

On uniqueness and continuous dependence in Thermoelasticity of micropolar bodies

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This paper is concerned with some basic theorems of linear Thermoelastodynamics for micropolar bodies. The uniqueness theorem and continuous dependence theorems are proved with the aid of Lagrange identity, with no definiteness assumptions on the thermoelastic coefficients. So, we establish the uniqueness and continuous dependence of the solutions with respect to the body forces, body couples and heat supply, for previous problems. We also deduce the continuous dependence of solutions of our problems with respect to initial data and, at last, to thermoelastic coefficients. The results are obtained for the bounded regions of the Euclidian three-dimensional space. We point out, again, that the results are obtained without recourse to an energy conservation law or to any boundedness assumptions on the thermoelastic coefficients and avoid the use of definiteness assumptions on the thermoelastic coefficients. Let the body occupy, at time $t = 0$, a properly regular region B of the Euclidian three-dimensional space. We denote the closure of B by ∂B and suppose that the boundary ∂B is a closed, bounded and piece-wise smooth surface. We use a fixed system of rectangular Cartesian axes and adopt the Cartesian tensor notation. Points of B are denoted by (x) and $t \in [0, t_0]$ is time. The usual summation and differentiation conventions are employed: Latin subscripts are understood to range over the integers $(1, 2, 3)$, summation over repeated subscripts is implied and a subscripts j precede by a comma denotes partial differentiation with respect to the Cartesian Co-ordinate x_j . The basic equations of the linear theory of micropolar thermoelasticity are

- the equations of motion

$$(1) \quad \begin{aligned} t_{ij,j} + \rho F_i &= \rho \ddot{u}_i, \\ m_{ij,j} + \epsilon_{ijk} t_{jk} + \rho M_i &= I_{ij} \ddot{\varphi}_j; \end{aligned}$$

- the equation of energy

$$(2) \quad T_0 \dot{\eta} = q_{i,i} + \rho r;$$

- the constitutive equations

$$(3) \quad \begin{aligned} t_{ij} &= A_{ijmn} \varepsilon_{mn} + B_{ijmn} \gamma_{mn} - E_{ij} \theta, \\ m_{ij} &= B_{ijmn} \varepsilon_{mn} + C_{ijmn} \gamma_{mn} - D_{ij} \theta, \\ \eta &= E_{ij} \varepsilon_{ij} + D_{ij} \gamma_{ij} + \frac{\alpha}{T_0} \theta, \quad q_i = k_{ij} \theta_{,j}; \end{aligned}$$

- the geometrical equations

$$(4) \quad \varepsilon_{ij} = u_{j,i} + \varepsilon_{jik} \varphi_k, \quad \gamma_{ij} = \varphi_{j,i}.$$

In these relations we have used the following notations: ρ -the constant mass density; T_0 -the constant absolute temperature of the body in its reference state; u_i -components of the displacement; φ_i -components of the microrotation; θ -the temperature measured from the temperature T_0 ; ε_{ij} , γ_{ij} -kinematic characteristics of the strain; t_{ij} -components of the stress tensor; m_{ij} -components of couple-stress tensor; F_i -components of the body force; M_i -the components of the body couple; r - the heat supply per unit mass; η -the entropy per unit mass; q_i -components of heat flux; ε_{ijk} -alternative symbol; I_{ij} -coefficients of microinertia; α , A_{ijmn} , B_{ijmn} , C_{ijmn} , E_{ij} , D_{ij} , k_{ij} -the characteristics of the material, and they obey the symmetry relations

$$(5) \quad A_{ijmn} = A_{mnij}, \quad C_{ijmn} = C_{mnij}, \quad k_{ij} = k_{ji}, \quad I_{ij} = I_{ji}.$$

We suppose that there exists a positive constant λ_0 such that

$$I_{ij} \xi_i \xi_j \geq \lambda_0 \xi_i \xi_i, \quad \forall \xi_i.$$

Also, the spatial argument and the time argument of a function will be omitted when there is no likelihood of confusion. The second law of thermodynamics implies that the conductivity tensor is positive semi-definite

$$k_{ij} \theta_{,i} \theta_{,j} \geq 0.$$

Along with Eqs.(1) to (4), we shall assume that the following standard initial conditions hold

$$(6) \quad \begin{aligned} u_i(x, 0) &= a_i(x), \quad \dot{u}_i(x, 0) = b_i(x), \quad \varphi_i(x, 0) = c_i(x), \\ \dot{\varphi}_i(x, 0) &= d_i(x), \quad \theta(x, 0) = \theta_0(x), \quad (x) \in B, \end{aligned}$$

where the functions a_i, b_i, c_i, d_i and θ_0 are prescribed. The mixed initial-boundary value problem in the linear thermoelasticity of micropolar bodies is complete if we adjoin the boundary conditions

$$(7) \quad \begin{aligned} u_i &= \bar{u}_i \text{ on } \partial B_1 \times [0, T), \quad t_i \equiv t_{ij}n_j = \bar{t}_i \text{ on } \partial B_1^c \times [0, T), \\ \varphi_i &= \bar{\varphi}_i \text{ on } \partial B_2 \times [0, T), \quad m_i \equiv m_{ij}n_j = \bar{m}_i \text{ on } \partial B_2^c \times [0, T), \\ \theta &= \bar{\theta} \text{ on } \partial B_3 \times [0, T), \quad q \equiv q_{,i}n_i = \bar{q} \text{ on } \partial B_3^c \times [0, T), \end{aligned}$$

where n_i are the components of the outward unit normal vector on ∂B ; ∂B_i with their complements ∂B_i^c are fixed subsets of ∂B such that

$$\begin{aligned} \partial B_1 \cap \partial B_1^c &= \partial B_2 \cap \partial B_2^c = \partial B_3 \cap \partial B_3^c = \emptyset, \\ \partial B_1 \cup \partial B_1^c &= \partial B_2 \cup \partial B_2^c = \partial B_3 \cup \partial B_3^c = \partial B, \end{aligned}$$

and the functions $\bar{u}_i, \bar{\varphi}_i, \bar{t}_i, \bar{m}_i, \bar{\theta}$ and \bar{q} are prescribed. By introducing (3) into (1) and (2), the following system of equations is derived

$$(8) \quad \begin{aligned} \rho \ddot{u}_i &= (A_{ijmn} \varepsilon_{mn})_{,j} + (B_{ijmn} \gamma_{mn})_{,j} - (E_{ij} \theta)_{,j} + \rho F_i, \\ I_{ij} \ddot{\varphi}_i &= (B_{mni,j} \varepsilon_{mn})_{,j} + (C_{ijmn} \gamma_{mn})_{,j} - (D_{ij} \theta)_{,j} + \\ &\quad + \varepsilon_{ijk} (A_{jkmn} \varepsilon_{mn} + B_{jkmn} \gamma_{mn} - E_{jk} \theta) + \rho M_i, \\ \alpha \dot{\theta} + T_0 (E_{ij} \dot{\varepsilon}_{ij} + D_{ij} \dot{\gamma}_{ij}) &= (k_{ij} \theta_{,j})_{,i} + \rho r. \end{aligned}$$

By a solution of the mixed initial-boundary value problem of the micropolar thermoelasticity in the cylinder $B \times [0, T)$, we mean an ordered array (u_i, φ_i, θ) which satisfies the system (8) for all $(x, t) \in B \times [0, T)$, the initial conditions (6) and the boundary conditions (7).

Throughout this paper it is assumed that a twice continuous differentiable solution (u_i, φ_i, θ) exists. Let $U_i(x, t)$ and $V_i(x, t)$ be functions assumed to be twice continuous differentiable with respect to the time variable. We have the following Lagrange identity

$$(9) \quad \begin{aligned} &\int_B \rho(x) [U_i(x, t) \dot{V}_i(x, t) - \dot{U}_i(x, t) V_i(x, t)] dV = \\ &= \int_0^t \int_B \rho(x) [U_i(x, s) \ddot{V}_i(x, s) - \ddot{U}_i(x, s) V_i(x, s)] dV ds + \\ &\quad + \int_B \rho(x) [U_i(x, 0) \dot{V}_i(x, 0) - \dot{U}_i(x, 0) V_i(x, 0)] dV. \end{aligned}$$

Let us denote by $(u_i^{(\alpha)}, \varphi_i^{(\alpha)}, \theta^{(\alpha)})$, $(\alpha = 1, 2)$, two solutions of the initial boundary value problem (8), (6), (8) corresponding to the same boundary and initial data but to the body forces, body couples and heat supplies $(F_i^{(\alpha)}, M_i^{(\alpha)}, r^{(\alpha)})$, $(\alpha = 1, 2)$, respectively. We introduce the notations

$$v_i = u_i^{(2)} - u_i^{(1)}, \quad \psi_i = \varphi_i^{(2)} - \varphi_i^{(1)}, \quad \chi = \theta^{(2)} - \theta^{(1)}.$$

Because of linearity, the difference (v_i, ψ_i, χ) represent the solution of an initial-boundary value problem analogous to (8), (6) and (7), in which, we have $G_i = F_i^{(2)} - F_i^{(1)}$, $L_i = M_i^{(2)} - M_i^{(1)}$, $P = r^{(2)} - r^{(1)}$, $\varepsilon_{ij} = \varepsilon_{ij}^{(2)} - \varepsilon_{ij}^{(1)}$, and so on, and the relations (6) and (7) become homogeneous.

By setting $U_i(x, s) = v_i(x, s)$, $V_i(x, s) = v_i(x, 2t - s)$, $s \in [0, 2t]$, $t \in [0, \frac{T}{2}]$, then (9), after some straightforward calculus, becomes

$$(10) \quad \begin{aligned} & 2 \int_B \varrho \varphi_i(t) \dot{\varphi}_i(t) dV = \\ & = \int_0^t \int_B \varrho [\varphi_i(2t - s) \ddot{\varphi}_i(s) - \ddot{\varphi}_i(2t - s) \varphi_i(s)] dV ds. \end{aligned}$$

We shall eliminate the inertial terms on the right side of (10). In view of (1)₁, we have

$$\begin{aligned} & \varrho [v_i(2t - s) \ddot{v}_i(s) - \ddot{v}_i(2t - s) v_i(s)] = \\ & = [v_i(2t - s) t_{ij}(s) - v_i(s) t_{ij}(2t - s)]_{,j} + [t_{ij}(2t - s) v_{j,i}(s) - \\ & \quad - t_{ij}(s) v_{j,i}(2t - s)] - \varrho [G_i(s) v_i(2t - s) - G_i(2t - s) v_i(s)], \end{aligned}$$

and then, with aid of Eqs.(4) and (3), we conclude

$$(11) \quad \begin{aligned} & \varrho [v_i(2t - s) v_i(s) - v_i(2t - s) v_i(s)] = \\ & = [v_i(2t - s) t_{ij}(s) - v_i(s) t_{ij}(2t - s)] + \varrho [G_i(s) v_i(2t - s) - \\ & \quad - G_i(2t - s) v_i(s)] + A_{ijmn} \varepsilon_{jik} [\varepsilon_{mn}(s) \psi_k(2t - s) - \varepsilon_{mn}(2t - s) \psi_k(s)] + \\ & \quad + B_{ijmn} [\gamma_{mn}(2t - s) \varepsilon_{ij}(s) - \gamma_{mn}(s) \varepsilon_{ij}(2t - s)] + E_{ij} [\chi(s) \varepsilon_{ij}(2t - s) - \\ & \quad - \chi(2t - s) \varepsilon_{ij}(s)] + B_{ijmn} \varepsilon_{jik} [\gamma_{mn}(s) \psi_k(2t - s) - \gamma_{mn}(2t - s) \psi_k(s)] + \\ & \quad + E_{ij} \varepsilon_{jik} [\chi(2t - s) \psi_k(s) - \chi(s) \psi_k(2t - s)]. \end{aligned}$$

On the other hand, because of symmetric relations of I_{ij} , it follows

$$I_{ij} \frac{d}{dt} [W_i(t) \dot{\psi}_j(t) - \dot{W}_i(t) \psi_j(t)] = I_{ij} [W_i(t) \ddot{\psi}_j(t) - \ddot{W}_i(t) \psi_j(t)].$$

Based on this equality, in view of homogeneous initial conditions the following identity is obtained

$$(12) \quad \int_B I_{ij} [W_i(t) \dot{\psi}_j(t) - \dot{W}_i(t) \psi_j(t)] dV = \\ = \int_0^t \int_B I_{ij} [W_i(s) \ddot{\psi}_j(s) - \ddot{W}_i(s) \psi_j(s)] dV ds.$$

By substituting $W_i(s)$ with $\psi_i(2t - s)$, $s \in [0, 2t]$, $t \in [0, \frac{T}{2})$, we get

$$(13) \quad 2 \int_B I_{ij} \psi_i(t) \dot{\psi}_j(t) dV = \\ = \int_0^t \int_B I_{ij} [\psi_i(2t - s) \ddot{\psi}_j(s) - \ddot{\psi}_i(2t - s) \psi_j(s)] dV ds.$$

We next eliminate the inertial terms on the right side of the relation (13) by means of (1)₂

$$(14) \quad I_{ij} [\psi_i(2t - s) \ddot{\psi}_j(s) - \ddot{\psi}_i(2t - s) \psi_j(s)] = \\ = [m_{ij}(s) \psi_i(2t - s) - m_{ij}(2t - s) \psi_i(s)]_{,j} + \rho [L_i(s) \psi_i(2t - s) - \\ - L_i(2t - s) \psi_i(s)] + m_{ij}(2t - s) \gamma_{ij}(s) - m_{ij}(s) \gamma_{ij}(2t - s) + \\ + \varepsilon_{kij} [t_{ij}(s) \psi_k(2t - s) - t_{ij}(2t - s) \psi_k(s)].$$

Next, with the aid of Eqs.(4) and (3), we conclude

$$(15) \quad I_{ij} [\psi_i(2t - s) \ddot{\psi}_i(s) - \ddot{\psi}_i(2t - s) \psi_i(s)] = \\ = [m_{ij}(s) \psi_i(2t - s) - m_{ji}(2t - s) \psi_i(s)]_{,j} + \rho [L_i(s) \psi_i(2t - s) - \\ - L_i(2t - s) \psi_i(s)] + B_{ijmn} [\varepsilon_{ij}(2t - s) \gamma_{mn}(s) - \varepsilon_{ij}(s) \gamma_{mn}(2t - s)] + \\ + D_{ij} [\chi(s) \gamma_{ij}(2t - s) - \chi(2t - s) \gamma_{ij}(s)] + A_{ijmn} \varepsilon_{kij} [e_{mn}(s) \psi_k(2t - s) - \\ - e_{mn}(2t - s) \psi_k(s)] + B_{ijmn} \varepsilon_{kij} [\gamma_{mn}(s) \psi_k(2t - s) - \gamma_{mn}(2t - s) \psi_k(s)] + \\ + E_{ij} \varepsilon_{kij} [\chi(2t - s) \psi_k(s) - \chi(s) \psi_k(2t - s)].$$

We now integrate Eqs.(11) and (15) over $B \times [0, t]$ and, with the aid of the divergence theorem and initial and boundary values (for difference), it follows, in view of (10) and (13),

$$\begin{aligned}
 & \int_B (\varrho v_i \dot{v}_i + I_{ij} \psi_i \dot{\psi}_j) dV = \\
 & = \int_0^t \int_B \{ \varrho [G_i(s) v_i(2t-s) + L_i(s) \psi_i(2t-s) - G_i(2t-s) v_i(s) - \\
 & \quad - L_i(2t-s) \psi_i(s)] + E_{ij} [\chi(s) \varepsilon_{ij}(2t-s) - \chi(2t-s) \varepsilon_{ij}(s)] + \\
 (16) \quad & \quad + D_{ij} [\chi(s) \gamma_{ij}(2t-s) - \chi(2t-s) \gamma_{ij}(s)] \} dV ds.
 \end{aligned}$$

We turn our attention to the constitutive equation of entropy and to the energy equation. After some straightforward calculations, we obtain

$$\begin{aligned}
 (17) \quad & E_{ij} [\chi(s) \varepsilon_{ij}(2t-s) - \chi(2t-s) \varepsilon_{ij}(s)] + D_{ij} [\chi(s) \gamma_{ij}(2t-s) \\
 & \quad - \chi(2t-s) \gamma_{ij}(s)] = \eta(2t-s) \chi(s) - \eta(s) \chi(2t-s).
 \end{aligned}$$

It is easily to verify that

$$\begin{aligned}
 & \eta(2t-s) \chi(s) - \eta(s) \chi(2t-s) = \\
 (18) \quad & = \frac{1}{T_0} k_{ij} [\chi(s) \int_0^{2t-s} \chi_j(\xi) d\xi - \chi(2t-s) \int_0^s \chi_j(\xi) d\xi + \\
 & \quad + \frac{1}{T_0} k_{ij} [\chi_i(2t-s) \int_0^s \chi_j(\xi) d\xi - \chi_i(s) \int_0^{2t-s} \chi_j(\xi) d\xi] + \\
 & \quad + \frac{\varrho}{T_0} [\chi(s) \int_0^{2t-s} P(\xi) d\xi - \chi(2t-s) \int_0^s P(\xi) d\xi].
 \end{aligned}$$

Based on the symmetry of k_{ij} , we can write

$$\begin{aligned}
 (19) \quad & \int_0^t \int_B \frac{1}{T_0} k_{ij} \frac{d}{ds} \left(\int_0^s \chi_i(\xi) d\xi \right) \left(\int_0^{2t-s} \chi_j(\xi) d\xi \right) dV ds = \\
 & = \int_B \frac{1}{T_0} k_{ij} \left(\int_0^{2t-s} \chi_i(\xi) d\xi \right) \left(\int_0^{2t-s} \chi_j(\xi) d\xi \right) dV.
 \end{aligned}$$

On the other hand, integrating by parts,

$$\begin{aligned}
 & \int_0^t \int_B \frac{1}{T_0} k_{ij} \left[\theta_i(s) \int_0^{2t-s} \theta_j(\xi) d\xi - \theta_j(2t-s) \int_0^{2t-s} \theta_i(\xi) d\xi \right] dV ds \\
 (20) \quad & = \int_0^t \int_B \frac{1}{T_0} k_{ij} \frac{d}{ds} \left(\int_0^{2t-s} \theta_i(\xi) d\xi \right) \left(\int_0^{2t-s} \theta_j(\xi) d\xi \right) dV ds,
 \end{aligned}$$

and then, with aid of (19) we get

$$(21) \quad \int_0^t \int_B \frac{1}{T_0} k_{ij} \left[\chi_{j,i}(s) \int_0^s \chi_{i,i}(\xi) d\xi - \chi_{i,i}(2t-s) \int_0^{2t-s} \chi_{j,j}(\xi) d\xi \right] dV ds \\ = - \int_0^t \int_B \frac{1}{T_0} k_{ij} \left(\int_0^t \chi_{i,i}(\xi) d\xi \right) \left(\int_0^t \chi_{j,j}(\xi) d\xi \right) dV ds.$$

Thus (16), with the help of Eqs.(17)-(21), may be restated as follows

$$(22) \quad 2 \int_B [\rho v_i(t) \dot{v}_i(t) + I_{ij} \psi_i(t) \dot{\psi}_j(t)] dV + \\ + \int_B \frac{1}{T_0} k_{ij} \left(\int_0^t \chi_{i,i}(\xi) d\xi \right) \left(\int_0^t \chi_{j,j}(\xi) d\xi \right) dV = \\ = \int_0^t \int_B \rho [G_i(s) v_i(2t-s) + L_i(s) \psi_i(2t-s) - \\ - G_i(2t-s) v_i(s) - L_i(2t-s) \psi_i(s)] dV ds + \\ + \int_0^t \int_B \frac{\rho}{T_0} \left(\chi(s) \int_0^{2t-s} P(\xi) d\xi - \chi(2t-s) \int_0^s P(\xi) d\xi \right) dV ds.$$

Combining the above assertions, we obtain the following theorem

Theorem 1. Let $(u_i^{(\alpha)}, \varphi_i^{(\alpha)}, \theta^{(\alpha)})$, $(\alpha = 1, 2)$ be two solutions of our initial boundary value problem and their difference

$$v_i = u_i^{(2)} - u_i^{(1)}, \psi_i = \varphi_i^{(2)} - \varphi_i^{(1)}, \chi = \theta^{(2)} - \theta^{(1)},$$

which corresponds to the null initial and boundary data. Then the Lagrange identity has the form (22).

Based on the identity (22), we shall prove the uniqueness and continuous dependence results. We first proceed to obtain the uniqueness of solution.

Theorem 2. Assume that the conductivity tensor k_{ij} is positive definite in the sens that there exists a positive constant k_0 such that

$$(23) \quad k_{ij} \xi_i \xi_j \geq k_0 \xi_i \xi_i, \forall \xi_i.$$

If ∂B_3 is not empty or $a(x) \neq 0$ on B , then the mixed problem of thermoelastodynamics in linear theory of micropolar bodies, has at most one solution.

Proof. In other words, we shall prove that

$$(24) \quad v_i(x, t) = 0, \quad \psi_i(x, t) = 0, \quad \chi(x, t) = 0, \quad \forall (x, t) \in B \times [0, T).$$

Because the solutions $(u_i^{(\alpha)}, \varphi_i^{(\alpha)}, \theta_i^{(\alpha)})$, correspond to the same body force, body couple and heat supply, it results that their differences correspond to the null body force, body couple and heat supply. Thus (22) may be restated as follows

$$2 \int_B [\rho v_i \dot{v}_i + I_{ij} \psi_i \dot{\psi}_j] dV + \int_B \frac{1}{T_0} k_{ij} \left(\int_0^t \chi_{,i}(\xi) d\xi \right) \left(\int_0^t \chi_{,j}(\xi) d\xi \right) dV = 0,$$

and then, by integrating on $[0, s]$, $s \in [0, \frac{T}{2})$,

$$\begin{aligned} & \int_B \rho v_i(s) v_i(s) dV + \int_B I_{ij} \psi_i(s) \psi_j(s) dV + \\ & + \int_0^s \int_B \frac{1}{T_0} k_{ij} \left(\int_0^\tau \chi_{,i}(\xi) d\xi \right) \left(\int_0^\tau \chi_{,j}(\xi) d\xi \right) dV d\tau = 0, \end{aligned}$$

which prove that

$$(25) \quad v_i(x, t) = 0, \quad \psi_i(x, t) = 0, \quad \chi_{,i}(x, t) = 0 \text{ on } B \times [0, \frac{T}{2}).$$

If ∂B_3 is not empty, from (7) we deduce that (24) hold. If $\alpha(x) \neq 0$, from the energy equation we get $\dot{\chi} = 0$. But χ vanishes initially, so that (24) again hold true. If T is infinite, then the proof of Theorem 2 is complete. If T is finite, then we set

$$v_i(x, \frac{T}{2}) = \dot{v}_i(x, \frac{T}{2}) = \psi_i(x, \frac{T}{2}) = \dot{\psi}_i(x, \frac{T}{2}) = \chi(x, \frac{T}{2}) = 0,$$

and repeat the above procedure on $[\frac{T}{2}, \frac{T}{2} + \frac{T}{4}]$ in order to extend the conclusion (24) on $B \times [0, \frac{3T}{4})$, and so on. \square

We are now ready to state and prove the continuous dependence theorem upon body force, body couple and heat supply on the compact subintervals of $[0, T)$ for the solutions of the initial-boundary value problems defined by (8), (6) and (7).

Theorem 3. Let $(u_i^{(\alpha)}, \varphi_i^{(\alpha)}, \theta^{(\alpha)})$, $(\alpha = 1, 2)$ be two solutions of our problem corresponding to the same boundary and initial data, but to the body forces, body couples and heat supplies $(F_i^{(\alpha)}, M_i^{(\alpha)}, r^{(\alpha)})$, $(\alpha = 1, 2)$ respectively, where

$$F_i^{(2)} = F_i^{(1)} + G_i, \quad M_i^{(2)} = M_i^{(1)} + L_i, \quad r^{(2)} = r^{(1)} + P.$$

Then we have the following estimate

$$\begin{aligned} & \int_B [\rho v_i(s)v_i(s) + I_{ij}\psi_i(s)\psi_j(s)]dV + \\ (26) \quad & + \int_0^s \int_B \frac{1}{T_0} k_{ij} \left(\int_0^t \chi_i(\xi)d\xi \right) \left(\int_0^t \chi_j(\xi)d\xi \right) dV ds \leq \\ & \leq t_* M \left[\int_0^{t_*} \int_B \rho G_i(t)G_i(t)dV dt \right]^{\frac{1}{2}} + t_* N \left[\int_0^{t_*} \int_B \frac{\rho}{T_0} \left(\int_0^t P(\xi)d\xi \right)^2 dV dt \right]^{\frac{1}{2}} + \\ & + t_* Q \left[\int_0^{t_*} \int_B \rho L_i(t)L_i(t)dV dt \right]^{\frac{1}{2}}, \quad s \in [0, \frac{t_*}{2}], \end{aligned}$$

provided there exists $t_* \in [0, T)$ so that

$$\begin{aligned} (27) \quad & \int_0^{t_*} \int_B \rho G_i(t)G_i(t)dV dt \leq M_1^2, \quad \int_0^{t_*} \int_B \rho L_i(t)L_i(t)dV dt \leq M_2^2, \\ & \int_0^{t_*} \int_B \frac{\rho}{T_0} \left(\int_0^t P(\xi)d\xi \right)^2 dV dt \leq M_3^2, \quad \int_0^{t_*} \int_B \rho v_i(t)v_i(t)dV dt \leq M^2, \\ & \int_0^{t_*} \int_B \frac{\rho}{T_0} \chi^2(\xi)dV dt \leq N^2, \quad \int_0^{t_*} \int_B I_{ij}\psi_i(t)\psi_j(t)dV dt \leq Q^2. \end{aligned}$$

Proof. According to the Schwarz's inequality, it follows

$$\begin{aligned} & \int_0^t \int_B \rho v_i(2t-s)G_i(s)dV ds \leq \\ & \leq \left[\int_0^t \int_B \rho G_i(s)G_i(s)dV ds \right]^{\frac{1}{2}} \left[\int_t^{2t} \int_B \rho v_i(s)v_i(s)dV ds \right]^{\frac{1}{2}} \leq \\ & \leq M \left[\int_0^t \int_B \rho G_i(s)G_i(s)dV ds \right]^{\frac{1}{2}}, \end{aligned}$$

where, at last, we use the substitution $2t - s \rightarrow s$. We proceed analogically on the other integrals in (22) and then by integrating over $[0, s]$, $s \in [0, \frac{t}{2}]$ we obtain estimate (26) and the proof of Theorem 3 is complete. \square

The estimate (26) will be used to obtain a continuous result upon initial data.

Theorem 4. Let $(u_i^{(1)}, v_i^{(1)}, \theta^{(1)})$, $(u_i^{(1)} + w_i, \varphi_i^{(1)} + \gamma_i, \theta^{(1)} + \sigma)$ be two solutions of our problem (8), (6) and (7), which correspond to the same body force, body couple, heat supply and boundary data, but to the initial data $(a^{(1)}, b^{(1)}, c^{(1)}, d^{(1)}, \theta_0^{(1)})$ and $(a_i^{(1)} + \alpha_i, b_i^{(1)} + \beta_i, c_i^{(1)} + \gamma_i, d_i^{(1)} + \delta_i, \theta_0^{(1)} + \varphi)$, where the perturbations $(\alpha_i, \beta_i, \gamma_i, \delta_i, \varphi)$ obey the following restrictions

$$\int_B \rho(\alpha_i \alpha_i + \beta_i \beta_i) dV \leq M_4^2, \int_B \rho(\gamma_i \gamma_i + \delta_i \delta_i) dV \leq M_5^2, \int_B \frac{T_0}{\rho} \eta_0^2 dV \leq M_6^2,$$

where we used the notation

$$\eta_0(x) = E_{i,j}(x)[\alpha_{j,i}(x) + \varepsilon_{j,k} \gamma_k(x)] + D_{i,j}(x) \gamma_{j,i}(x) + \frac{\alpha}{T_0} \varphi(x).$$

If we define

$$v_i(x, t) = \int_0^t \int_0^s w_i(x, t) d\tau ds,$$

$$(28) \quad \psi_i(x, t) = \int_0^t \int_0^s \omega_i(x, t) d\tau ds, \quad \chi(x, t) = \int_0^t \int_0^s \sigma(x, t) d\tau ds,$$

then the estimate

$$(29) \quad \int_B [\rho v_i(t) v_i(t) + I_{ij} \psi_i(t) \psi_j(t)] dV + \\ + \int_0^t \int_B \frac{1}{T_0} K_{ij} \left(\int_0^s \chi_{i,i}(\xi) d\xi \right) \left(\int_0^s \chi_{j,j}(\xi) d\xi \right) dV ds \leq \\ \leq t_* M \left((t_* + \frac{t_*^2}{2}) \int_B \rho a_i a_i dV + (\frac{t_*^2}{2} + \frac{t_*^2}{3}) \int_B \rho b_i b_i dV \right)^{\frac{1}{2}} + \\ + t_* Q \left((t_* + \frac{t_*^2}{2}) \int_B \rho c_i c_i dV + (\frac{t_*^2}{2} + \frac{t_*^2}{3}) \int_B I_{ij} d_i d_i dV \right)^{\frac{1}{2}} + \\ + N t_*^{\frac{1}{2}} (20)^{-\frac{1}{2}} \left(\int_B \frac{T_0}{\rho} \eta_0^2 dV \right)^{\frac{1}{2}},$$

hold, provided the perturbations (v_i, ψ_i, χ) (from (28)) satisfy (27).

Proof. An integration by parts shows that (v_i, ψ_i, χ) defined in (28) may be

restated, equivalently

$$v_i(x, t) = \int_0^t (t - s)w_i(x, s)ds, \quad \psi_i(x, t) = \int_0^t (t - s)\omega_i(x, s)ds,$$

$$\chi(x, t) = \int_0^t (t - s)\sigma(x, s)ds.$$

It is easily to prove that the functions (w_i, ω_i, σ) satisfy the equations of motion and of energy with $F_i = M_i = r = 0$, and the initial conditions

$$w_i(x, 0) = \alpha_i(x), \quad \dot{w}_i(x, 0) = \beta_i(x), \quad \omega_i(x, 0) = \gamma_i(x),$$

$$\dot{\omega}_i(x, 0) = \delta_i(x), \quad \sigma(x, 0) = \varphi(x).$$

A straightforward calculus proves that the functions (v_i, ψ_i, θ) defined in (28) satisfy the equations of motion and the energy equation, in which

$$F_i(x, t) = \alpha_i(x) + t\beta_i(x), \quad M_i(x, t) = \gamma_i(x) + t\delta_i(x),$$

$$r(x, t) = T_0[E_{ij}(x)\alpha_{j,i}(x) + \varepsilon_{jik}E_{ij}(x)\gamma_k(x) + D_{ij}\gamma_{j,i}(x) + \frac{\alpha}{T_0}\varphi(x)]t.$$

With these specifications, the estimate (29) follows from (27). \square

Finally, we obtain a continuous dependence result of solution of our problem, upon the thermoelastic coefficients, again as a consequence of the Theorem 3.

Theorem 5. *We consider the specific case when $\partial B_1^c = \partial B_2^c = \partial B_3^c = 0$. Let $(u_i^{(\alpha)}, \varphi_i^{(\alpha)}, \theta^{(\alpha)})$ be two solutions of our problem with above assumptions, corresponding to the same boundary data, initial data, same body force, body couple and heat supply, but to the thermoelastic coefficients*

$$(A_{ijmn}^{(1)}, B_{ijmn}^{(1)}, C_{ijmn}^{(1)}, E_{ij}^{(1)}, D_{ij}^{(1)}, k_{ij}^{(1)}, a^{(1)}),$$

($A_{ijmn}^{(1)} + \mathcal{A}_{ijmn}, B_{ijmn}^{(1)} + \mathcal{B}_{ijmn}, C_{ijmn}^{(1)} + \mathcal{C}_{ijmn}, E_{ij}^{(1)} + \mathcal{E}_{ij}, D_{ij}^{(1)} + \mathcal{D}_{ij}, a^{(1)} + \alpha, k_{ij}^{(1)} + \kappa_{ij}$) respectively. If the perturbations (v_i, ψ_i, χ) of solutions satisfy (27), then any solution of our problem, for which

$$(30) \quad \int_B (u_{i,j}u_{i,j} + u_{i,jk}u_{i,jk} + \varphi_{i,j}\varphi_{i,j} + \theta_{,j}\theta_{,j} + \theta_{,jk}\theta_{,jk} + \dot{u}_{i,j}\dot{u}_{i,j} + \dot{\varphi}_{i,j}\dot{\varphi}_{i,j} + \theta^2)dV ds \leq M_7^2,$$

depends continuously on the thermoelastic coefficients on $[0, \frac{t_2}{2}]$, in

$$\int_B [\rho v_i(t)v_i(t) + I_i \psi_i(t)\psi_j(t)] dV + \\ + \int_0^t \int_B \frac{1}{T_0} k_{ij} \left(\int_0^s \chi_{,i}(\xi) d\xi \right) \left(\int_0^s \chi_{,j}(\xi) d\xi \right) dV ds.$$

Proof. In the usual way, we can prove that the perturbations (v_i, ψ_i, χ) of two solutions, verify the equations of motion and the equation of energy with the following body force, body couple and heat supply

$$F_i = A_{ijmn} \varepsilon_{mn}^{(2)} + B_{ijmn} \gamma_{mn}^{(2)} - \varepsilon_{ij} \theta^{(2)}, \quad M_i = B_{mni j} \varepsilon_{mn}^{(2)} + C_{ijmn} \gamma_{mn}^{(2)} - D_{ij} \theta^{(2)}, \\ \frac{\rho}{T_0} r = \varepsilon_{ij} \dot{\varepsilon}_{ij}^{(2)} + D_{ij} \dot{\gamma}_{ij}^{(2)} + \frac{\rho}{T_0} \dot{\theta}^{(2)} - \frac{1}{T_0} [\kappa_{ij} \theta_{,j}^{(2)}]_{,i}.$$

Thus the problem is analogous to the above problem, from Theorem 4. So, according to (29) and (26), we obtain the desired result. \square

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