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FRAMED (2m+3)-DIMENSIONAL RIEMANNIAN MANIFOLDS ENDOWED WITH A KENMOTSU ALMOST CONTACT 3-STRUCTURE

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Abstract

A class of framed manifolds (which we call Kenmotsu almost contact 3- structured) is studied, showing that such a manifold is the local product of a Kählerian submanifold and a space form. Moreover, some reults about concircular exterior recurrent vector fields are obtained.

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1 Introduction

Framed manifolds and f-structures have been initiated by K. Yano and M. Kon [Y.K.1] and also studied by some other authors as for instance [Be], [T.Y], [G.R], [R1]. We recall that if $M(\phi,\Omega,\xi_r,\eta^r,g)$ is a $(2m+q)$-dimensional $(r=2m+1,\ldots,2m+q)$ such manifold, then ξ_r are the *Reeb vector fields* (in the large sense) of the f- structure and η^r are their corresponding covectors. One has the following structure equations:

$$\phi^2=-Id+\eta^r\otimes\xi_r,\ \phi(\xi_r)=0,\ \eta^r\circ\phi=0,\ \eta^s(\xi_t)=\delta^s_t \tag{1.1}$$

where $r,s,t\in\{2m+1,\ldots,2m+q\}$.

If ϕ is skew symmetric with respect to g, i.e., $g(X,\phi(Y))+g(\phi(X),Y)=0$ then the 2-form Ω of rank $2m$ such that

$$\Omega(X,Y)=g(\phi(X),Y),\ \Omega^m\wedge\eta^{2m+1}\wedge\ldots\wedge\eta^{2m+q}\neq 0 \tag{1.2}$$

is called the *fundamental form* of the framed manifold (or of the associated f-structure).

In the present paper we consider a $(2m+3)$-dimensional framed manifold M such that $d\Omega = 0$ and the Reeb vector fields satisfy

$$\nabla\xi_r = a_r dp^{\top} + \xi_s \wedge \xi_t \tag{1.3}$$

where r, s, t are ciclic permutations of $2m+1, 2m+2, 2m+3$, and $dp^{\top}$, $\wedge$ and a_r denote the horizontal component of the soldering form dp (i.e., dp is identity (1,1) tensor field on M), the wedge product and the scalar fields (called the structure scalars) respectively.

The unit vector field $\xi = \sum a_r\xi_r$ is called the *principal vector field* of M and $da_r = 0$ defined a closed Pfaff system, and the following structure equation is satisfied:

$$(\nabla\phi)Z = -\eta(Z)\phi dp^{\top} - (\phi(Z))^{\flat}\otimes\xi \Rightarrow (\nabla_{Z'}\phi)Z = -\eta(Z)\phi(Z') - g(\phi(Z), Z')\xi \tag{1.4}$$

where $Z, Z' \in \mathcal{X}\mathcal{M}$. Since the above equation is similar to that of standard Kenmotsu manifolds [K] we agree to say that the f-manifolds under consideration is endowed with a *Kenmotsu 3-structure* (abr. K 3- structure). The follwing theorem will be proved: *Let $M(\phi, \Omega, \xi_r, \eta^r, a_r, g)$ be a framed $(2m+3)$-dimensional manifold endowed with a Kenmotsu 3-structure. Then any such M is a framed CR- manifold [Be] and may be viewed as the local Riemannian product $M = M^{\top}\times M^{\perp}$ such that: (i) $M^{\top}$ is a $2m$-dimensional Kählerian submanifold having $\Omega\,|_{M^{\top}}$ as symplectic form; (ii) $M^{\perp}$ is a 3-dimensional space form of curvature $+1$ tangent to the Reeb vector fields ξ_r and the immersion $\kappa : M^{\perp} \to M$ is totally geodesic.*

In addition, different properties of some vector fields carried by such a manifold are pointed out. As for instance:

(i) ξ is an *exterior concurrent* vector field [R.2], [M.R.V], i.e., $d^{\nabla}(\nabla\xi) = \nabla^2\xi = \eta\wedge dp$, where $\eta = \xi^{\flat}$ and $\Re(\xi,\xi) = -2(m+1)$, where $\Re$ is the Ricci tensor and M is not compact.

(ii) the structure scalars a_r are *isoparametric* functions, and ∇a_r are conformal vector fields on $M^{\top}$. If U is a *concircular exterior recurrent* vector field having ξ as recurrence vector, that is $\nabla U = -U^{\flat}\otimes\xi$; $dU^{\flat} = -U^{\flat}\wedge\eta$, then this property is invariant under the action of ϕ. Both forms $U^{\flat}$ and $(\phi(U))^{\flat}$ are in terms of cohomology $d^{-\eta}$-*exact* and one has: $\Re(U,U) = \Re(\phi(U),\phi(U)) = -2m \parallel U \parallel^2 = const.$ Finally, if one considers the vector valued 1-form $F = U\wedge\phi(U) \in A^1(M, TM)$ then $d^{\nabla^3}F$ is invariant by U and $\phi(U)$.

2 Preliminaries

Let (M, g) be a Riemannian C^∞-manifold and let ∇ be the covariant differential operator defined by the metric tensor g. We assume in the following that M is oriented and ∇ is the Levi-Civita connection. Let $\Gamma(TM) = \mathcal{X}M$ and $\flat : TM \to T^*M$ the set of sections of the tangent bundle TM and the *musical isomorphism* defined by g, respectively. Following [P] we set $A^q(M, TM) = Hom(\Lambda^q TM, TM)$ and notice that the elements of $A^q(M, TM)$ are vector valued q-forms. Denote by $d^\nabla : A^q(M, TM) \to A^{q+1}(M, TM)$ the exterior covariant derivative operator with respect to ∇. It should be noticed that generically $d^{\nabla^2} = d^\nabla \circ d^\nabla \neq 0$ unlike $d^2 = d \circ d = 0$. If $p \in M$ then the vector valued 1-form $dp \in A^1(M, TM)$ is the identity vector valued 1-form of M and is called the *soldering form* of M [D]. Since ∇ is symmetric one has $d^\nabla(dp) = 0$. The operator

$$d^\omega = d + e(\omega) \tag{2.1}$$

acting on ΛM is called the *cohomology operator* [GL], where $e(\omega)$ means the exterior product by the closed 1-form $\omega \in \Lambda^1 M$, i.e.,

$$d^\omega u = du + \omega \wedge u \tag{2.2}$$

for any $u \in \Lambda M$. One has

$$d^\omega \circ d^\omega = 0 \tag{2.3}$$

A form u such that $d^\omega u = 0$, u is said to be *d^ω-closed*, and if ω is exact, then u is said to be *d^ω-exact*. A vector field $X \in \mathcal{X}M$ such that

$$d^\nabla(\nabla X) = \nabla^2 X = \pi \wedge dp \in A^2(M, TM),\ \pi \in \Lambda^1 M \tag{2.4}$$

is defined to be an *exterior concurrent* vector field [R1], [PRV]. In (2.4) π is called the *concurrence* form and is defined by

$$\pi = \lambda X^\flat,\ \lambda \in \Lambda^0 M. \tag{2.5}$$

In this case if $\Re$ is the Ricci tensor of ∇, then by (2.4) one has

$$\Re(X, Z) = -(n-1)\lambda g(X, Z),\ \forall Z \in \mathcal{X}M\ (n = dim M) \tag{2.6}$$

A function $f : R^n \to R$ is *isoparametric* if $\| \nabla f \|^2$ and $div(\nabla f)$ are functions of f.

Let $\mathcal{O} = vect\{e_A; A \in 1, \ldots, n\}$ be a local field of adapted vectorial frames over M and let $\mathcal{O}^* = covect\{\omega^A\}$be the associated coframe. Then the soldering form dp is expressed by

$$dp = \omega^A \otimes e_A \tag{2.7}$$

and E. Cartan's structure equations can be written in indexless manner as

$$\nabla e = \vartheta \otimes e \tag{2.8}$$

$$d\omega = -\vartheta \wedge \omega \tag{2.9}$$

$$d\vartheta = -\vartheta \wedge \vartheta + \Theta. \tag{2.10}$$

In the above equations ϑ (resp. Θ) are the local connection forms in the tangent bundle TM (resp. the curvature 2-form on M).

3 The main theorem

Let $M(\phi, \Omega, \xi_r, \eta^r, g)$ be a $(2m+3)$-dimensional C^∞-manifold with soldering form dp and carrying an f-structure ϕ [YK], that is, ϕ is a tensor field of type (1,1) and of rank $2m$ which satisfies

$$\phi^3 + \phi = 0 \tag{3.1}$$

$$\phi^2 = -Id + \sum \eta^r \otimes \xi_r,\ \phi(\xi_r) = 0,\ \eta^r \circ \phi = 0 \tag{3.2}$$

$$g(Z, Z') = g(\phi Z, \phi Z') + \sum \eta^r(Z)\eta^r(Z') \tag{3.3}$$

where Id is the identity morphism of M.

If in addition the fundamental 2-form Ω of M satisfies

$$\Omega(Z, Z') = g(\phi Z, Z'),\ \Omega^m \wedge \eta^{2m+1} \wedge \ldots \wedge \eta^{2m+q} \neq 0 \tag{3.4}$$

then $M(\phi, \Omega, \xi_r, \eta^r, g)$ is as known [WK] defined as a *framed f-manifold* (see also [M]).

With respect to the cobasis $\mathcal{O}^* = covect\{\omega^A, \eta^r\}$ of $\mathcal{O} = vect\{e_A, \xi_r;\ A = 1, \ldots, 2m;\ r = 2m+1, 2m+2, 2m+3\}$ the 2-form Ω is expressed by the standard form

$$\Omega = \sum_{a=1}^{m} \omega^a \wedge \omega^{a^*};\ a^* = a + m \tag{3.5}$$

Next making use of (2.2) one finds by (3.1), (3.2) the known Kählerian conditions

$$\vartheta^a_b = \vartheta^{a^*}_{b^*}; \; \vartheta^{a^*}_b = \vartheta^{b^*}_a . \tag{3.6}$$

In the present paper we assume that the structure vector fields ξ_r (called also by abusing language the Reeb vector fields) satisfy

$$\nabla \xi_r = a_r dp^\top + \xi_s \wedge \xi_t ; \; \forall \widehat{r,s,t} \tag{3.7}$$

where $dp^\top = dp - \sum \eta^r \otimes \xi_r$, and a_r, $\wedge$ and $\widehat{r,s,t}$ denote scalar functions, the wedge product of vector fields and the ciclic permutations of $2m+1, 2m+2, 2m+3$, respectively.

Next with the help of the structure equations (2.2) we derive by (3.7)

$$\Theta^A_r = a_r \omega^A ; \; A = 1, \ldots, 2m \tag{3.8}$$

and

$$\Theta^r_s = \eta^t ; \; \forall \widehat{r,s,t} \tag{3.9}$$

and with the help of the structure equations (2.9) one has

$$d\eta^r = -2\eta^s \wedge \eta^t ; \; \forall \widehat{r,s,t} \tag{3.10}$$

By (3.10) it follows

$$\eta^r \wedge d\eta^r \neq 0 ; \; \eta^r \wedge (d\eta^r)^2 = 0 \tag{3.11}$$

which shows that the Reeb covectors η^r are of *class* 3. In addition we agree to call

$$\xi = \sum a_r \xi_r \tag{3.12}$$

the *principal vector field* on M, and we assume that its local form $\eta = \xi^\flat = a_r \eta^r$ is closed. This implies $d(\mathcal{L}_\xi \eta) = 0$, that is η is a *relatively integral invariant* of ξ. Then since $d\eta = 0$ one derives taking account (3.10)

$$da_r = 2(a_s \eta^t - a_t \eta^s ; \; \forall \widehat{r,s,t} \tag{3.13}$$

In the following we agree to call $a_r \in C^\infty(M)$ the *structure scalars* of M, and on behalf of (3.10) it is seen by (3.13) that they define a Pfaffian system.

One may split the tangent space $T_p(M)$ of as $T_p(M) = D^\top_p \oplus D^\perp_p$, where $D^\top_p$ and $D^\perp_p$ are two complementary distributions defined by $\{e_A\}$ and $\{\xi_r\}$, i.e.,

$D_p^\top = \{e_A\}$, $D_p^\perp = \{\xi_r\}$ and called the *horizontal* and the *vertical* distributions of M respectively.

This being it quickly follows from (3.7)

$$< \nabla_{Z'^\top}\xi_r, Z^\top > + < \nabla_{Z^\top}\xi_r, Z'^\top >= 2a_r < Z^\top, Z'^\top > \tag{3.14}$$

where $Z^\top, Z'^\top \in D_p^\top$ are any horizontal vector fields. Hence from above, one may say that the Reeb vector fields ξ_r are $D^\top$-*conformal* vector fields and we notice that "mutatis mutandis" this situation is similar to that of Kenmotsu manifolds [K]. This using, by (3.2), (3.6) and (3.8) one deduces the following structure equations on M:

$$(\nabla\phi)Z = -\eta(Z)\phi dp^\top - (\phi Z)^\flat \otimes \xi \Rightarrow (\nabla_{Z'}\phi)Z = -\eta(Z)\phi(Z') - g(\phi Z, Z')\xi \tag{3.15}$$

which is similar to that of Kenmotsu manifolds having ξ as Reeb vector field. Therefore we agree to say that the f-manifold under consideration is endowed with a *Kenmotsu 3-structure* (abr. K 3-structure). Next denote by

$$\varphi^\top = \omega^1 \wedge \ldots \wedge \omega^{2m} \tag{3.16}$$

and

$$\varphi^\perp = \eta^{2m+1} \wedge \eta^{2m+2} \wedge \eta^{2m+3} \tag{3.17}$$

the simple unit forms which correspond to the distributions $D_p^\top$ and $D_p^\perp$ respectively. Making use of (3.8) and (3.10) and remembering that $\xi^\flat = \eta = \sum a_r\eta^r$ one finds with the help of the structure equations (2.9)

$$d\varphi^\top = 2m\eta \wedge \varphi^\top;\ d\varphi^\perp = 0. \tag{3.18}$$

Hence following known definitions, the above equations shows that $\varphi^\top$ is a *conformal integral invariant* of $D_p^\top$ and that $\varphi^\perp$ is an*integral invariant* of $D_p^\perp$. Therefore, by Fröbenius Theorem, we conclude that both distributions $D_p^\top$ and $D_p^\perp$ are *involutive*. Recall now that the *torsion tensor field* S of an f-structure is the vector valued 2- form defined by

$$S = N_\phi + 2\sum d\eta^r \otimes \xi_r \tag{3.19}$$

where

$$N_\phi(Z, Z') = [\phi Z, \phi Z'] + \phi^2[Z, Z'] - \phi[Z, \phi Z'] - \phi[\phi Z, Z'] \tag{3.20}$$

is the Nijenhuis tensor field of ϕ. In the case under discussion it is seen by (3.2), (3.4), (3.6) and (3.15) that S vanishes on the horizontal distribution

$D_p^{\top}$. Hence following a known definition (see also [Be]) the f-structure on a K 3-structure is $D^{\top}$-*normal* and consequently the manifold $M(\phi, \Omega, \xi_r, \eta^r, a_r, g)$ is a *framed CR manifold* (in the sense of [Be]).

In addition it is seen by (3.7) that one has

$$\nabla_{Z^{\perp}} Z'^{\perp} \in D_p^{\perp} \tag{3.21}$$

for any vertical $Z^{\perp}, Z'^{\perp} \in D_p^{\perp}$ vector fields. This as is known proves that $D_p^{\perp}$ is an *autoparallel foliation* and that the leaves $M^{\perp}$ of $D_p^{\perp}$ are *totally geodesic submanifolds* of M.

On the other hand, on behalf of (3.5) and (3.6), one easily finds $d\Omega = 0$, which shows that the horizontal submanifold $M^{\top}$ of M is a Kählerian manifold having $D_p^{\top}$ as symplectic vector space. Finally denote by Θ_s^r the curvature forms of $M^{\top}$. Then by reference to (3.9) and with the help of the structure equations (2.10) one derives

$$\Theta_s^r = \eta^r \wedge \eta^s. \tag{3.22}$$

Hence on behalf of a known result concerning the curvature forms on a Riemannian manifold, equations (3.22) prove the meaningfull fact that $M^{\perp}$ is a *space form* of curvature $+1$.

Summarizing, we state the

Theorem 3.1 *Let Let $M(\phi, \Omega, \xi_r, \eta^r, a_r, g)$ be a framed $(2m+3)$-dimensional manifold endowed with a Kenmotsu 3-structure. Then any such M is a framed CR-manifold and may be viewed as the local Riemannian product $M = M^{\top} \times M^{\perp}$ such that:*

(i) $M^{\top}$ is a $2m$-dimensional Kählerian submanifold having $\Omega\mid_{M^{\top}}$ as symplectic form;

(ii) $M^{\perp}$ is a 3-dimensional space form of curvature $+1$ tangent to the Reeb vector fields ξ_r and the immersion $\kappa : M^{\perp} \to M$ is totally geodesic.

4 Some consequences

In this section we shall point out some additional properties of the framed manifold $M(\phi, \Omega, \xi_r, \eta^r, a_r, g)$ defined in Section 3.

From (3.13) one gets at once $\sum(a_r)^2 = c = const.$ and without loss of generality one may take $c = 1$, that is, the principal vector field $\xi = \sum a_r \xi_r$ is an unit vector field. Next taking the covariant derivative of ξ, one gets by (3.7) and (3.13)

$$\nabla \xi = dp^{\top} + \frac{1}{2} \sum da_r \otimes \xi_r \tag{4.1}$$

and infers after some calculations

$$d^{\nabla}(\nabla\xi) = \nabla^2\xi = \eta \wedge dp \tag{4.2}$$

wich proves the significative fact that ξ is an *exterior concurrent* (abr. E.C.) vector field (see (2.4)).

On behalf of (4.2) one may write

$$\Re(\xi, Z) = -2(m+1)g(\xi, Z) \Rightarrow \Re(\xi,\xi) = -2(m+1) \tag{4.3}$$

where $\Re$ denotes the Ricci tensor field of ∇. Since $\Re(\xi,\xi)$ is negative definite, it follows according to a known theorem (see [Y]) that the considered K 3-structured manifold can not be compact (as the standard K-manifolds).

The following properties emerge also from (3.7) and (4.1)

$$\nabla_{\xi_r}\xi_r = 0;\ \nabla_{\xi}\xi = 0 \tag{4.4}$$

$$dp\,\xi = 2m \tag{4.5}$$

and

$$< \nabla_Z\xi, Z' > = < \nabla_{Z'}\xi, Z > \tag{4.6}$$

which proves that ξ_r, ξ are *geodesics*, that ξ defines an *infinitesimal homothety* on M and that $\xi^\flat = \eta$ is an *exact form.*

Next by (3.13) one has

$$\nabla a_r = 2(a_s\xi_t - a_t\xi_s);\ \forall \widehat{r,s,t} \tag{4.7}$$

where $\nabla a_r = grad\, a_r$. And since $\| \xi \|^2 = 1$ one derives from (4.7)

$$\| \nabla a_r \|^2 = 4(1 - a_r^2). \tag{4.8}$$

On the other hand since in general $div\, Z = tr(\nabla Z);\ Z \in \mathcal{X}M$ one derives from (3.7) and (4.7)

$$div\,\xi_r = 2ma_r;\ div\,(\nabla a_r) = -8a_r \Rightarrow \Delta a_r = 8a_r \tag{4.9}$$

Since $\| \nabla a_r \|^2$ and $div\,(\nabla a_r)$ are both functions of a_r, then following a known definition (see also [W]) that the structure scalars are *isopaametric functions* (we notice also that a_r are *eigenfunctions* of Δ, with 2^3 as associated eigenvalue).

The following fact is also worth to be outlined: By (3.22) and (3.10) it is easily seen that the vertical curvature forms Θ^r_s are closed. Accordingly since one may consider Θ^r_s as presymplectic forms on the submanifold $M^\perp$

we set on behalf of a known notation $Z \to -i_Z\Theta^r_s = \overset{\flat}{Z}$. Then if $Z = \xi$ one quickly finds $\overset{\flat}{\xi}_{(r,s)} = -i_\xi\Theta^r_s = -2da_t \Rightarrow \mathcal{L}_\xi\Theta^r_s = 0$ and we may formulate the following meaningfull fact: The principal vector field ξ defines an infinitesimal automorphism of the curvature form of $M^\perp$.

One has the

Theorem 4.1 *Let $M(\phi, \Omega, \xi_r, \eta^r, a_r, g)$ be the framed f-manifold defined in section 3, and let a_r and ξ be the structure scalars and the principal vector field on M. Then one has the following properties:*

(i) ξ is an exterior concurrent vector field, i.e., $\nabla^2\xi = \eta \wedge dp$; $\eta = \xi^\flat$, which implies $\Re(\xi, Z) = -2(m+1)g(\xi, Z) \Rightarrow \Re(\xi, \xi) = -2(m+1)$ and since η is closed, M is foliated by hypersurfaces normal to ξ;

(ii) ξ and the Reeb vector fields ξ_r are geodesics and ξ defines an infinitesimal homothety on M;

(iii) a_r are isoparametric functions and eigenfunctions od Δ;

(iv) the curvature forms of $M^\perp$ are closed and are invariant by ξ

5 Concircular exterior recurrent vector fields

In the present section we shall discuss some properties of a *concicular exterior recurrent* (abr. C.E.R.) vector field on $M(\phi, \Omega, \xi_r, \eta^r, a_r, g)$, that is, following R. Rosca [R3], if U is such that

$$\nabla U = cU^\flat \otimes V; \; dU^\flat = cU^\flat \wedge V^\flat; \; c = const. \tag{5.1}$$

where V is a certain vector field, called the *recurrence vector* of U.

Since in the case under discussion one has by (3.8) and the structure equations (2.8)

$$de_A = \Theta^B_A \otimes e_B - \omega^A \otimes \xi; \; A, B \in \{1, \ldots, 2m\} \tag{5.2}$$

we assume that $U \in D^\top$ is an horizontal vector field and has the principal vector field ξ as recurrence vector. Therefore by reference to (5.1) one has

$$\nabla U = -U^\flat \otimes \xi; \; dU^\flat = -U^\flat \wedge \eta. \tag{5.3}$$

Setting $U = \sum U^A e_A$, it follows at once from (5.3)

$$dU^A + U^B\omega^A_B = 0. \tag{5.4}$$

In addition by (3.6) and (3.15), one finds that (5.3) implies

$$\nabla\phi U = -(\phi U)^\flat \otimes \xi; \; d(\phi U)^\flat = -(\phi U)^\flat \wedge \eta \tag{5.5}$$

and one may say that the property of C.E.R. is invariant under the action of the operator ϕ. We notice that by (5.3) and (5.5) one may write in terms of d^{ω}-cohomology

$$d^{-\eta}U^{\flat} = 0;\ d^{-\eta}(\phi U)^{\flat} = 0 \tag{5.6}$$

Consequently, since η is an exact form, one may say that the 1-forms $U^{\flat}$ and $(\phi U)^{\flat}$ are $d^{-\eta}$-exact.

Since $g(U,\phi U) = 0$ one finds at once by (5.3) and (5.5)

$$[U,\phi U] = 0;\ \nabla_{\phi U}U + \nabla_U \phi U = 0 \tag{5.7}$$

which shows that U and ϕU define a *commutative anti-invariant pairing*. Next by (3.7), (5.3) and (5.4) one derives

$$[\xi_r, U] = -a_r U;\ [\xi_r, \phi U] = -a_r \phi U \tag{5.8}$$

and

$$\mathcal{L}_{\xi}U^{\flat} = U^{\flat};\ \mathcal{L}_{\xi}(\phi U)^{\flat} = (\phi U)^{\flat} \tag{5.9}$$

and the above relations show that the Reeb vector fields ξ_r and the principal vector field ξ define conformal transformations of the vector fields U and ϕU, and conformal transformations of their dual forms $U^{\flat}$ and $(\phi U)^{\flat}$ respectively.

Next, clearly by (5.4) one has

$$\| U \|^2 = \| \phi U \|^2 = 2l = const. \tag{5.10}$$

and one also derives from (5.3), (5.5) and (5.10)

$$div\, U = 0;\ div\,(\nabla_U U) = -4lm \tag{5.11}$$

From (5.3) and the above, and making use of the general formula (2.8) one deduces by a short calculation

$$\Re(U,U) = -4lm \Rightarrow Ric\,(U) = -2m \tag{5.12}$$

and by similar derives

$$\Re(\phi U,\phi U) = -4lm \Rightarrow Ric\,(\phi U) = -2m \tag{5.13}$$

Therefore one may say that $M(\phi,\Omega,\xi_r,\eta^r,a_r,g)$ is U and ϕU Ricci constant. It also should be noticed that the above formulas are involving the fact the manifold under consideration is not compact.

Operating now on ∇U and $\nabla \phi U$ by d^{∇^2}, one derives after some calculations

$$\left\{\begin{array}{l} d^{\nabla^2}U = \nabla^3 U = (U^\flat \wedge \eta) \wedge dp \in A^3(M,TM) \\ d^{\nabla^2}\phi U = \nabla^3(\phi U) = ((\phi U)^\flat \wedge \eta) \wedge dp \in A^3(M,TM) \end{array}\right\} \tag{5.14}$$

which reveals that U and ϕU are *2-exterior concurrent* with closed concurrence form, that is $\nabla^4 U$ and $\nabla^4 \phi U$ are 0-elements of $A^4(M,TM)$.

Further consider the vector valued 1-form

$$F = U \wedge \phi U \Leftrightarrow F = (\phi U)^\flat \otimes U - U^\flat \otimes \phi U \tag{5.15}$$

Since in general $d^\nabla \circ d^\nabla \neq 0$, operating 3-times on F by d^∇ one gets

$$d^{\nabla^3} F = 2\varphi \wedge dp + \eta \wedge d^{\nabla^2} F \tag{5.16}$$

where φ is the closed 3-form defined by

$$\varphi = \eta \wedge (\phi U)^\flat \wedge U^\flat . \tag{5.17}$$

Hence as a generalization of definition given in [PRL] we agree to say that the vector valued 1-form F defines a d^{∇^3}*-torse forming* vector valued form.

By reference to (4.1) one finds

$$\mathcal{L}_U \nabla\xi = -U^\flat \otimes \xi - \eta \otimes U;\ \mathcal{L}_{\phi U} \nabla\xi = -(\phi U)^\flat \otimes \xi - \eta \otimes \phi U \tag{5.18}$$

and since one has

$$d^{\nabla^2} F = \varphi \otimes \xi + (\phi U)^\flat \wedge U^\flat \otimes \nabla\xi \tag{5.19}$$

one finally derives

$$\mathcal{L}_U d^{\nabla^3} F = \varphi \wedge \mathcal{L}_U \nabla\xi = 0;\ \mathcal{L}_{\phi U} d^{\nabla^3} F = \varphi \wedge \mathcal{L}_{\phi U} \nabla\xi = 0$$

which shows that the vector valued 4-form $d^{\nabla^3}F$ is invariant by U and ϕU.

Theorem 5.1 *If the framed f-manifold* $M(\phi, \Omega, \xi_r, \eta^r, a_r, g)$ *defined in section 3 carries a C.E.R. vector field* U *having a recurrence vector* ξ, *i.e.,* $\nabla U = -U^\flat \otimes \xi$, *then this property is invariant under the action of* ϕ, *i.e.,* $\nabla(\phi U) = -(\phi U)^\flat \otimes \xi$, *and both 1-forms* $U^\flat$ *and* $(\phi U)^\flat$ *are* η*-exact.*

In addition, the following properties emerge:

(i) $\| U \|^2 = \| \phi U \|^2 = 2l (= const.)$ *and the Reeb vector fields* ξ_r *define infinitesimal transformations of both* U *and* ϕU;

(ii) $\Re(U,U) = \Re(\phi U, \phi U) = -4ml$ *and* U *and* ϕU *are 2- exterior concurrent vector fields, i.e.,* $\nabla^3 U = (U^\flat \wedge \eta) \wedge dp$; $\nabla^3 \phi U = ((\phi U)^\flat \wedge \eta) \wedge dp$ *and* $\nabla^4 U$, $\nabla^4 \phi U$ *are 0-elements of* $A^4(M,TM)$;

(iii) the vector valued 1-form $F = U \wedge \phi U$ *defines a closed* d^{∇^3}*-torse forming, that is* $d^{\nabla^3} F = 2\varphi \wedge dp + \eta \wedge d^{\nabla^2} F$, $d^{\nabla^4} F = 0$*; and* $d^{\nabla^3} F$ *is invariant by* U *and* ϕU

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