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Some remarks on analytical functions

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1. Introduction

In our study, we consider a function $g(s) = a(s) + i\ b(s)$, defined for $s \in [-\pi, \pi]$, where a and b are not conjugated. We find a complex function $f(z)$ defined and analytical on the unit disc, $|z| < 1$, except certain singular points, such that $f(e^{is}) = a(s) + i\,b(s)$, on the boundary of the disc.

Our following considerations are suggested by the well known Milne-Thompson's result, (see [1]):

Theorem 1. *Let $f(z)$ be an analytical function defined on the unit disc $|z| < 1$, except certain singular points. If the function f is continuous on $|z| \leq 1$, except certain singular points, and $Im\ f(z) = 0$, then we have:*

$$f(z) = f_p(z) + \overline{f_p}(\frac{1}{\bar{z}}) + k,$$

where $f_p(z)$ represents the principal part of the function f and k is constant.

Clearly, the function $f(z)$ from the above theorem, determines the function $Re\ f(e^{i\,s}) = a(s)$, which together with $Im\ f(z) = 0$ (by assumption) are defined for $s \in [-\pi, \pi]$.

In the following we intend to give the answer to next two problems:

(P_1). May a function $f(z)$ be constructed, as in theorem 1, such that $Re\ f(e^{i\,s}) = a(s)$ and $Im\ f(z) = 0$, for $s \in [-\pi, \pi]$?

(P_2). May a function $f(z)$ be constructed which is analytical on $|z| < 1$, continuous on $|z| \leq 1$, except certain singular points, such that $f(e^{is}) = a(s) + i\,b(s)$, where the functions a and b are prescribed ?

Of course, the answer at the first problem will be obtained by particularizing the result of the second problem.

2. Main results

We start the study of the mentioned problems (P_1) and (P_2), considering the following result, (see [3]):

Proposition 1. *Let us consider the function $g(s) = a(s)+ib(s)$, $s \in [-\pi, \pi]$ which is continuous and satisfies the condition $g(-\pi) = g(\pi)$. Then the function g is the limit of a function $f(z)$, holomorphic on $|z| < 1$, continuous on $|z| \leq 1$ and satisfying the condition $f(e^{is}) = a(s) + i\,b(s)$, if and only if we have*

$$b(s) = \frac{1}{2\pi i}\int_{-\pi}^{\pi} a(t)\ \cot\frac{s-t}{2}dt,$$
$$a(s) = -\frac{1}{2\pi\, i}\int_{-\pi}^{\pi} b(t)\ \cot\frac{s-t}{2}dt. \tag{1}$$

Without restricte the generality, the relations (1) are obtained assuming the conditions $\int_{-\pi}^{\pi} a(s)ds = \int_{-\pi}^{\pi} b(s)ds = 0$. Also, for the existence of the integrals in (1) assuming that a and b are Hölder functions. We say that the functions a and b are *inside conjugated.*

Using the Cauchy's formula

$$z \longrightarrow f(z) = \frac{1}{2\pi i}\oint_{|\zeta|=1} \frac{a(s) + i\,b(s)}{\zeta - z}d\zeta,\ \ \zeta = e^{i\,s}, \tag{2}$$

we obtain that the function $f(z)$ is holomorphic inside of the disc and on the boundary we have:

$$\lim_{z\to e^{is},\, z\ int} f(z) = f(e^{is}) = a(s) + i\,b(s).$$

For z outside of the disc, the value of the integral vanishes.
In other words, we have:

$$f(z)=\begin{cases}\frac{1}{2\pi i}\oint_{|\zeta|=1}\frac{a(s)+i\,b(s)}{\zeta-z}d\zeta, & for\ |z|<1,\\ a(s)+i\,b(s), & for\ z=e^{is},\ as\ a\ value\ from\ inside,\\ 0, & for\ |z|>1.\end{cases} \tag{3}$$

An analogous result, as in Proposition 1, but for the outside of the disc, it is obtained by considering the functions:

$$b(s)=-\frac{1}{2\pi i}\int_{-\pi}^{\pi}a(t)\ \cot\frac{s-t}{2}dt,$$
$$a(s)=\frac{1}{2\pi i}\int_{-\pi}^{\pi}b(t)\ \cot\frac{s-t}{2}dt.$$

In this case a and b are called *outside conjugated.*
With the aid of the above results, we tackle the problem P_2.

Let us consider a and b, a pair of Hölder functions, defined for $s\in[-\pi,\pi]$ and satisfying the closing conditions $a(-\pi)=a(\pi)$, $b(-\pi)=b(\pi)$. It is necessary to outline that the functions a and b are not conjugated.

Theorem 2. *We have the decomposition*

$$a(s)=a_1(s)+a_2(s),$$
$$b(s)=b_1(s)+b_2(s), \tag{4}$$

where $a_1(s)$, $a_2(s)$ *are inside conjugated and* $b_1(s)$, $b_2(s)$ *are outside conjugated.*

Proof. It is easy to see that

$$a_1(s)=\frac{1}{2}\left[a(s)-\frac{1}{2\pi i}\int_{-\pi}^{\pi}b(t)\ \cot\frac{s-t}{2}dt\right],$$
$$b_1(s)=\frac{1}{2}\left[b(s)+\frac{1}{2\pi i}\int_{-\pi}^{\pi}a(t)\ \cot\frac{s-t}{2}dt\right], \tag{5}$$

respectively,

$$a_2(s) = \frac{1}{2}\left[a(s) + \frac{1}{2\pi i}\int_{-\pi}^{\pi} b(t)\ \cot\frac{s-t}{2}dt\right],$$

$$b_2(s) = \frac{1}{2}\left[b(s) - \frac{1}{2\pi i}\int_{-\pi}^{\pi} a(t)\ \cot\frac{s-t}{2}dt\right]. \tag{6}$$

By means of (2), for the pair (a_1, b_1) it results that the function

$$f_1(z) = \frac{1}{2\pi i}\oint_{|\zeta|=1}\frac{a(s)+i\,b(s)}{\zeta - z}d\zeta,\ \ \zeta = e^{is}, \tag{7}$$

is holomorphic on $|z| < 1$, continuous on $|z| \leq 1$ and satisfies the equality:

$$f_1(e^{is}) = a_1(s) + i\,b_1(s).$$

Analagous, for the pair (a_2, b_2) we obtain the function $f_2(z)$ defined by:

$$f_2(z) = \frac{1}{2\pi i}\oint_{|\zeta|=1}\frac{a_2(s)+i\,b_2(s)}{\zeta - z}d\zeta,\ \ \zeta = e^{is}, \tag{8}$$

which is holomorphic outside of the disc and satisfies the condition:

$$f_2(e^{is}) = a_2(s) + i\,b_2(s).$$

Remark. In view of (3), we deduce that the function $f_1(z)$ from (7) is defined inside of the disc and outside of the disc and, also, on the boundary. More exactly, we have:

$$f_1(z) = \begin{cases} \frac{1}{2\pi i}\oint_{|\zeta|=1}\frac{a_1(s)+i\,b_1(s)}{\zeta-z}d\zeta, & for\ |z|<1, \\ a(s)+i\,b(s), & for\ z=e^{is},\ as\ a\ value\ from\ inside, \\ 0, & for\ |z|>1. \end{cases}$$

Analogous, for the function $f_2(z)$, defined in (8), we have:

$$f_2(z) = \begin{cases} \frac{1}{2\pi i}\oint_{|\zeta|=1}\frac{a_2(s)+i\,b_2(s)}{\zeta-z}d\zeta, & for\ |z|<1 \\ a(s)+i\,b(s), & for\ z=e^{is}\ as\ a\ value\ from\ inside, \\ 0, & for\ |z|>1, \end{cases}$$

such that we obtain that the function

$$F(z) = f_1(z) + f_2(z),$$

will be analytical both, inside and outside of the disc. Moreover, the function $F(z)$ is defined also on the boundary of the disc.

It is easy to see that our function $F(z)$ coincide with the function $F(z)$ obtained by using the Cauchy's formula:

$$F(z) = \frac{1}{2\pi i}\oint_{|\zeta|=1} \frac{a(s) + i\, b(s)}{\zeta - z} d\zeta.$$

Moreover, the boundary values of the function $F(z)$ (which appeared in the well known Sokhotskii-Plemelj's relations), denoted by $F_i(e^{is})$ and $F_e(e^{is})$, respectively, can be expressed by means of the pairs (a_1, b_1) and (a_2, b_2), defined in (5) and (6).
More exactly, we have:

$$F_i(e^{is}) = a_1(s) + i\, b_1(s),$$

$$F_e(e^{is}) = a_2(s) + i\, b_2(s).$$

These considerations show that when a function $g(s) = a(s) + i\ b(s)$ is given and the integral of the Cauchy type (1) is defined, we can estimate directly the boundary values $F_i(e^{is})$ and $F_e(e^{is})$, (which appeared in the Sokhotskii-Plemelj's relations), i.e., without calculation of the Cauchy's integral in the sense of main value.

Let us denote by $f_2^p(z)$ the inside analytical extension of the function $f_2(z)$, for z outside of the disc. The extension $f_2^p(z)$ will have, compulsory, certain singular points.

To the contrary, the restriction of the function $f_2(z)$ to the outside of the disc together with its inside extension should determine an integer function.

By means of all above considerations, we deduce that the function $f(z) = f_1(z) + f_2^p(z)$ gives the answer to the problem (P_2).

In the particular case when $a = a(s)$ and $b = b(s) = 0$, we obtain the answer to the problem (P_1). Moreover, in this case the relations (5) and (6) become:

$$a_1(s) = \frac{1}{2}a(s),\ b_1(s) = \frac{1}{2}\frac{1}{2\pi}\int_{-\pi}^{\pi} a(t)\ \cot\frac{s-t}{2}dt,$$
$$a_2(s) = \frac{1}{2}a(s),\ b_2(s) = -\frac{1}{2}\frac{1}{2\pi}\int_{-\pi}^{\pi} a(t)\ \cot\frac{s-t}{2}dt.$$

3. Examples

(1°). Let us determine the function $f(z)$, analytical on $|z| < 1$, except certain singular points, which on the unit circle becomes $f(e^{is}) = (1+k^2)\cos s + i\ (1-k^2)\sin s$. It is easy to see that the functions a and b are not conjugated.
By using the relations (5) and (6), we obtain:

$$a_1 = \frac{1}{2}\left[(1+k^2)\cos s + (1-k^2)\cos s\right] = \cos s,$$
$$b_1 = \frac{1}{2}\left[(1-k^2)\sin s + (1+k^2)\sin s\right] = \sin s,$$

respectively,

$$a_2(s) = k^2\cos s,$$
$$b_2(s) = -k^2\sin s.$$

Of course, the pair (a_1, b_1) determines the function $f_1(z) = z$ and the pair (a_2, b_2) determines the function $f_2(z) = k^2/z$.
In this way we obtain that the function $f(z)$ becomes:

$$f(z) = f_1(z) + f_2^p(z) = z + k^2/z,$$

i.e., the well known Jukovschi's function.

(2°). Let us determine the function $f(z)$, which is analytical on $|z| < 1$, except certain singular points, such that on the boundary $f(z)$ becomes

$f(e^{is}) = a(s) + i\,b(s)$, where $a(s) = \cos s + 2\sin s$, $b(s) = 0$. With the aid of the relations (9), we obtain:

$$a_1(s) = \frac{\cos s + 2\sin s}{2},$$

$$b_1(s) = \frac{1}{2}\frac{1}{2\pi}\int_{-\pi}^{\pi}(\cos s + 2\sin 3s)ds = \frac{1}{2}(\sin s - 2\cos 3s),$$

respectively,

$$a_2(s) = \frac{\cos s + 2\sin 3s}{2},$$

$$b_2(s) = \frac{\sin s - 2\cos 3s}{2}.$$

After some simple calculations, it results:

$$f_1(z) = \frac{1}{2}\left(z - 2iz^3\right),$$

$$f_2(z) = \frac{1}{2}\left(\frac{1}{z} + i\frac{2}{z^3}\right).$$

Because to the extension of $f_2(z)$ from outside of the disc to inside of the disc we have the same expression, it results:

$$f(z) = f_1(z) + f_2^p(z) = \frac{1}{2}\left(z - 2iz^3\right) +$$

$$+\frac{1}{2}\left(\frac{1}{z} + i\frac{2}{z^3}\right) = \frac{1}{2}\left(z + \frac{1}{z}\right) + i\left(\frac{1}{z^3} - z^3\right),$$

which satisfies the desired conditions from the beginning of this example.

References

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