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ON SOME RELATIONSHIPS BETWEEN THE SHAPLEY VALUE AND THE BANZHAF VALUE

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A potential function has been introduced for the Shapley value by S. Hart and A. Mas-Colell (1988, 1989); it has been used for proving an axiomatic characterization for the Shapley value. In our paper, we introduce a potential function for the Banzhaf value by requiring that the sum of its marginals equals the sum of the components of this value. A formula for the potential in terms of coalitional form of the game is obtained by combinatorial manipulations and again by combinatorial reasoning we show that each marginal of the potential equals a component of the Banzhaf value. Now, we define a so-called "power game" associated with any game and we show that the Shapley value of the power game equals the Banzhaf value of the given game.

The potential function of the Banzhaf value is allowing us to determine a basis of the null space of this value; a basis for the null space of the Shapley value has been determined in a recent paper of the author (1991). These two bases are used to prove that the 3-person and the 4-person games for which the Shapley value equals the Banzhaf value are those games obtained by adding an additive game to a self-dual game. We call a game a self-dual game if $v(S) = v(N - S),\ \forall S \subseteq N$. For games with more than four players, beside these games there are other games possessing the property. Note that a new axiomatic definition for the Banzhaf value based on the reduced game property relative to a new reduced game has been given in an earlier paper of the author (1995).

1. The power game and the values.

Let N be a fixed set of players and G^N be the space of cooperative T.U. games in coalitional form on N. Throughout this paper a game is thought of as a $2^n - 1$-dimensional vector, where $n = |N|$. The nonempty subsets S of N, the coalitions in the game, will index the components of the vector; however, for a game $v \in G^N$, a component will be denoted by $v(S)$, as usual in Game Theory. Any functional $\psi: G^N \to R^n$ is called a value and the components

$\psi_i(v)$, $\forall i \in N$, will be the payoffs to players $i \in N$ in game v. Obviously, a functional depends also on N and to emphasize this fact we shall denote it by $\psi(N, v)$; for $S \subseteq N$, $S \neq \emptyset$, and $v \in G^N$, we shall denote by $\psi(S, v)$ the corresponding functional from G^S to R^S, $s = \mid S \mid$, where v is kept to mean the subgame obtained by restricting to G^S the original game v. In the paper, we discuss only two values: the Shapley value given by

$$Sh_i(N, v) = \sum_{S: i \in S \subseteq N} \tfrac{(s-1)!(n-s)!}{n!}[v(S) - v(S - \{i\})],\ \forall i \in N, \tag{1.1}$$

(see [9] and [10]), and the Banzhaf value given by

$$B_i(N, v) = \tfrac{1}{2^{n-1}} \sum_{S: i \in S \subseteq N} [v(S) - v(S - \{i\})],\ \forall i \in N \tag{1.2}$$

(see [1] and [8]). There is a growing literature describing the relationships between these values (see [3], [4] and [5]). Let us mention only the fact that the two values can be defined axiomatically by means of the axioms: linearity, symmetry, dummy player, and efficiency for the first, and semiefficiency for the second. For $T \subseteq N$, $T \neq \emptyset$, and $v \in G^N$, denote

$$\pi(T, v) = \sum_{i \in T} B_i(T, v); \tag{1.3}$$

we called semiefficiency the equality (1.3) for $T = N$, while efficiency of a value is the fact that the sum of components makes $v(N)$. Clearly, formula (1.3) defines a game $\pi \in G^N$ associated with each $v \in G^N$; the game π will be called the power game of v. It is convenient to take $\pi(\emptyset, v) = 0$, just as we take $v(\emptyset) = 0$. The following result shows that the correspondence between π and v is a one-to-one correspondence:

Lemma 1.1: For any game $v \in G^N$, the power game $\pi \in G^N$ defined by (1.3) can be expressed in terms of the coalitional form of v as

$$\pi(T, v) = \tfrac{1}{2^{t-1}} \sum_{U \subseteq T} (2u - t) v(U),\ \forall T \subseteq N, T \neq \emptyset, \tag{1.4}$$

where $t = \mid T \mid$ and $u = \mid U \mid$.

Proof: For a fixed $U \subseteq T$, terms $v(U)$ occur in u components $B_i(T, v)$ with a positive sign and (if $u < t$) in $t - u$ components with a negative sign. Therefore, we have

$$\pi(T, v) = \sum_{i \in T} B_i(T, v) = \tfrac{1}{2^{t-1}} \sum_{U \subseteq T} [u v(U) - (t - u) v(U)],$$

from which (1.4) follows.

Note that we can separate the term $v(T)$ and rewrite formula (1.4) as

$$v(T) = \tfrac{1}{t}[2^{t-1} \pi(T, v) - \sum_{U \subset T} (2u - t) v(U)],\ \forall T \subseteq N, T \neq \emptyset. \tag{1.5}$$

This shows that the power game defined recursively the game from which has been obtained. For example, if $\pi(2) = \pi(3) = \pi(2,3) = 0$ and $\pi(1) = \pi(1,2) = \pi(1,3) = \pi(1,2,3) = 1$, then from (1.5) for $\mid T \mid = 1$ we get $v(1) = 1, v(2) = v(3) = 0$, for $\mid T \mid = 2$ and the already found values we get $v(1,2) = v(1,3) = 1$ and $v(2,3) = 0$, finally for $\mid T \mid = N$ and the values already found we get $v(1,2,3) = 1$.

In the following, we intend to show an interesting relationship between the Banzhaf value of a game and the Shapley value of the associated power game. In order to do this, we introduce the potential of the Banzhaf value. Following S. Hart and A. Mas-Colell ([6], [7]), we define the potential of the Banzhaf value for a game $v \in G^N$ as a function Q satisfying

$$\sum_{i \in N} [Q(N,v) - Q(N - \{i\}, v)] = \pi(N,v). \tag{1.6}$$

If $\mathcal{P}(N)$ is the set of subsets of N, then (1.6) is equivalent $Q\colon \mathcal{P}(N) \rightarrow R$ recursively defined by $Q(\emptyset, v) = 0$ and

$$Q(T,v) = \tfrac{1}{t}[\pi(T,v) + \sum_{j \in T} Q(T - \{j\}, v)], \forall T \subseteq N, T \neq \emptyset. \tag{1.7}$$

Recall that for the Shapley value, the recursive definition can be obtained from (1.7) by substituting $v(T)$ for $\pi(T,v)$. To get the announced relationship, we need two results interesting by themselves.

Theorem 1.2: If $Q\colon \mathcal{P}(N) \rightarrow R$ is the potential of the Banzhaf value for the game $v \in G^N$, then we have

$$Q(T,v) = \tfrac{1}{2^{t-1}} \sum_{S \subseteq T} v(S), \forall T \subseteq N, T \neq \emptyset, \tag{1.8}$$

where $t = \mid T \mid$.

Proof: We prove by induction over the size of T. If $\mid T \mid = 1$, that is $T = \{i\}$, then from (1.7) we get $Q(\{i\}, v) = \pi(\{i\}, v)$ and from (1.2) we get $\pi(\{i\}, v) = v(\{i\})$, hence (1.8) holds. Assume that (1.8) holds for all coalitions of size $t - 1$ and compute the sum occuring in the bracket in (1.7); we have

$$\sum_{j \in T} Q(T - \{j\}, v)] = \tfrac{1}{2^{t-2}} \sum_{U \subset T} (t - u) v(U),$$

because any coalition $U \subset T$ belongs to $t - u$ coalitions $T - \{j\}$, $j \in T$. The Lemma 1.1 and the last equality give

$$\pi(T,v)+\sum_{j\in T}Q(T-\{j\},v)=\tfrac{1}{2^{t-1}}[tv(T)+\sum_{U\subset T}(2u-t)v(U)]+\tfrac{1}{2^{t-2}}\sum_{U\subset T}(t-u)v(U)$$
$$=\tfrac{1}{2^{t-1}}t\sum_{U\subseteq T}v(U)$$

and (1.7) shows that (1.8) holds for a coalition of size t; hence (1.8) holds in general.

Recall that hte potential $P(N,v)$ of the Shapley value can be expressed in terms of the coalitional form of the game by

(1.9) $$P(N,v)=\sum_{S\subseteq N}\tfrac{(s-1)!(n-s)!}{n!}v(S).$$

Now, we use Theorem 1.2 to justify why the function defined by (1.6) or (1.7) is called the potential of the Banzhaf value.

Theorem 1.3: Let $B(N,v)\in R^n$ be the Banzhaf value of the game $v\in G^N$, and $Q\colon \mathcal{P}(N)\to R$ be the function recursively defined by (1.7). Then, we have

(1.10) $$Q(N,v)-Q(N-\{i\},v)=B_i(N,v),\ \forall i\in N.$$

Proof: By (1.8) for $T=N$ and $T=N-\{i\}$, we have

$$Q(N,v)-Q(N-\{i\},v)=\tfrac{1}{2^{n-1}}\Big\{\sum_{S:i\in S\subseteq N}v(S)-\sum_{S\subseteq N-\{i\}}v(S)\Big\}-\tfrac{1}{2^{n-2}}\sum_{S\subseteq N-\{i\}}v(S)$$
$$=\tfrac{1}{2^{n-1}}\sum_{S:i\in S\subseteq N}[v(S)-v(S-\{i\})]=B_i(N,v).$$

Recall that the potential $P(N,w)$ of the Shapley value for $w\in G^N$ satisfies

(1.11) $$P(N,w)-P(N-\{i\},w)=Sh_i(N,w),\ \forall i\in N.$$

Theorem 1.4: Let $B(N,v)\in R^n$ be the Banzhaf value of the game $v\in G^N$ and $Sh(N,w)\in R^n$ be the Shapley value of the game $w\in G^N$. If $w(S)=\pi(S,v)$, $\forall S\subseteq N$, then we have

(1.12) $$B(N,v)=Sh(N,w).$$

Proof: If $w(S)=\pi(S,v)$, $\forall S\subseteq N$, then the potential of the Shapley value denoted $P(N,w)$ is recursively defined by $P(\emptyset,w)=0$ and

$$P(T,w)=\tfrac{1}{t}[w(T)+\sum_{j\in T}P(T-\{j\},w)],\ \forall T\subseteq N, T\neq\emptyset,$$

which is the same as (1.7). Now, the uniqueness of the solution proves $P(N,w)=Q(N,v)$ and $P(N-\{i\},w)=Q(N-\{i\},v)$, $\forall i\in N$, and (1.12) follows from Theorem 1.3 and (1.11).

Note that this result may be useful for computing the Shapley value, by computing the Banzhaf value via formulae (1.5) and (1.12).

2. The potential bases and the equality of the values.

Let $\psi: G^N \to R^n$ be a value which has a potential and $Z = \{z_S \in G^N : S \subseteq N, S \neq \emptyset\}$ be a basis for G^N. The basis Z is a potential basis for G^N relative to the value ψ if the expansion $v = \sum_{S \subseteq N} \alpha_S z_S$ of any $v \in G^N$ will have α_S equal to the potential of the subgame generated by S for all $S \subseteq N$, $S \neq \emptyset$. In an earlier paper of the author (see [2]), a potential basis for G^N relative to the Shapley value has been determined, namely $W = \{w_S \in G^N : S \subseteq N, S \neq \emptyset\}$, where for all $S \subseteq N$

$$w_S(S) = s,\ w_S(S \cup \{j\}) = -1,\ j \notin S,\ w_S(T) = 0 \text{ otherwise,} \tag{2.1}$$

and

$$w_N(N) = n,\ w_N(T) = 0, \text{ otherwise.} \tag{2.2}$$

For example, if $|N| = 3$ the potential basis relative to the Shapley value comprises the games

$$\begin{array}{ll} & w_{1,2} = (0,0,0,2,0,0,-1), \\ w_1 = (1,0,0,-1,-1,0,0), & \\ & w_{1,3} = (0,0,0,0,2,0,-1), \\ w_2 = (0,1,0,-1,0,-1,0), & \\ & w_{2,3} = (0,0,0,0,0,2,-1), \\ w_3 = (0,0,1,0,-1,-1,0), & \\ & w_{1,2,3} = (0,0,0,0,0,0,3), \end{array} \tag{2.3}$$

when the components of the game are taken in the order $\{1\}$, $\{2\}$, $\{3\}$, $\{1,2\}$, $\{1,3\}$, $\{2,3\}$, and $\{1,2,3\}$.

In the same paper, it has been shown that the null space of the Shapley value is generated by the linearly independent vectors w_S with $|S| \leq n-2$ and $w_N + \sum_{j \in N} w_{N-\{j\}}$; hence these vectors form a basis and $dim(Nu(Sh)) = 2^n - n - 1$.

In the following, we intend to discuss the problem of determining the set E of games $v \in G^N$ for which $B(N,v) = Sh(N,v)$. Therefore we shall determine first a potential basis for G^N relative to the Banzhaf value and a basis for the null space of this value. Consider the games c_S for $S \subseteq N$, $S \neq \emptyset$, defined by

$$c_S(T) = (-1)^{t-s} \cdot 2^{s-1} \text{ if } T \supset S,\ c_S(T) = 0 \text{ otherwise.} \tag{2.4}$$

For example, if $|N| = 3$ then

$$\begin{array}{ll} & c_{1,2} = (0,0,0,2,0,0,-2), \\ c_1 = (1,0,0,-1,-1,0,1), & \\ & c_{1,3} = (0,0,0,0,2,0,-2), \\ c_2 = (0,1,0,-1,0,-1,1), & \\ & c_{2,3} = (0,0,0,0,0,2,-2), \\ c_3 = (0,0,1,0,-1,-1,1), & \end{array} \tag{2.5}$$

$$c_{1,2,3} = (0,0,0,0,0,0,4).$$

It is obvious that they are linearly independent and the set has the cardinality 7, hence they form a basis for G^N with $|N| = 3$. We can check that they form a potential basis for G^N relative to the Banzhaf value by computing the coordinates of any game in this basis and the potentials. We get this fact instead from the result:

Theorem 2.1: The set of games $C = \{c_S \in G^N : S \subseteq N, S \neq \emptyset\}$, defined by (2.4) forms a potential basis for G^N relative to the Banzhaf value.

Proof: It is easy to see that the set C is linearly independent, hence it forms a basis for G^N, because $|C| = 2^n - 1 = dim(G^N)$. We should show that in the expansion $v = \sum\limits_{T \subseteq N} \alpha_T(v) c_T$ we have $\alpha_T(v) = Q(T,v)$, $\forall T \subseteq N$, where Q is the potential of the Banzhaf value. From the expansion, we get componentwise $v(S) = \sum\limits_{U \subseteq N} \alpha_U(v) c_U(S)$, $\forall S \subseteq N$. For an arbitrary $T \subseteq N$, by adding up as shown in formula (1.8), we get that $Q(T,v) = \sum\limits_{U \subseteq N} \alpha_U(v) Q(T, c_U)$. One can prove by induction that the above games (2.4) satisfy $Q(T, c_U) = 0$, $\forall U \neq T$ and $Q(T, c_T) = 1$. Taking into account this fact, in the last sum there is only one nonzero term so that the sum reduces to $Q(T,v) = \alpha_T(v)$. As T was arbitrary, the result follows.

To determine the null space of the Banzhaf value we need:

Lemma 2.2: The Banzhaf values of the games in the potential basis C are

$$B(N, c_S) = 0, \forall S \subset N, \ |S| \leq n-2,$$

(2.6)

$$B(N, c_{N-\{j\}}) = -e_j, \forall j \in N, B(N, c_N) = e,$$

where $e_j \in R^n$ are the vectors of the standard basis and $e \in R^n$ has all components one.

Proof: To compute the Banzhaf value of c_S we shall use (1.10) and the fact shown in the proof of Theorem 2.1 that $Q(T, c_S) = 0$, if $T \neq S$ and $Q(S, c_S) = 1$. In

$$B_i(N, c_S) = Q(N, c_S) - Q(N - \{i\}, c_S), \forall i \in N, \tag{2.7}$$

obtained from Theorem 1.3 for $v = c_S$, if $|S| \leq n-2$ then both N and $N - \{i\}$ are different of S so that the first formula (2.6) holds. In (2.7), if $S = N$ we have $Q(N, c_N) = 1$ and $Q(N - \{i\}, c_N) = 0$, $\forall i \in N$, so that the last formula (2.6) holds. Finally, if $S = N - \{j\}$, then in (2.7) we have $Q(N, c_{N-\{j\}}) = 0$ and $Q(N - \{i\}, c_{N-\{j\}}) = \delta_i^j$, where $\delta_i^j = 0$ whenever $j \neq i$ and $\delta_i^i = 1$, hence the result follows.

Theorem 2.3: The null space $Nu(B)$ of the Banzhaf value is generated by the set of linearly independent games

$$g_0 = \{c_S \in G^N : S \subset N, S \neq \emptyset, \mid S \mid \ \leq n-2\} \cup \{c_N + \sum_{j \in N} c_{N-\{j\}}\},$$

hence $dim(Nu(B)) = 2^n - n - 1$.

Proof: From Lemma 2.2, we see that each game g_0 has a null Banzhaf value, hence any linear combination of vectors in g_0 is in the null space due to the linearity of B, and g_0 generate a subspace $N_0 \subseteq Nu(B)$. We should prove that in this inclusion we have in fact equality. Take any $v \in G^N$ for which $B(N, v) = 0$; by Theorem 1.3 we have $Q(N, v) = Q(N - \{i\}, v)$, $\forall i \in N$, and using these equalities in the expansion relative to the potential basis, we get

$$v = \sum_{S \subseteq N} Q(S, v) c_S = \sum_{S \subset N, |S| \leq n-2} Q(S, v) c_S + Q(N, v)(c_N + \sum_{j \in N} c_{N-\{j\}}),$$

that is $v \in N_0$, hence $Nu(B) \subseteq N_0$ and the equality $N_0 = Nu(B)$ follows. The set g_0 is linearly independent, hence it represents a basis for $Nu(B)$ and the dimension follows.

Recall that the null space of the Shapley value $Nu(Sh)$ has also $dim(Nu(Sh)) = 2^n - n - 1$.

Let denote $E = \{v \in G^N : B(N, v) = Sh(N, v)\}$; note that E is also a subspace of G^N due to the linearity of both values. Note also that the subspace of all additive games in G^N is included in E because the sum of coefficients in both (1.1) and (1.2) makes one. However, it is easy to give an example of a nonadditive game belonging to E. To determine E, at least for games of lower sizes, we need:

Lemma 2.4: Let $I = Nu(B) \cap Nu(Sh)$ and for any game $v \in E$ denote $w = v - b$ where $b \in G^N$ is the additive game for which $b(\{i\}) = B_i(N, v) = Sh_i(N, v)$, $\forall i \in N$. If $v \in E$ then $w \in I$, and if $w \in I$ and b is any additive game then $v = w + b \in E$.

Proof: If $v \in E$, then by linearity and additivity of b we have $B(N, w) = B(N, v) - b = Sh(N, v) - b = Sh(N, w) = 0$, hence $w \in I$. Conversely, if $w \in I$ and b is any additive game then $B(N, v) = b = Sh(N, v)$, hence $v \in E$.

Lemma 2.4 shows that to determine E is enough to determine I and we shall get E as the direct sum of the subspace I and the subspace of additive games. Obviously, no additive game except the null game is in I. A starting point in determining I is to consider the set SD of self-dual games in G^N. A game is self-dual if $v(S) = v(N - S)$, $\forall S \subseteq N$, including the empty set.

Theorem 2.5: The set SD of self-dual games in G^N is a subspace of G^N with $dim(SD) = 2^{n-1} - 1$ and $SD \subseteq I = Nu(B) \cap Nu(Sh)$.

Proof: The equations of SD are $v(S) = v(N - S)$, $\forall S \subset N$, $S \neq \emptyset$, while $v(N) = v(\emptyset) = 0$; there are $2^n - 2$ equations and a half of them are linearly independent. It is straightforward to check that SD is a subspace of G^N, hence its dimension is $dim(SD) = 2^{n-1} - 1$. To prove that $SD \subseteq I$ we should show that each self-dual game has a null Banzhaf and a null Shapley value. Indeed, in the sum (1.2) we can pair the terms $v(S)$ with $v(N - S)$ which come with oposite signs, because one contains i and the other does not; if v is self-dual then each group $v(S) - v(N - S)$ makes zero, hence the Banzhaf value is zero. Also, in the sum (1.1), we can pair the terms $v(S)$ with $v(N - S)$; the coefficients have equal absolute value because the coalitions are complementary, and they come with oposite signs, hence the Shapley value is also zero if v is self-dual.

In the following, we intend to show that for $n = 3$ and $n = 4$ we have $SD = I$, but for $n \geq 5$ the inclusion $SD \subseteq I$ is strict; to do this we use the methods of linear algebra. We know $dim(SD) = 2^{n-1} - 1$ and we can compute $dim(I)$, because we know $dim(Nu(B)) = dim(Nu(Sh)) = 2^n - n - 1$ and we are able to compute $dim(Nu(B) + Nu(Sh))$ and to apply the well known equality between the dimensions of the two subspaces and the dimensions of their sum and intersection: if S_1 and S_2 are subspaces, then

$$dim(S_1) + dim(S_2) = dim(S_1 + S_2) + dim(S_1 \cap S_2). \tag{2.8}$$

As $SD \subseteq I$, if $dim(SD) = dim(I)$, then $SD = I$; this will happen for $n = 3$ and $n = 4$. Otherwise, we have strict inequality between the dimensions and this implies strict inclusion; this will happen for $n \geq 5$.

Theorem 2.6: Let I be the subspace of G^N of those games having null Banzhaf and Shapley values; if $\mid N \mid = 3$, then this is the subspace of self-dual games.

Proof: If $\mid N \mid = 3$, then we have $dim(SD) = 3$, $dim(Nu(B)) = dim(Nu(Sh)) = 4$ and we compute $dim(Nu(B) + Nu(Sh))$ by computing the rank of the matrix

$$(c_1, c_2, c_3, c_{123} + c_{12} + c_{13} + c_{23}, w_1, w_2, w_3, w_{123} + w_{12} + w_{13} + w_{23})$$

comprising the basic vectors of the two null subspaces, $Nu(B)$ and $Nu(Sh)$ where the first four columns are obtained from (2.5) written as columns, and the last four columns have been obtained from (2.3) written as columns. If we operate on the last four columns as shown below, we get a matrix

$$(c_1, c_2, c_3, c_{123} + c_{12} + c_{13} + c_{23}, w_1 - c_1, w_2 - c_2, w_3 - c_3,$$
$$w_{123} + w_{12} + w_{13} + w_{23} - c_{123} - c_{12} - c_{13} - c_{23})$$

of rank 5, hence $dim(Nu(B) + Nu(Sh)) = 5$. From (2.8) for $S_1 = Nu(B)$ and $S_2 = Nu(Sh)$ we compute $dim(Nu(B) \cap Nu(Sh)) = dim(I) = 3$. Hence $SD = I$.

Lemma 2.4 and Theorem 2.6 show that the set of 3-person games for which the Banzhaf and Shapley value are equal can be represented as

$$v(1) = a_1 + b_1,\ v(2) = a_2 + b_2,\ v(3) = a_3 + b_3,$$
$$v(1,2) = a_3 + b_1 + b_2,\ v(1,3) = a_2 + b_1 + b_3,\ v(2,3) = a_1 + b_2 + b_3,$$
$$v(1,2,3) = b_1 + b_2 + b_3,$$

where $b = (b_1, b_2, b_3)$ is the vector of the common equal values.

Theorem 2.7: Let I be the subspace of G^N of those games having null Banzhaf and Shapley values; if $\mid N \mid\ = 4$, then this is the subspace of self-dual games.

Proof: If $\mid N \mid\ = 4$, then we have $dim(SD) = 7$, $dim(Nu(B)) = dim(Nu(Sh)) = 11$, and we can compute $dim(Nu(B) + Nu(Sh))$ by computing the rank of the matrix comprising the vectors of the bases for the two null spaces, i.e. the vectors shown in Theorem 2.3 and formula (2.4), and those given by the similar result for the Shapley value and formula (2.1). This matrix can be written as

$$(c_1, c_2, c_3, c_4, c_{12}, c_{13}, c_{14}, c_{23}, c_{24}, c_{34}, c_{1234} + c_{123} + c_{124} + c_{134} + c_{234},$$
$$w_1, w_2, w_3, w_4, w_{12}, w_{13}, w_{14}, w_{23}, w_{24}, w_{34}, w_{1234} + w_{123} + w_{124} + w_{134} + w_{234}).$$

If we operate on the last eleven columns as shown below, we get a matrix

$$(c_1, c_2, c_3, c_4, c_{12}, c_{13}, c_{14}, c_{23}, c_{24}, c_{34}, c_{1234} + c_{123} + c_{124} + c_{134} + c_{234},$$
$$w_1 - c_1, w_2 - c_2, w_3 - c_3, w_4 - c_4, w_{12} - c_{12}, w_{13} - c_{13},$$
$$w_{14} - c_{14}, w_{23} - c_{23}, w_{24} - c_{24}, w_{34} - c_{34},$$
$$w_{1234} + w_{123} + w_{124} + w_{134} + w_{234} - \tfrac{4}{3}(c_{1234} + c_{123} + c_{124} + c_{134} + c_{234})),$$

of rank 15, hence $dim(Nu(B) + Nu(Sh)) = 15$. Again fron (2.8) we compute $dim(Nu(B) \cap Nu(Sh)) = dim(I) = 7$. Hence $SD = I$.

Lemma 2.4 and Theorem 2.7 show that the set of 4-person games for which the Banzhaf ans the Shapley values are equal can be represented as

$$v(1) = a_1 + b_1,\ v(2) = a_2 + b_2,\ v(3) = a_3 + b_3,\ v(4) = a_4 + b_4,$$
$$v(1,2) = d + b_1 + b_2,\ v(1,3) = e + b_1 + b_3,\ v(1,4) = f + b_1 + b_4,\ v(2,3) = f + b_2 + b_3,$$
$$v(2,4) = e + b_2 + b_4,\ v(3,4) = d + b_3 + b_4,\ v(1,2,3) = a_4 + b_1 + b_2 + b_3,$$

$$v(1,2,4)=a_3+b_1+b_2+b_4,\ v(1,3,4)=a_2+b_1+b_3+b_4,\ v(2,3,4)=a_1+b_2+b_3+b_4,$$
$$v(1,2,3,4)=b_1+b_2+b_3+b_4,$$

where the vector $b=(b_1,b_2,b_3,b_4)$ is the vector of the common equal values.

Now, we intend to apply the same procedure used in the proofs of Theorems 2.6 and 2.7 to the case $\mid N \mid\ \geq 5$, in order to explain where the differences in results are coming from.

Theorem 2.8: If G^N is the space of T.U. games and $\mid N \mid\ \geq 5$, then $SD \subset I$, where SD is the subspace of self-dual games and I is the intersection $I=Nu(B)\cap Nu(Sh)$.

Proof: By Theorem 2.5, we have $dim(SD)=2^{n-1}-1$ and $SD\subseteq I$, hence $dim(I)\geq 2^{n-1}-1$; by Theorem 2.3 and the analogue result for the Shapley value we have $dim(Nu(B))=dim(Nu(Sh))=2^n-n-1$. The matrix whose rank equals $dim(Nu(B)+Nu(Sh))$ is a $(2^n-1)\times(2^{n+1}-2n-2)$ matrix, hence its rank can not be greater than 2^n-1 which is the smaller between the two numbers. Therefore, we have

$$dim(Nu(B))+dim(Nu(Sh))-dim(I)\leq 2^n-1$$

form which we get $dim(I)\geq 2^n-2n-1$. Obviously, we have $2^n-2n-1>2^{n-1}-1$ for $n\geq 5$, and this can be proved by induction for all $n\geq 5$, hence $SD\subset I$.

Note that for $n=4$ the strict inequality becomes an equality and the rank equals really the smaller size of the matrix, and for $n=3$ the rank is smaller than the smaller size; this explains the difference between the cases $n=3$, 4 and $n\geq 5$.

Note that Theorem 2.8 says that beside the self-dual games there are other games with null Banzhaf and Shapley values. To show such a game we computed fully the case $n=5$ by using results of linear algebra. We have found the equation of the null subspace of the Banzhaf value, then the equations of the null subspace of the Shapley value, then they have been put together to get the equations of the intersection of the two subspaces, i.e. the equations of I. We obtained 10 independent equations in 31 variables, which have been solved for 10 variables. To present them in short form here, we take $w(S)=v(S)-v(N-S)$, $\forall S\subseteq N$, and the system can be written as

$$\begin{aligned}
&w(\emptyset)=0,\ w(1)=w(2)=w(3)=w(4)=w(5)\\
&\qquad\qquad=-\tfrac{1}{9}(w(2,3)+w(2,4)+w(2,5)+w(3,4)+w(3,5)+w(4,5)),\\
&w(1,2)=-\tfrac{1}{3}(w(2,3)+w(2,4)+w(2,5)-2w(3,4)-2w(3,5)-2w(4,5)),\\
&w(1,3)=-\tfrac{1}{3}(w(2,3)-2w(2,4)-2w(2,5)+w(3,4)+w(3,5)-2w(4,5)),\\
&w(1,4)=-\tfrac{1}{3}(-2w(2,3)+w(2,4)-2w(2,5)+w(3,4)-2w(3,5)+w(4,5)),
\end{aligned}$$

$$w(1,5) = -\tfrac{1}{3}(-2w(2,3) - 2w(2,4) + w(2,5) - 2w(3,4) + w(3,5) + w(4,5)).$$

Obviously, $w(S) = 0, \forall S \subseteq N$, is a solution, that is the self-dual games belong to I, but there are other games as well because this system has also nontrivial solutions. For example, it is easy to see that $v(1) = v(2) = v(3) = v(4) = v(5) = 1$, $v(1,2) = v(1,3) = 3$, $v(1,4) = v(1,5) = -6$, $v(2,3) = -9$ and $v(S) = 0$ otherwise, is a solution. Indeed, we get $w(1) = w(2) = w(3) = w(4) = w(5) = 1$, $w(1,2) = w(1,3) = 3$, $w(1,4) = w(1,5) = -6$, $w(2,3) = -9$, $w(2,4) = w(2,5) = w(3,4) = w(3,5) = w(4,5) = 0$, which satisfy the above system. One can check that both the Banzhaf and the Shapley value of this 5-person game are zero; obviously, this is not a self-dual game. An arbitrary additive game should be added to get a game with equal Banzhaf anf Shapley values.

The above results make us to say that there is here an open problem: characterize the n-person cooperative games with transferable utilities for which the Shapley value and the Banzhaf value are equal. Such a characterization should include the results shown in this paper for $n = 3$, 4 and 5.

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