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COUNTEREXAMPLES IN MAXIMAL DYNAMIC FLOWS

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Abstract. FORD and FULKERSON have shown that a stationary maximal dynamic flow can be obtained by solving a transhipment problem associated with the static network and thereby finding the maximal temporally repeated dynamic flow. This flow is known to be an optimal dynamic flow. Howerer, this result cannot be extended to other problems of dynamic flows. This paper presents four counterexamples for certain problems of dynamic flows.

1. STATIONARY MAXIMAL DYNAMIC FLOWS

Let $G = (N, A)$ be a connected digraph with $N = \{s, \ldots, i, \ldots, s'\}, |N| = n$ the node set and $A = \{(i,j)|i,j \in N\}, |A| = m \leq n \cdot n$ the arc set. Let $\mathcal{N}$ be the set of natural numbers and $P = \{0, 1, \ldots, p\}$ the set of periods. Let us state the time function $h : A \to \mathcal{N}$ and the capacity function $c : A \to \mathcal{N}$, where $h(i,j), c(i,j)$ represent the arc transit time and the arc capacity, $(i,j) \in A$. Let us designate by s the source and by s' the sink of the network $G = (N, A, h, c)$.

The stationary maximal dynamic flows problem for p time periods may be formulated as follows. Let us determine the function $f : A \times P \to \mathcal{N}$, which should satisfy

$$\sum_{P}\left(\sum_{j} f(i,j;t) - \sum_{j} f(j,i;\bar{t})\right) = v(p), \qquad i = s, \qquad (1)$$

$$\sum_j f(i,j;t) - \sum_j f(j,i;\bar{t})) = 0, \qquad i \neq s, s'; \quad t \in P, \tag{2}$$

$$\sum_P (\sum_j f(i,j;t) - \sum_j f(j,i;\bar{t})) = -v(p), \qquad i = s', \tag{3}$$

$$0 \leq f(i,j;t) \leq c(i,j), \qquad (i,j) \in A, \quad t \in P, \tag{4}$$

$$\max v(p), \tag{5}$$

where $\bar{t} = t - h(j,i)$.

A dynamic flow problem is said to be stationary if the network parameters such as capacities, arc traversal times, and so on, are constant over time, i.e., $h : A \to \mathcal{N}, c : A \to \mathcal{N}$ and so on.

The fact that a dynamic flow for the p time periods in the network $G = (N, A, h, c)$ is equivalent to a static flow within an extended network $G(p) = (N(p), A(p), c)$ was pointed out in the reference [8], or an extended network $G^*(p) = (N^*(p), A^*(p), c^*)$ in the references [2], [3].

The network $G(p)$ may be constructed from network G as follows:

$$\begin{aligned} N(p) &= \{i(t) | i \in N, \quad t \in P\} \\ A(p) &= \{(i(t), j(t^*)) | (i,j) \in A; t, t^* \in P; t^* = t + h(i,j)\} \\ c(i(t), j(t^*)) &= c(i,j), \qquad (i,j) \in A; \quad t, t^* \in P \end{aligned}$$

The network $G^*(p)$ may be constructed as follows. Let $d(s,i)$ be the length of a shortest route from the source s to the node i, regarding to $h(i,j)$ and $d(i,s')$ be the length of a shortest route from the node i to the sink s', regarding to $h(i,j)$. Computation of $d(s,i)$ and $d(i,s')$ for all $i \in N$ is performed by means of the usual shortest path algorithms and is not discussed in this paper. Let

$$\begin{aligned} P(i) &= \{t | t \in P, d(s,i) \leq t \leq p - d(i,s')\}, \quad i \in \mathcal{N} \\ P(i,j) &= \{t | t \in P, d(s,i) \leq t \leq p - (h(i,j) + d(j,s'))\}, (i,j) \in A. \end{aligned}$$

The network $G'(p) = (N'(p), A'(p), h', c')$ may be constructed as follows:

$$\begin{aligned} N'(p) &= \{i | i \in N, P(i) \neq \emptyset\} \\ A'(p) &= \{(i,j) | (i,j) \in A, P(i,j) \neq \emptyset\} \end{aligned}$$

and h', c' are the restrictions of h and c, respectively, to $A'(p)$.

The network $G^*(p) = (N^*(p), A^*(p), c^*)$ is constructed from the network $G'(p)$ in the following way

$$\begin{aligned} N^*(p) &= \{i(t) | i \in N'(p), t \in P(i)\} \\ A^*(p) &= \{(i(t), j(t^*)) | (i,j) \in A'(p), t \in P'(i,j), \\ &\qquad t^* = t + h(i,j)\} \\ c^*(i(t), j(t^*)) &= c(i,j), \qquad (i,j) \in A'(p); t \in P(i,j). \end{aligned}$$

In the stationary case one does not require the construction of the time-expanded network $G(p)$ or $G^*(p)$ for solving this problem for any p. An efficient method of solving the stationary maximal dynamic flow problem is developed in the references [8], [3], [7]. The maximal flow in $G(p)$ or $G^*(p)$ can be obtained by solving a transhipment problem associated with the static network G or $G'(p)$ for any given p. A dynamic flow can then be generated from a chain decomposition of f in G or $f'(p)$ in $G'(p)$ by starting each chain flow at time zero and repeating each so long as there is enough time left in the p periods for the flow along the chain to arrive at the sink. The flow constructed above is referred to as a maximal temporally repeated flow. However temporally repeated flow may not be optimal for other problem of dynamic flows.

2. STATIONARY UNIVERSAL MAXIMAL DYNAMIC FLOWS

Gale [9] introduced the term universal maximal dynamic flow, i. e., a single dynamic flow that provides a solution to the maximal problem, not only for the allotted time periods, but also for any smaller number of time periods. Gale [9] answered the question as to the existence of such a flow, not only for the stationary case, but also for the case in which the capacities may vary with time.

In seeking the universal dynamic solution, we alter the problem by changing the objective function (5) as follows:

$$\max v(t) \qquad t = 0, 1, \ldots, p \tag{6}$$

In the example given in the paper, we illustrate the case where no temporally repeated flow solution is an universal maximum flow.

Consider the network $G = (N, A, h, c)$ of Fig. 1, where node 1 is the source, 4 is the sink, and 2, 3 are the intermediate nodes. The arc transit times are given as the first members and the capacities of the arc are given as the second members of the pairs of numbers written adjacent to the arcs of Fig. 1.

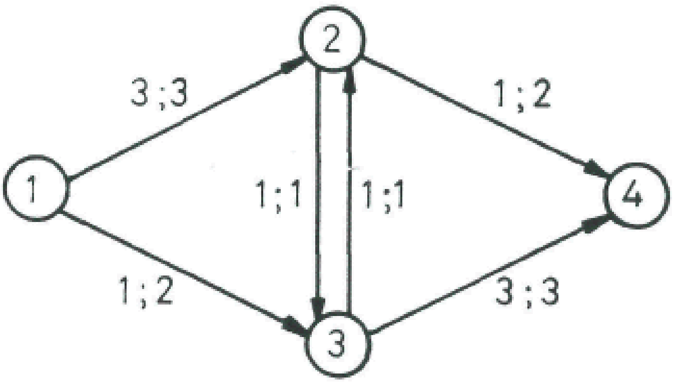

Fig. 1.

Let $p = 5$. The maximal temporally repeated flow is shown in network $G^*(5)$ of Fig. 2.

The universal maximum flow is shown in network $G^*(5)$ of Fig. 3.

Clearly, no temporally repeated flow solution is a stationary universal maximum flow.

3. MAXIMAL DYNAMIC FLOWS WITH NONSTATIONARY CAPACITY FUNCTION

In seeking the maximal dynamic flows with nonstationary capacity function, we alter the problem by changing the conditions (4) as follows

$$0 \leq f(i, j; t) \leq c(i, j; t), \qquad (i, j) \in A, t \in P \tag{7}$$

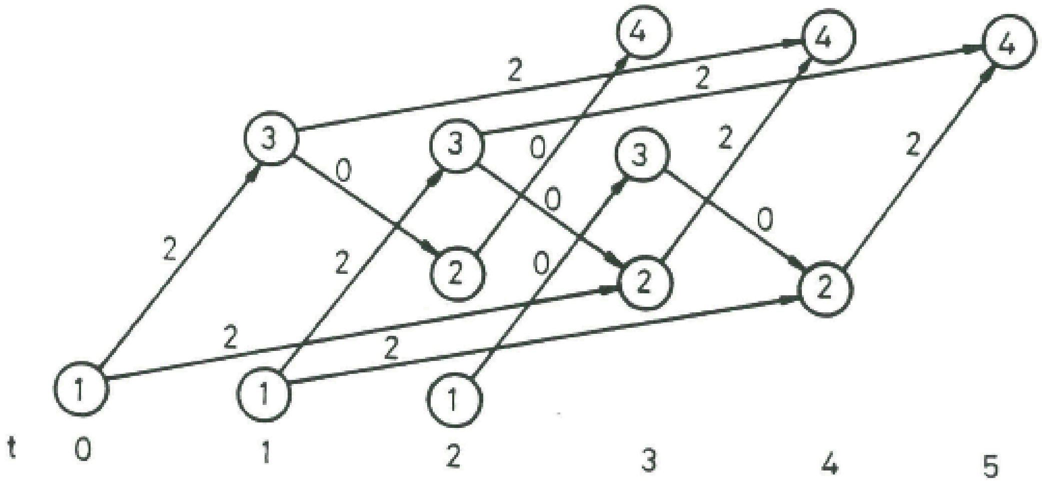
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Fig. 2.

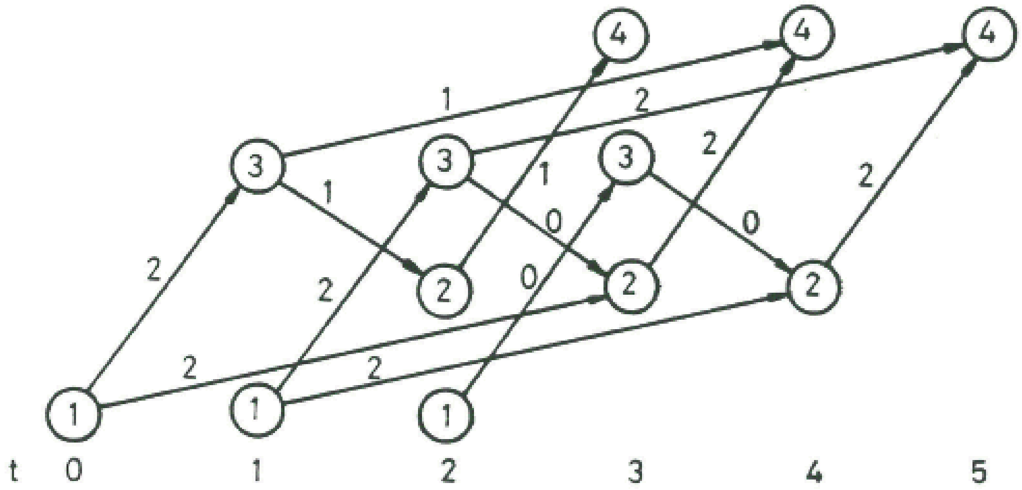
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Fig. 3.

In the example given on, we ilustrate that temporally repeated flows may not be optimal for a maximal dynamic flow with nonstationary capacity function.

Let us consider the network of Fig. 1, where the arc transit times are given as the first members of pairs of numbers written adjacent to the arcs of the network. The capacities for $p = 6$ are shown in Table 1. A maximal dynamic flow is illustrated by the network $G^*(6)$ of Fig. 4. Clearly in this case, no temporally repeated flow solution is a maximal dynamic flow with nonstationary capacity function.

c	t					
	0	1	2	3	4	5
c(1,2;t)	2	2	3	4	4	4
c(1,3;t)	1	2	3	4	4	3
c(2,3;t)	3	1	2	2	1	3
c(2,4;t)	3	2	3	2	3	4
c(3,2;t)	4	4	3	2	2	3
c(3,4;t)	2	2	1	4	4	4

Table 1: The network capacities

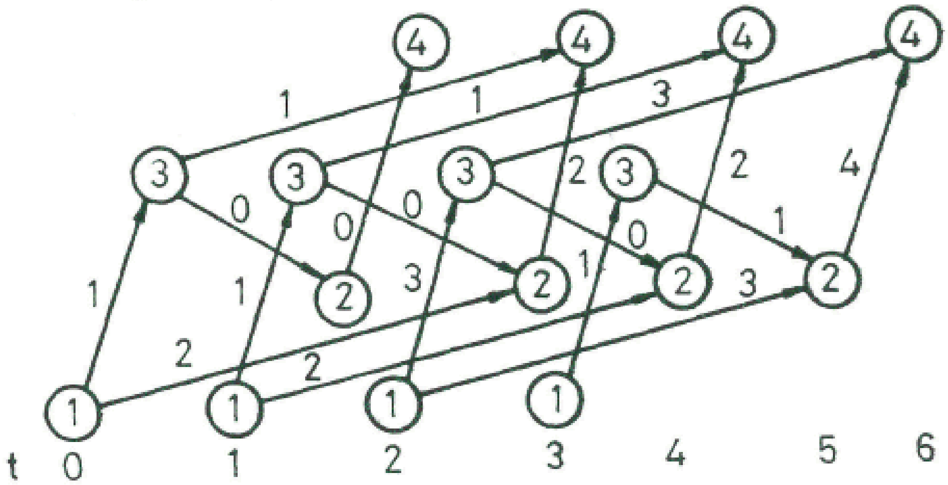

Fig. 4

4. MAXIMAL DYNAMIC FLOWS OF MINIMAL COST

Let $b : A \rightarrow \mathcal{N}$ be the cost function, where $b(i,j)$ is the transport unit cost for the arc $(i,j) \in A$. The maximal dynamic flow problem of minimal cost consists in determining the function f which satisfies the conditions (1)-(5) and

$$\min \sum_{P} \sum_{A} b(i,j) f(i,j;t) \tag{8}$$

We'll show through an example that temporally repeated flows may be not optimal for a maximal dynamic flow of minimal cost.

Consider the network $G = (N, A, h, c)$ of Fig. 1. The transport unit costs of the arcs are given by

$$b(1,2) = 2, \quad b(1,3) = 1, \quad b(2,3) = 4$$
$$b(2,4) = 1, \quad b(3,2) = 4, \quad b(3,4) = 6$$

A maximal dynamic flow of minimal cost for $p = 4$ is illustrated by the network $G^{*}(4)$ of Fig. 5.

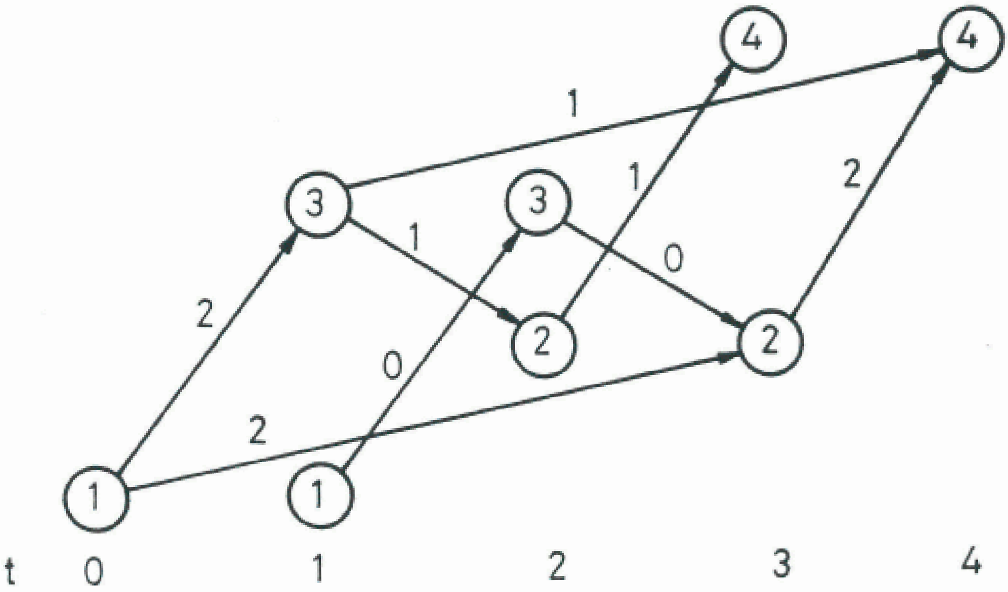

Fig.5

Clearly, no temporally repeated flow solution is a nonstationary ma-ximal dynamic flow of minimal cost.

5. MULTICOMMODITY MAXIMAL DYNAMIC FLOW

For single commodity flow problems, the node-arc formulation leads to algorithms which are much more efficient, and this model is therefore used commonly. For multicommodity flow problems the arc-chain formulation leads to a resonable sulution approach. To model the multicommodity dynamic flow problem we need other notions. For the arc-chain formulation multicommodity dynamic flow problem see reference [1]. In this reference the next counterexample is presented.

Consider the three-commodity network shown in Fig. 6.

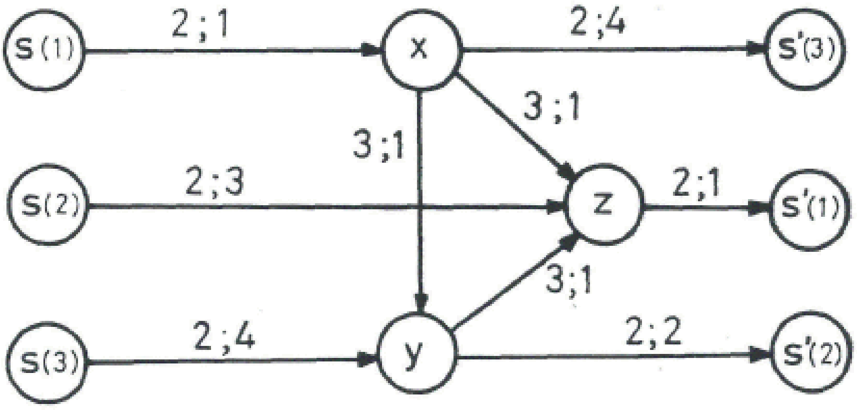

Fig. 6

The first number in the pairs is the capacity of the arc and the second number is the arc traversal time. For $p = 20$, the maximal temporally repeated flow is 63 units, whereas the maximal dynamic flow is 65 units.

6. REMARKS

1. In all the flow problems discussed in the paper, we ignored any consideration of holdover flow at the node i from time point t to $t+1$. In this case we have:

$$h : A \cup N \to \mathcal{N}, \qquad c : A \cup N \to \mathcal{N},$$
$$b : A \cup N \to \mathcal{N}, \qquad f : (A \cup N) \times P \to \mathcal{N},$$

where

$h(i,i) = 1$	represents the fact that flow held over at node i from time point t to $t+1$;
$c(i,i)$	represents the maximum amount of flow that can be held over at node i;
$b(i,i)$	represents the holdover unit cost at node i;
f(i,i;t)	represents the amount of flow held over node i from time point t to $t+1$.

This flow model is called holdover dynamic flow. In this case, the dynamic networks $\overline{G}(p) = (\overline{N}(p), \overline{A}(p), \overline{c})$ and $\overline{G}^*(p) = (\overline{N}^*(p), \overline{A}^*(p), \overline{c}^*)$ may be constructed as follows:

$$\begin{aligned}
\overline{N}(p) &= N(p) \\
\overline{A}(p) &= A(p) \cup \{(i(t), i(t+1)) | t \in P, t \neq p, i \in N\} \\
\overline{c}(i(t), j(t^*)) &= c(i(t), j(t^*)), \quad (i(t), j(t^*)) \in A(p) \\
\overline{c}(i(t), i(t+1)) &= c(i,i), \qquad i \in N,
\end{aligned}$$

respectively

$$\begin{aligned}
\overline{N}^*(p) &= N^*(p) \\
\overline{A}^*(p) &= A^*(p) \cup \{(i(t), i(t+1)) | t \in P(i), i \in N'(p)\} \\
\overline{c}^*(i(t), j(t^*)) &= c(i(t), j(t^*)), \quad (i(t), j(t^*)) \in A^*(p) \\
\overline{c}^*(i(t), i(t+1)) &= c(i,i), \qquad i \in N'(p).
\end{aligned}$$

Since holders at nodes are not required in the temporally repeated flow solution, we need not include arcs $(i(t), i(t+1))$.

In the case $c : A \times P \to \mathcal{N}$, in reference [4] is proved that

$$v^*(p) \leq \overline{v}^*(p),$$

where

$v^*(p)$ is the value of maximal flow in network $G^*(p)$
$\overline{v}^*(p)$ is the value of maximal flow in network $\overline{G}^*(p)$

Clearly, no temporally repeated flow solution is a holdover nonstationary maximal dynamic flow.

2. Also, in all the flow problems discussed in this paper, we ignored the case of dynamic flows with nonstationary time function. In this case the dynamic flow model is complicated and may be presented in other papers.

3. The dynamic flows may be classified in two classes:

- stationary;
- nonstationary.

In each class may be considerred the problems of:

- maximal flow;
- universal maximal flow;
- maximal flow of minimal cost;
- etc.

For the stationary maximum value dynamic flow problem, Ford and Fulkerson [8] introduced a remarkably simple and effective algorithm which determine a maximum temporally repeated flow. However temporally repeated flow may not be optimal for other problems of dynamic flows. For this problems, the maximal dynamic flow can be obtained from the network $G^*(p)$. The network $G^*(p)$ is a partial subnetwork of $G(p)$. But, unfortunately, if p is large, even a relatively modest network G expanded through p periods can lead to an enormous time-expanded network $G^*(p)$.

The literature on algorithms for the maximum value dynamic flow problem is less vast than for the maximum value static flow problem. Unfortunately, efficient techniques such as the one mentioned for the stationary dynamic maximum value flow problem are not known at the moment for other dynamic network flow problems.

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