbitrary intersections of sets of $\mathcal{W}_{\sigma}(\mathcal{L})$. It is this topological space $I_R^{\sigma}(\mathcal{L}), \tau\mathcal{W}_{\sigma}(\mathcal{L})$, and lattice $\mathcal{W}_{\sigma}(\mathcal{L})$ which we will consider here and subsequent sections.Analogously, we also consider $I_{\sigma}(\mathcal{L})$ and $\mathcal{V}_{\sigma}(\mathcal{L})$; here we do not need the assumption of disjunctiveness on $\mathcal{L}$, and $\mathcal{V}_{\sigma}(\mathcal{L}) = \{V_{\sigma}(L) / L \in \mathcal{L}\}$ where $V_{\sigma}(A) = \{\mu \in I_{\sigma}(\mathcal{L}) / \mu(A)=1\}$,$A \in \mathcal{A}(\mathcal{L})$. $\mathcal{V}_{\sigma}(\mathcal{L})$ is prime complete, and is a base for the closed sets $\tau\mathcal{V}_{\sigma}(\mathcal{L})$ of $I_{\sigma}(\mathcal{L})$.

<u>Theorem 2.1</u> a). Consider $I_R^{\sigma}(\mathcal{L})$ and $\mathcal{W}_{\sigma}(\mathcal{L})$ with $\mathcal{L}$ disjunctive. $\mathcal{W}_{\sigma}(\mathcal{L})$ is regular iff for all $\mu_1, \mu_2 \in I(\mathcal{L})$ and $\nu \in I_R^{\sigma}(\mathcal{L})$, if $\mu_1 \leq \mu_2(\mathcal{L})$ and $\mu_1 \leq \nu\ (\mathcal{L})$ then $\mu_2 \leq \nu\ (\mathcal{L})$.

b). The topological space $I_R^{\sigma}(\mathcal{L}), \tau\mathcal{W}_{\sigma}(\mathcal{L})$ with $\mathcal{L}$ disjunctive is considered. Then the space is T_2 iff for $\mu \in I(\mathcal{L})$ and $\mu \leq \nu_1(\mathcal{L})$, $\mu \leq \nu_2(\mathcal{L})$ where $\nu_1, \nu_2 \in I_R^{\sigma}(\mathcal{L})$ it follows that $\nu_1 = \nu_2$.

c). Consider $I_\sigma(\mathcal{L})$ and $\mathcal{V}_\sigma^t(\mathcal{L})$. $\mathcal{V}_\sigma^t(\mathcal{L})$ is regular iff for all $\mu_1, \mu_2 \in I(\mathcal{L})$ and $\nu \in I_\sigma(\mathcal{L})$ if $\mu_1 \le \mu_2(\mathcal{L})$ and $\mu_1 \le \nu(\mathcal{L})$ then $\mu_2 \le \nu(\mathcal{L})$.

d). Consider the topological space $I_\sigma(\mathcal{L})$, $\tau\mathcal{V}_\sigma^t(\mathcal{L})$. This space is T_2 iff for $\mu \in I(\mathcal{L})$ with $\mu \le \nu_1(\mathcal{L})$ and $\mu \le \nu_2(\mathcal{L})$ where $\nu_1, \nu_2 \in I_\sigma(\mathcal{L})$, it follows $\nu_1 = \nu_2$.

Proof. The proofs for a) and c) and for b) and d) are similar We just prove a) and b).

a). Let $\mu_1, \mu_2 \in I(\mathcal{L})$ such that $\mu_1 \le \mu_2(\mathcal{L})$. Then there exist $\mu_1', \mu_2' \in I(\mathcal{W}_\sigma^t(\mathcal{L}))$ and $\mu_1'(W_\sigma(L)) = \mu_1(L)$, $\mu_2'(W_\sigma(L)) = \mu_2(L)$ for all $L \in \mathcal{L}$. $\mu_1(L) \le \mu_2(L) \Rightarrow \mu_1' \le \mu_2'$ on $\mathcal{W}_\sigma^t(\mathcal{L})$.
Suppose $\mathcal{W}_\sigma^t(\mathcal{L})$ is regular . Then $S(\mu_1') = S(\mu_2')$ where $S(\mu_1') = \bigcap \{W_\sigma(L) \in \mathcal{W}_\sigma^t(\mathcal{L}) / \mu_1'(W_\sigma(L) = 1, L \in \mathcal{L}\}$

Let now $\nu \in I_R^\sigma(\mathcal{L})$ with $\mu_1 \le \nu$. We have $\nu' \in I_R^\sigma(\mathcal{W}_\sigma^t(\mathcal{L}))$ and $\mu_1' \le \nu'$ on $\mathcal{W}_\sigma^t(\mathcal{L})$, therefore $S(\nu') \subset S(\mu_1') = S(\mu_2')$; hence $\mu_2' \le \nu'$ on $\mathcal{W}_\sigma^t(\mathcal{L})$ i.e. $\mu_2 \le \nu$ on $\mathcal{L}$.
Conversely, let μ_1, $\mu_2 \in I(\mathcal{L})$ and $\nu \in I_R^\sigma(\mathcal{L})$ such that if $\mu_1 \le \mu_2(\mathcal{L})$ and $\mu_1 \le \nu(\mathcal{L})$ then $\mu_2 \le \nu(\mathcal{L})$. Let now $\lambda_1, \lambda_2 \in I(\mathcal{W}_\sigma^t(\mathcal{L}))$ and assume $\lambda_1 \le \lambda_2$ on $\mathcal{W}_\sigma^t(\mathcal{L})$. Then $\lambda_1 = \mu_1'$ and $\lambda_2 = \mu_2'$ where $\mu_1, \mu_2 \in I(\mathcal{L})$ and $\mu_1' \le \mu_2'(\mathcal{W}_\sigma^t(\mathcal{L}))$ i.e. $\mu_1 \le \mu_2(\mathcal{L})$.
Now $S(\mu_2') \subset S(\mu_1')$. If $\lambda \in S(\mu_1')$, then clearly $\lambda \in I_R^\sigma(\mathcal{L})$ and $\mu_1 \le \lambda(\mathcal{L})$. Hence by the assumption $\mu_2 \le \lambda(\mathcal{L})$ which implies $\lambda \in S(\mu_2')$.

b) Suppose $I_R^\sigma(\mathcal{L})$, $\tau\mathcal{W}_\sigma^t(\mathcal{L})$ is T_2 which implies that $\mathcal{W}_\sigma^t(\mathcal{L})$ is T_2, and let μ, ν_1, ν_2 as above. Then $\mu' \le \nu_1'$ on $\mathcal{W}_\sigma^t(\mathcal{L})$ where $\mu' \in I(\mathcal{W}_\sigma^t(\mathcal{L}))$ and $\nu_1' \in I_R^\sigma(\mathcal{W}_\sigma^t(\mathcal{L}))$, which implies $\nu_1 \in S(\nu_1') \subset S(\mu')$. Also $\mu' \le \nu_2'$ on $\mathcal{W}_\sigma^t(\mathcal{L})$ where $\nu_2' \in I_R^\sigma(\mathcal{W}_\sigma^t(\mathcal{L}))$,which implies $\nu_2 \in S(\nu_2') \subset S(\mu')$. Recall that $\mathcal{L}$ is T_2 iff for each $\mu \in I(\mathcal{L})$, $S(\mu) = \emptyset$ or a singleton, hence since $\mathcal{W}_\sigma^t(\mathcal{L})$ is T_2 it follows $\nu_1 = \nu_2$.
Conversely, assume that for $\mu \in I(\mathcal{L})$ and $\nu_1, \nu_2 \in I_R^\sigma(\mathcal{L})$, if $\mu \le \nu_1(\mathcal{L})$ and $\mu \le \nu_2(\mathcal{L})$ then $\nu_1 = \nu_2$. Suppose $S(\mu') \ne \emptyset$, where $\mu \in I(\mathcal{L})$, $\lambda \in I(\mathcal{W}_\sigma^t(\mathcal{L}))$ and $\lambda = \mu'$. If $\nu_1, \nu_2 \in S(\mu')$ then $\mu \le \nu_1(\mathcal{L})$ and $\mu \le \nu_2(\mathcal{L})$ i.e. $\nu_1 = \nu_2$. Therefore $\mathcal{W}_\sigma^t(\mathcal{L})$ is T_2 and thus $\tau\mathcal{W}_\sigma^t(\mathcal{L})$ is T_2.

Theorem 2.2 Consider $I_\sigma(\mathcal{L})$ and $\mathcal{V}_\sigma^t(\mathcal{L})$. $\mathcal{V}_\sigma^t(\mathcal{L})$ is regular iff $I_\sigma(\mathcal{L})=I_R^\sigma(\mathcal{L})$.

Proof. Suppose $I_\sigma(\mathcal{L})=I_R^\sigma(\mathcal{L})$. Then $\mathcal{V}_\sigma^t(\mathcal{L})=\mathcal{W}_\sigma^t(\mathcal{L})$. Now let $\mu_1,\mu_2\in I(\mathcal{L})$, $\nu\in I_\sigma(\mathcal{L})$ and $\mu_1\leq\mu_2(\mathcal{L})$, $\mu_1\leq\nu(\mathcal{L})$. Then, since $I_\sigma(\mathcal{L})=I_R^\sigma(\mathcal{L})$, $\mu_1\in I_R^\sigma(\mathcal{L})$ so $\mu_1=\mu_2$ and $\mu_1=\nu$.
Conversely, suppose $\mathcal{V}_\sigma^t(\mathcal{L})$ is regular and let $\mu\in I_\sigma(\mathcal{L})$; there exists $\nu\in I_R(\mathcal{L})$ such that $\mu\leq\nu$ $(\mathcal{L})$ i.e. $\mu'\leq\nu'$ $(\mathcal{V}_\sigma^t(\mathcal{L}))$, where $\mu',\nu'\in I_\sigma(\mathcal{V}_\sigma^t(\mathcal{L}))$. But $S(\mu')=S(\nu')$ since $\mathcal{V}_\sigma^t(\mathcal{L})$ is regular. Hence $\mu\in S(\nu')$ i.e. $\nu\leq\mu(\mathcal{L})$. It follows $\mu=\nu$ and then $\mu\in I_R^\sigma(\mathcal{L})$.

3. ON NORMAL, SLIGHTLY NORMAL, MILDLY NORMAL AND LINDELOF LATTICES

In this section we wish to consider normality and related questions as well as Lindelöf properties concerning the lattices $\mathcal{W}_\sigma^t(\mathcal{L})$ in $I_R^\sigma(\mathcal{L})$

Definition 3.1

a) $\mathcal{L}$ is slightly normal if for all $\mu\in I_\sigma(\mathcal{L}')$, there exists a unique $\nu\in I_R'(\mathcal{L})$ such that $\mu\leq\nu$ $(\mathcal{L})$.

b) $\mathcal{L}$ is mildly normal if for all $\mu\in I_\sigma(\mathcal{L})$, there exists a unique $\nu\in I_R(\mathcal{L})$ such that $\mu\leq\nu(\mathcal{L})$.

c) $\mathcal{L}$ is almost countably compact if $\mu\in I_R(\mathcal{L}')$ implies $\mu\in I_\sigma(\mathcal{L})$.

Theorem 3.1 Suppose $\mathcal{L}$ is disjunctive. Then

a) Consider $I_R^\sigma(\mathcal{L})$ and $\mathcal{W}_\sigma^t(\mathcal{L})$ and suppose $\mathcal{L}$ is Lindelöf and satisfies the condition: for all $\mu_1,\mu_2\in I(\mathcal{L})$ and $\nu\in I_R^\sigma(\mathcal{L})$, if $\mu_1\leq\mu_2(\mathcal{L})$ and $\mu_1\leq\nu(\mathcal{L})$, then $\mu_2\leq\nu$ $(\mathcal{L})$. Then $\mathcal{W}_\sigma^t(\mathcal{L})$ is slightly and mildly normal.

b) If $\mathcal{L}$ is complement generated then $\mathcal{W}_\sigma^t(\mathcal{L})$ is slightly normal.

c) If $\mathcal{L}$ is almost countably compact and mildly normal then $\mathcal{W}_\sigma^t(\mathcal{L})$ is normal.

Proof. a) $\mathcal{L}$ disjunctive and Lindelöf implies $\mathcal{W}_\sigma^t(\mathcal{L})$ Lidelöf Also, by Theorem 2.1 it follows that $\mathcal{W}_\sigma^t(\mathcal{L})$ is regular. Then $\mathcal{W}_\sigma^t(\mathcal{L})$ is slightly and mildly normal (see [4]).

b) $\mathcal{L}$ complement generated implies $L=\bigcap_1^\infty L_n'$, L and $L_n\in\mathcal{L}$, all n. $W_\sigma(L)=W_\sigma(\bigcap_1^\infty L_n')=\bigcap_1^\infty W_\sigma(L_n')=\bigcap_1^\infty [W_\sigma(L_n)]'$. Hence $\mathcal{W}_\sigma^t(\mathcal{L})$ complement generated which implies $\mathcal{W}_\sigma^t(\mathcal{L})$ slightly normal(see [4]).

c) By the assumption, for any $\mu \in I_R(\mathcal{L}')$ it follows $\mu \in I_\sigma(\mathcal{L})$ and then there exists a unique $\nu \in I_R(\mathcal{L})$ such that $\mu \leq \nu$ $(\mathcal{L})$. Let $\mu \in I(\mathcal{L})$ such that $\mu \leq \lambda(\mathcal{L}')$ with $\lambda \in I_R(\mathcal{L}')$. Also $\lambda \in I_\sigma(\mathcal{L})$ and $\lambda \leq \mu \leq \nu_1$ on $\mathcal{L}$ with $\nu_1 \in I_R(\mathcal{L})$, unique. Therefore if $\mu \leq \nu_2(\mathcal{L})$ with $\nu_2 \in I_R(\mathcal{L})$ then $\lambda \leq \mu \leq \nu_2(\mathcal{L})$, and so $\nu_1 = \nu_2$. Hence $\mathcal{L}$ is normal and also $\mathcal{W}_\sigma^{\leq}(\mathcal{L})$ is normal.

Remark. Consider $I_\sigma(\mathcal{L})$ and $\mathcal{V}_\sigma^{\leq}(\mathcal{L})$ with $\mathcal{L}$ Lindelöf. If for all $\mu_1, \mu_2 \in I(\mathcal{L})$ and $\nu \in I_\sigma(\mathcal{L})$ such that if $\mu_1 \leq \mu_2(\mathcal{L})$ and $\mu_1 \leq \nu$ $(\mathcal{L})$ it follows that $\mu_2 \leq \nu(\mathcal{L})$, then $\mathcal{V}_\sigma^{\leq}(\mathcal{L})$ is slightly and mildly normal.

Proof. Similar to a) of Theorem 3.1.

We next consider the following condition:

(1) For any $\pi \in \widetilde{\Pi}_\sigma(\mathcal{L})$, there exists $\nu \in I_\sigma(\mathcal{L})$ such that $\pi \leq \nu$ $(\mathcal{L})$.

Theorem 3.2.

a) If condition (1) is satisfied and if $\mathcal{L}$ is prime complete then $\mathcal{L}$ is Lindelöf.

b) If $\mathcal{L}$ is Lindelöf then condition (1) holds.

c) $\mathcal{L}$ satisfies condition (1) iff $I_\sigma(\mathcal{L})$, $\tau\mathcal{V}_\sigma^{\leq}(\mathcal{L})$ is Lindelöf.

Proof. a) Let $\pi \in \widetilde{\Pi}_\sigma(\mathcal{L})$ be an $\mathcal{L}$-filter with the countable intersection property. By condition (1) there exists $\nu \in I_\sigma(\mathcal{L})$ and $\pi \leq \nu$ $(\mathcal{L})$. $\mathcal{L}$ prime complete implies $S(\nu) \neq \emptyset$ and then $S(\pi) \neq \emptyset$.

b) Let $\pi \in \widetilde{\Pi}_\sigma(\mathcal{L})$. Since $\mathcal{L}$ is Lindelöf, $S(\pi) \neq \emptyset$ and therefore there exists $x \in X$ such that $x \in S(\pi)$. Then $\pi \leq \mu_x(\mathcal{L})$ and $\mu_x \in I_\sigma(\mathcal{L})$.

c) Suppose that $\mathcal{L}$ satisfies condition (1). Let $\pi' \in \widetilde{\Pi}_\sigma(\mathcal{V}_\sigma^{\leq}(\mathcal{L}))$ and define $\pi(L) = \pi'(V_\sigma(L))$, $L \in \mathcal{L}$. If $L_n \downarrow \emptyset$, $L_n \in \mathcal{L}$ then $V_\sigma(L_n) \downarrow \emptyset$ and $\pi(L_n) = \pi'(V_\sigma(L_n)) \longrightarrow 0$, i.e. $\pi \in \widetilde{\Pi}_\sigma(\mathcal{L})$. By condition (1), there exists $\nu \in I_\sigma(\mathcal{L})$ such that $\pi \leq \nu$ $(\mathcal{L})$. Hence $\nu' \in I_\sigma(\mathcal{V}_\sigma^{\leq}(\mathcal{L}))$ and $\pi' \leq \nu'$ on $\mathcal{V}_\sigma^{\leq}(\mathcal{L})$, where $\nu'(V_\sigma(L)) = \nu(L)$, $L \in \mathcal{L}$. Therefore $\mathcal{V}_\sigma^{\leq}(\mathcal{L})$ satisfies condition (1). Next, we show that $\mathcal{V}_\sigma^{\leq}(\mathcal{L})$ is prime complete. For this, let $S(\nu') = \bigcap_{\alpha \in A} \{ V_\sigma(L_\alpha) / \nu'(V_\sigma(L_\alpha)) = 1, L_\alpha \in \mathcal{L} \}$. But

$\nu'(V_\sigma(L))=1$ iff $\nu(L)=1$, iff $\nu \in V_\sigma(L)=\{\mu \in I_\sigma(\mathcal{L}) / \mu(L)=1, L\in\mathcal{L}\}$. Hence $V_\sigma(L)\neq\emptyset$ which implies $S(\nu')\neq\emptyset$. Now, $\mathcal{V}_\sigma^t(\mathcal{L})$ satisfies condition (1) and prime complete implies $\mathcal{V}_\sigma^t(\mathcal{L})$ Lindelöf and then $\tau\mathcal{V}_\sigma^t(\mathcal{L})$ is Lindelöf.

Conversely, let $(I_\sigma(\mathcal{L}), \tau\mathcal{V}_\sigma^t(\mathcal{L}))$ be Lindelöf. Let $\pi\in\tilde{I}_\sigma(\mathcal{L})$ and define $\pi'(V_\sigma(L))=\pi(L), L\in\mathcal{L}$. Then $V_\sigma(L_n)\downarrow\emptyset$ implies $L_n\downarrow\emptyset$ and $\pi(L_n)=\pi'(V_\sigma(L_n))\longrightarrow 0$, hence $\pi'\in\tilde{I}_\sigma(\mathcal{V}_\sigma^t(\mathcal{L}))$. $\tau\mathcal{V}_\sigma^t(\mathcal{L})$ Lindelöf implies $\mathcal{V}_\sigma^t(\mathcal{L})$ Lindelöf and then $\mathcal{V}_\sigma^t(\mathcal{L})$ satisfies condition (1); hence there exists $\nu'\in I_\sigma(\mathcal{V}_\sigma^t(\mathcal{L}))$ such that $\pi'\leq\nu'$ $(\mathcal{V}_\sigma^t(\mathcal{L}))$, where $\nu'(V_\sigma(L))=\nu(L), L\in\mathcal{L}$. $\pi(L)=1$ implies $\pi'(V_\sigma(L))=1$ and then $\nu'(V_\sigma(L))=1$ i.e. $\nu(L)=1, L\in\mathcal{L}$. Hence $\pi\leq\nu$ $(\mathcal{L})$.

4. ON PRIME COMPLETE AND COUNTABLY COMPACT LATTICES

In this section we investigate the equivalence and consequences of stronger lattice completeneness assumption.

Theorem 4.1 Let $\mathcal{L}$ be a disjunctive lattice. $\mathcal{W}_\sigma^t(\mathcal{L})$ is prime complete iff for $\mu\in I_\sigma(\mathcal{L})$ there exists $\nu\in I_R^\sigma(\mathcal{L})$ such that $\mu\leq\nu$ $(\mathcal{L})$.

Proof. Let $\mu\in I_\sigma(\mathcal{L})$ and the associated $\mu'\in I_\sigma(\mathcal{W}_\sigma^t(\mathcal{L}))$ defined by $\mu'(W_\sigma(L))=\mu(L), L\in\mathcal{L}$. If $\mathcal{W}_\sigma^t(\mathcal{L})$ is prime complete, $S(\mu')\neq\emptyset$ and then there exists $\nu\in S(\mu')$, $\nu\in I_R^\sigma(\mathcal{L})$ and it follows that $\mu\leq\nu$ $(\mathcal{L})$. Conversely, let $\mu'\in I_\sigma(\mathcal{W}_\sigma^t(\mathcal{L}))$ and consider the associated $\mu\in I_\sigma(\mathcal{L})$ such that $\mu'(W_\sigma(L))=\mu(L)$. For $\mu\in I_\sigma(\mathcal{L})$, there exists $\nu\in I_R^\sigma(\mathcal{L})$ such that $\mu\leq\nu$ $(\mathcal{L})$. Therefore $\nu'\in I_R^\sigma(\mathcal{W}_\sigma^t(\mathcal{L}))$ and $\mu'\leq\nu'$ $(\mathcal{W}_\sigma^t(\mathcal{L}))$ which implies $S(\nu')\subset S(\mu')$ and since $\mathcal{W}_\sigma^t(\mathcal{L})$ is replete, $S(\nu')\neq\emptyset$.

Theorem 4.2

a) Let $\mathcal{L}$ be disjunctive, almost countably compact and mildly normal and let $\mathcal{W}_\sigma^t(\mathcal{L})$ be prime complete. Then $\mathcal{L}$ is countably compact.

b) Let $\mathcal{L}$ be disjunctive, regular, Lindelöf, almost countably compact and let $\mathcal{W}_\sigma^t(\mathcal{L})$ be prime complete . Then $\mathcal{L}$ is countably compact.

Proof. a) Must show that $I_R(\mathcal{L})=I_R^{\sigma}(\mathcal{L})$. Let $\mu\in I_R(\mathcal{L})$; we have $\mu\le\nu$ $(\mathcal{L}')$ where $\nu\in I_R(\mathcal{L}')$. Since $\mathcal{L}$ is almost countably compact we have $\nu\le\mu(\mathcal{L})$ with $\mu\in I_R(\mathcal{L})$ and $\nu\in I_\sigma(\mathcal{L})$. But $\mathcal{W}_\sigma^L(\mathcal{L})$ is prime complete and by Theorem 4.1 there exists $\rho\in I_R^{\sigma}(\mathcal{L})$ such that $\nu\le\rho$ $(\mathcal{L})$. $\mathcal{L}$ almost countably compact and mildly normal implies $\mathcal{L}$ normal (see [4]). By the normality of $\mathcal{L}$ the $\mathcal{L}$-regular measure μ such that $\nu\le\mu$ must be unique, hence $\mu=\rho\in I_R^{\sigma}(\mathcal{L})$.
b) $\mathcal{L}$ regular and Lindelöf implies $\mathcal{L}$ mildly normal and by the above result, it follows that $\mathcal{L}$ is countably compact.

<u>Theorem 4.3</u> Suppose $I_\sigma(\mathcal{L})$, $\mathcal{V}_\sigma^L(\mathcal{L})$ is T_1 and $\mathcal{L}$ disjunctive and $\mathcal{W}_\sigma^L(\mathcal{L})$ prime complete. Then $I_\sigma(\mathcal{L})=I_R^{\sigma}(\mathcal{L})$.

Proof. Since $I_\sigma(\mathcal{L})$, $\mathcal{V}_\sigma^L(\mathcal{L})$ is T_1, given $\mu_1\ne\mu_2$ with $\mu_1,\mu_2\in I_\sigma(\mathcal{L})$, there exist $L_1,L_2\in\mathcal{L}$ such that $\mu_1\in V_\sigma(L_1')$, $\mu_2\notin V_\sigma(L_1')$ and $\mu_2\in V_\sigma(L_2')$, $\mu_1\notin V_\sigma(L_2')$. Therefore $\mu_1(L_1')=1$, $\mu_2(L_1')=0$ or $\mu_1(L_1)=0$, $\mu_2(L_1)=1$ and $\mu_2(L_2')=1$, $\mu_1(L_2')=0$ or $\mu_2(L_2)=0$, $\mu_1(L_2)=1$. Since $\mathcal{W}_\sigma^L(\mathcal{L})$ prime complete, given $\mu\in I_\sigma(\mathcal{L})$ there exists $\nu\in I_R^{\sigma}(\mathcal{L})$ with $\mu\le\nu$ $(\mathcal{L})$. If $\mu\ne\nu$, by above there exists $L\in\mathcal{L}$ such that $\nu(L)=0$ and $\mu(L)=1$.
This is a contradiction, hence $\mu=\nu$, and $I_\sigma(\mathcal{L})=I_R^{\sigma}(\mathcal{L})$.

<u>Definition 4.1</u> Let $\mu\in I(\mathcal{L})$, $E\subset X$ and define
$\mu'(E)=\inf\{\sum_{i=1}^{n}\mu(L_i'),E\subset\bigcup_{i=1}^{n}L_i',L_i\in\mathcal{L}\}=\inf\{\mu(L'),E\subset L',L\in\mathcal{L}\}$.

<u>Definition 4.2</u> Let $\mu\in I_\sigma(\mathcal{L})$, $E\subset X$ and define
$\mu''(E)=\inf\{\sum_{i=1}^{\infty}\mu(L_i'),E\subset\bigcup_{i=1}^{\infty}L_i',\ L_i\in\mathcal{L}\}$.

Clearly, μ' is a finitely subadditive outer measure and μ'' is an outer measure (see [7]). Let $\mathcal{S}_{\mu''}$ be the set of μ''-measurable sets, where E is measurable with respect to μ'' if for any $A\subset X$, $\mu''(A)=\mu''(A\cap E)+\mu''(A\cap E')$

Theorem 4.4 Let $\mu \in I_\sigma(\mathcal{L})$. Suppose $\mathcal{L} \subset \mathcal{S}_{\mu''}$ and $\mathcal{L}$ semiseparates $\delta(\mathcal{L})$. Then $\mu \leq \mu''(\mathcal{L})$ and $\mu''|_{\mathcal{A}(\mathcal{L})} \in I_R^\sigma(\mathcal{L})$.

Proof. Let $\mu \in I_\sigma(\mathcal{L})$. Then we have $\mu \leq \mu''(\mathcal{L})$ and $\mathcal{L} \subset \mathcal{S}_{\mu''}$ which is closed under complement and countable unions (see [7]). Therefore $\mathcal{A}(\mathcal{L}) \subset \mathcal{S}_{\mu''}$. $\mu''|_{\mathcal{A}(\mathcal{L})}$ is then a measure on $\mathcal{A}(\mathcal{L})$. μ'' countably additive implies $\mu''|_{\mathcal{A}(\mathcal{L})} \in I^\sigma(\mathcal{L})$. To show that $\mu''|_{\mathcal{A}(\mathcal{L})} \in I_R(\mathcal{L})$, assume $\mu''(A')=1$, $A \in \mathcal{L}$. Then there exist $\{L_n\}$, $L_n \in \mathcal{L}$ such that $A' \supseteq \bigcap_n L_n$ and $\mu(L_n)=1$ for all n.

But $\bigcap_n L_n \in \delta(\mathcal{L})$ and $A \cap (\bigcap_n L_n)=\emptyset$.Hence by semiseparation there exists $\tilde{L} \in \mathcal{L}$ such that $A \cap \tilde{L}=\emptyset$ (or $\tilde{A} \subset A'$) and $\bigcap_n L_n \subset \tilde{L}$. May assume $L_n \downarrow$ and then $\mu''(\bigcap L_n)=1$. We then have $\bigcap_n L_n \subset \tilde{L} \subset A'$ which implies $\mu''(L)=1$ i.e. $\mu''|_{\mathcal{A}(\mathcal{L})} \in I_R(\mathcal{L})$.

5. STRONGLY σ- SMOOTH MEASURES

Here we consider another general Wallman space and analyze the relevant lattice in detail.

Definition 5.1 A measure $\mu \in I(\mathcal{L})$ is strongly σ-smooth on $\mathcal{L}$ iff for any sequence $\{L_n\}$, $L_n \in \mathcal{L}$,$L_n \downarrow$, if $\bigcap_n L_n \in \mathcal{L}$ then $\mu(\bigcap_n L_n)=\inf_n \mu(L_n)=\lim_{n\to\infty} \mu(L_n)$.
We denote $J(\mathcal{L})$ the set of strongly σ-smooth nontrivial zero-one valued measures on $\mathcal{L}$.

Definition 5.2 The lattice $\mathcal{L}$ is weakly prime complete if for $\mu \in J(\mathcal{L})$, $S(\mu) \neq \emptyset$.

Now define the following condition:

(2) For any $\pi \in \Pi_\sigma(\mathcal{L})$ there exists $\nu \in J(\mathcal{L})$ such that $\pi \leq \nu(\mathcal{L})$.

We summarize a few notes on σ-smoothness that will be used throughout this section for the reader's convenience (see [6])

a) $I^{\sigma}(\mathcal{L}) \subset J(\mathcal{L}) \subset I_{\sigma}(\mathcal{L})$

b) $\mathcal{L}$ normal and complement generated implies $J(\mathcal{L}) \subset I^{\sigma}_{R}(\mathcal{L})$

c) $\mu \in I_{\sigma}(\mathcal{L})$ and $\mu' = \mu''(\mathcal{L}')$ implies $\mu \in J(\mathcal{L})$.

d) Since $\mu \in I^{\sigma}_{R}(\mathcal{L})$ implies $\mu' = \mu''(\mathcal{L}')$, it follows that $\mu \in J(\mathcal{L})$ and then $I^{\sigma}_{R}(\mathcal{L}) \subset J(\mathcal{L})$.

Theorem 5.1

a) If condition (2) holds and if $\mathcal{L}$ is weakly prime complete then $\mathcal{L}$ is Lindelöf.

b) If $\mathcal{L}$ is Lindelöf then condition (2) holds.

Proof. Omitted.

Theorem 5.2 Define $\mathcal{V}_J(\mathcal{L}) = \{ V_J(L) \;/\; L \in \mathcal{L} \}$ where $V_J(L) = \{ \mu \in J(\mathcal{L}) / \mu(L) = 1,\ L \in \mathcal{L} \}$. Then $\mathcal{L}$ satisfies condition (2) iff $J(\mathcal{L}), \tau\mathcal{V}_J(\mathcal{L})$ is Lindelöf.

Proof. Suppose $\mathcal{L}$ satisfies condition (2) we show that $\mathcal{V}_J(\mathcal{L})$ satisfies condition (2). For this, let $\pi' \in \tilde{\Pi}_{\sigma}(\mathcal{V}_J(\mathcal{L}))$ and define $\pi(L) = \pi'(V_J(L))$, $L \in \mathcal{L}$. If $L_n \downarrow \emptyset$, $L_n \in \mathcal{L}$ then $V_J(L_n) \downarrow \emptyset$ and $\lim_n \pi(L_n) = \lim_n \pi'(V_J(L_n)) = 0$, hence $\pi \in \tilde{\Pi}_{\sigma}(\mathcal{L})$.
By condition (2) there exists $\nu \in J(\mathcal{L})$ such that $\pi \leq \nu\ (\mathcal{L})$.Hence $\nu' \in J(\mathcal{V}_J(\mathcal{L}))$ and $\pi' \leq \nu'$ on $\mathcal{V}_J(\mathcal{L})$ where $\nu'(V_J(L)) = \nu(L)$, $L \in \mathcal{L}$. For $\pi' \in \tilde{\Pi}_{\sigma}(\mathcal{V}_J(\mathcal{L}))$, there exists $\nu' \in J(\mathcal{V}_J(\mathcal{L}))$ such that $\pi' \leq \nu'$ on $\mathcal{V}_J(\mathcal{L})$.
Next we show that $\mathcal{V}_J(\mathcal{L})$ is weakly prime complete; let $S(\nu') = \bigcap_{\alpha \in A} \{ V_J(L_\alpha) / \nu'(V_J(L_\alpha)) = 1, L_\alpha \in \mathcal{L} \}$. $\nu'(V_J(L)) = 1$ iff $\nu(L) = 1$ iff $\nu \in V_J(L) = \{ \mu \in J(\mathcal{L}) / \mu(L) = 1, L \in \mathcal{L} \}$. Hence $V_J(L) \neq \emptyset$ implies $S(\nu') \neq \emptyset$. Therefore $\mathcal{V}_J(\mathcal{L})$ is Lindelöf, and then $\tau\mathcal{V}_J(\mathcal{L})$ is Lindelöf.
Conversely, assume $(J(\mathcal{L}), \tau\mathcal{V}_J(\mathcal{L}))$ is Lindelöf and let $\pi \in \tilde{\Pi}_{\sigma}(\mathcal{L})$
Define $\pi'(V_J(L)) = \pi(L)$, $L \in \mathcal{L}$. Then $V_J(L_n) \downarrow \emptyset$ which implies $L_n \downarrow \emptyset$ and $\lim_n \pi(L_n) = \lim_n \pi'(V_J(L_n)) = 0$ i.e. $\pi' \in \tilde{\Pi}_{\sigma}(\mathcal{V}_J(\mathcal{L}))$.
$\tau\mathcal{V}_J(\mathcal{L})$ Lindelöf, then $\mathcal{V}_J(\mathcal{L})$ Lindelöf, then $\mathcal{V}_J(\mathcal{L})$ satisfies condition (2). Hence there exists $\nu' \in J(\mathcal{V}_J(\mathcal{L}))$ such that $\pi' \leq \nu'$ on $\mathcal{V}_J(\mathcal{L})$, where $\nu'(V_J(L)) = \nu(L)$, $L \in \mathcal{L}$. Therefore $\pi \leq \nu\ (\mathcal{L})$.

<u>Theorem 5.3</u> Consider $J(\mathcal{L})$ and $\mathcal{V}_J(\mathcal{L})$. $\mathcal{V}_J(\mathcal{L})$ is regular iff for all $\mu_1, \mu_2 \in I(\mathcal{L})$ and $\nu \in J(\mathcal{L})$, if $\mu_1 \le \mu_2 (\mathcal{L})$ and $\mu_1 \le \nu$ $(\mathcal{L})$ then $\mu_2 \le \nu$ $(\mathcal{L})$.

Proof. For $\mu_1, \mu_2 \in I(\mathcal{L})$ we have $\mu_1', \mu_2' \in I(\mathcal{W}_\sigma(\mathcal{L}))$ and then $\mu_1', \mu_2' \in I(\mathcal{V}_J(\mathcal{L}))$, $\mu_1'(V_J(L)) = \mu_1(L)$ and $\mu_2'(V_J(L)) = \mu_2(L)$. If $\mathcal{V}_J(\mathcal{L})$ is regular then $S(\mu_1') = S(\mu_2')$, where $S(\mu_1') = \bigcap \{V_J(L) \in \mathcal{V}_J(\mathcal{L}) / \mu_1'(V_J(L) = 1$ $L \in \mathcal{L}\}$. Let $\nu \in J(\mathcal{L})$; $\nu' \in J(\mathcal{V}_J(\mathcal{L}))$ and $\mu_1' \le \nu'$ on $\mathcal{V}_J(\mathcal{L})$. Then $\nu \in S(\mu_2')$ i.e. $\mu_2 \le \nu$ $(\mathcal{L})$.
Conversely, suppose $\mu_1, \mu_2 \in I(\mathcal{L})$ and $\nu \in J(\mathcal{L})$ such that if $\mu_1 \le \mu_2 (\mathcal{L})$ and $\mu_1 \le \nu (\mathcal{L})$ then $\mu_2 \le \nu$ $(\mathcal{L})$. Let $\lambda_1, \lambda_2 \in I(\mathcal{V}_J(\mathcal{L}))$ and $\lambda_1 \le \lambda_2$ on $\mathcal{V}_J(\mathcal{L})$. Then $\lambda_1 = \mu_1'$ and $\lambda_2 = \mu_2'$ with $\mu_1, \mu_2 \in I(\mathcal{L})$. Thus $\mu_1' \le \mu_2'$ on $\mathcal{V}_J(\mathcal{L})$ which implies $\mu_1 \le \mu_2$ on $\mathcal{L}$, hence $S(\mu_2') \subset S(\mu_1')$. If $\lambda \in S(\mu_1')$ then clearly $\lambda \in J(\mathcal{L})$ and $\mu_1 \le \lambda$ $(\mathcal{L})$. By the condition of the ststement, $\mu_2 \le \lambda$ $(\mathcal{L})$ and then $\lambda \in S(\mu_2')$ Hence $S(\mu_2') = S(\mu_1')$ and $\mathcal{V}_J(\mathcal{L})$ is regular.

<u>Theorem 5.4</u> Consider $J(\mathcal{L})$, $\mathcal{V}_J(\mathcal{L})$. If $\mathcal{V}_J(\mathcal{L})$ is regular, then $J(\mathcal{L}) = I_R^\sigma(\mathcal{L})$.

Proof. Let $\mu \in J(\mathcal{L})$. Then there exists $\nu \in I_R(\mathcal{L})$ such that $\mu \le \nu$ $(\mathcal{L})$, hence $\mu' \le \nu'$ on $\mathcal{V}_J(\mathcal{L})$, where $\mu' \in J(\mathcal{V}_J(\mathcal{L}))$ and $\nu' \in I_R(\mathcal{V}_J(\mathcal{L}))$. $\mathcal{V}_J(\mathcal{L})$ regular implies $S(\mu') = S(\nu')$, therefore $\nu \le \mu (\mathcal{L})$. Then $\mu = \nu$ $(\mathcal{L})$ and since $\nu \in I_R(\mathcal{L})$, $J(\mathcal{L}) \subset I_\sigma(\mathcal{L})$ it follows that $\mu \in I_R(\mathcal{L}), I_\sigma(\mathcal{L})$ and then $\mu \in I_R^\sigma(\mathcal{L})$. Thus $J(\mathcal{L}) = I_R^\sigma(\mathcal{L})$.

<u>Theorem 5.5</u> Consider $J(\mathcal{L})$ and $\mathcal{V}_J(\mathcal{L})$, with $\mathcal{L}$ Lindelöf. If for all $\mu_1, \mu_2 \in I(\mathcal{L})$ and $\nu \in J(\mathcal{L})$ such that if $\mu_1 \le \mu_2 (\mathcal{L})$ and $\mu_1 \le \nu (\mathcal{L})$ then $\mu_2 \le \nu$ $(\mathcal{L})$ it follows that $\mathcal{V}_J(\mathcal{L})$ is slightly and mildly normal.

Proof. By Theorem 5.3 $\mathcal{V}_J(\mathcal{L})$ is regular. We show as in Remark of Theorem 3.1 that $\mathcal{V}_J(\mathcal{L})$ is Lindelöf and then, $\mathcal{V}_J(\mathcal{L})$ being regular and Lindelöf, it follows that it is also slightly and mildly normal.

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