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On Manifolds Endowed with a $\mathcal{T}$-Killing Almost Contact 2-Structure

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Abstract

A $(2m+2)$-dimensional Riemannian manifold $M(\xi_r, \eta^r, g)$ carrying two Reeb vector fields ξ_r which enjoy the property to be skew symmetric Killing vector fields and such that the class of their associated covectors η^r be not maximal is considered.

In the last decade, contact, almost contact or paracontact manifolds carrying $r(r > 2)$ Reeb vector fields ξ_r have been studied by a certain number of authors (see also K. Yano and M. Kon [YK2]).

On the other hand *skew symmetric Killing* (abr. *S.S.K.*) vector fields have been defined in [R1], [R2].

In the present paper we consider a $(2m+2)$- dimensional Riemannian manifold $M(\xi_r, \eta^r, g)$ carrying 2 Reeb vector fields ξ_r $(r = 2m+1, 2m+2)$ which enjoy the property to be skew symmetric Killing vector fields and such that the class of their associated covectors η^r be not maximal.

If ∇ (resp. $\wedge$) denotes the covariant differential operator defined by g (resp. the wedge product of vectors), such vector fields satisfy:

$$(0.1) \qquad \nabla\xi_{2m+1} = \xi_{2m+2} \wedge \mathcal{T} \quad , \quad \nabla\xi_{2m+2} = -\xi_{2m+1} \wedge \mathcal{T} \quad ,$$

where $\mathcal{T}$ is a *horizontal* vector field (i.e. $\eta^r(\mathcal{T}) = 0$) called the *generator* vector of the pairing $\{\xi_r\}$. It is proved that η^r are *exterior recurrent* forms [D] and are of *class* 3.

Accordingly we agree to say that $\{\xi_r\}$ defines a *$\mathcal{T}$-Killing almost contact 2-structure.* One may decompose the tangent space T_pM to M as $T_pM = D_p^{\top} \oplus D_p^{\perp}$, where $D_p^{\top}$ (resp. $D_p^{\perp}$) denotes the *horizontal* (resp. *vertical*) distribution on M.

In Section 2, it is proved that any such a manifold $M(\xi_r, \eta^r, g)$ is foliated by *totally geodesic* $2m$-dimensional submanifolds $M^{\top}$ tangent to $D^{\top}$. In addition, it is shown that $\{\xi_r\}$ defines a *left invariant* pairing of vector fields and that the simple unit form $\varphi = \eta^{2m+1} \wedge \eta^{2m+2}$ enjoys the property to be closed and conformal to the vertical curvature 2-form Θ_{2m+1}^{2m+2} of M.

In Section 3, we assume that the manifold $M(\xi_r, \eta^r, g)$ under discussion is equipped with a *f-structure* [YK2] (see also [M]) defined by the (1.1) tensor ϕ of square -1 and by a 2-form Ω tangent to $D^{\top}$ (the fundamental 2-form) and finally that $\mathcal{T}$ is a *ϕ-soldering S.S.K.* vector field, which is defined.

In this case $M^{\top}$ moves to a Kählerian submanifold of M and the dual form $\bar{\omega} = \flat(\phi\mathcal{T})$ of $\phi\mathcal{T}$ is closed. The following properties are also pointed out.

i) $\mathcal{T}$ defines an *infinitesimal automorphism* of Ω, i.e. $\mathcal{L}_{\mathcal{T}}\Omega = 0$;

ii) the relation $\mathcal{L}_{\phi\mathcal{T}}\Omega + 2\Theta_{2m+1}^{2m+2} = 0$ holds good;

iii) $\mathcal{T}$ and $\phi\mathcal{T}$ commute and $\|\mathcal{T}\|^2$ is an *isoparametric* function [W];

iv) with the *transversal curvature* 2-forms of M are associated two bimonomial 2-forms ψ_r which are *cohomologically* closed;

v) if $\mathcal{F} = \xi_{2m+1}\wedge\xi_{2m+2}$ and $Z^{\perp}$ denotes the vertical component of a vector field $Z \in \Gamma TM$, then the curvature tensor field R of M satisfies

$$R(Y,Z)X + \phi R(Y,Z)\phi X = g(X,\mathcal{T})(g(Z,\mathcal{T})Y^{\perp} - g(Y,\mathcal{T})Z^{\perp})$$

$$+f\{g(Z,\phi X)\mathcal{F}Y - g(Y,\phi X)\mathcal{F}Z\} + \varphi(Y,Z)\mathcal{F}X + \varphi(X,Y)\phi Z - \varphi(X,Z)\phi Y\,,$$

where $f \in C^{\infty}M$ and $X, Y, Z \in \Gamma TM$.

1 Preliminaries.

Let (M,g) be a (pseudo) Riemannian manifold and let ∇ be the covariant differential operator defined by the metric tensor g. Let $\Gamma TM = \mathcal{X}M$ and $\flat : TM \to T^*M$ be the set of sections of the tangent bundle TM and the *musical isomorphism* defined by g, respectively.

Following W.A. Poor [P], we set

$$A^q(M,TM) = \Gamma Hom(\wedge^q TM, TM)$$

and notice that the elements of $A^q(M,TM)$ are vector valued 1-forms. The vector valued 1-form $dp \in A^1(M,TM)$, where $p \in M$, is called the *soldering form* of M [Di] (dp is the canonical vector valued 1-form of M).

Next the operator

$$d^{\nabla} : A^q(M,TM) \to A^{q+1}(M,TM) \tag{1.1}$$

denotes the *exterior covariant derivative* with respect to ∇ (see also [P]). Notice that generally $d^{\nabla^2} = d^{\nabla} \circ d^{\nabla} \neq 0$, unlike $d \circ d = 0$.

A function $f : \mathbf{R}^n \to \mathbf{R}$ is *isoparametric* [W] if $\|\nabla f\|^2$ and div (∇f) (where $\nabla f =$ grad f) can be expressed as functions of f.

The operator $d^{\omega} = d + e(\omega)$, acting on $\wedge M$, where $e(\omega)$ means the exterior product by the closed 1-form ω, is called the *cohomology operator* [GL]. One has $d^{\omega} \circ d^{\omega} = 0$. Any form $u \in \wedge M$ such that $d^{\omega}u = 0$ is said to be d^{ω}*-closed*. In particular, if the cohomology form ω is exact, then u is said to be a d^{ω}*-exact* form.

Let $\mathcal{O} = vect\,\{e_A | \, A = 1, ..., n\}$ be a local field of orthonormal vectorial frames over M and let $\mathcal{O}^* = covect\,\{\omega^A\}$ be the associated coframe. Then E. Cartan's structure equations, written in indexless form are

$$\nabla e = \theta \otimes e \quad , \tag{1.2}$$

(1.3) $$d\omega = -\theta \wedge \omega \quad ,$$

(1.4) $$d\theta = -\theta \wedge \theta + \Theta \, .$$

In the above equations θ (resp. Θ) are the *local connection forms* in the tangent bundle TM (resp. the *curvature 2-forms* on M.

Proposition. Let α be a Pfaff form on a manifold M. Then the necessary and sufficient conditions in order that α be of *class* $2s+1$ on M are

(1.5) $$\alpha \wedge (d\alpha)^s \neq 0 \quad , \quad (d\alpha)^{s+1} = 0$$

(see also [G]).

2 Killing almost contact 2-structures.

Let $M(\xi_r, \eta^r, g)$ be a $(2m+2)$-dimensional oriented Riemannian manifold carrying 2 Reeb vector fields ξ_r ($r = 2m+1, 2m+2$), that is, if η^r are their associated dual forms, one has

(2.1) $$\eta^r(\xi_s) = \delta_{rs} \quad , \quad i_{\xi_r} d\eta^r = 0 \quad ; \, r, s \in \{2m+1, 2m+2\} \, .$$

One may decompose in a unique fashion the tangent space T_pM to M as

(2.2) $$T_pM = D_p^{\top} \oplus D_p^{\perp} \quad ,$$

where:

a) $D_p^{\perp}$ is the 2-distribution defined by $\{\xi_r\}$ which is called the *vertical* distribution on M;

b) $D_p^{\top}$ is the orthogonal complementary distribution to $D_p^{\perp}$ and is called the *horizontal* distribution on M ($D_p^{\top} = \{Z \in \Gamma TM \, ; \, \eta^r(Z) = 0\}$).

We assume in this paper that ξ_r are *skew symmetric Killing* (abr. *S.S.K.*) vector fields (R. Rosca [R2]) and satisfy

(2.3) $$\begin{cases} \nabla \xi_{2m+1} &= \xi_{2m+2} \wedge \mathcal{T} \\ \nabla \xi_{2m+2} &= -\xi_{2m+1} \wedge \mathcal{T} \end{cases} .$$

In the above equations $\wedge$ means the *wedge product* ($\wedge$ is skew symmetric with respect to g) and $\mathcal{T} \in D_p^{\top}$ is an horizontal vector field, called the *generator* of the pairing $\{\xi_r\}$.

Denoting by $\omega = \mathcal{T}^{\flat}$ the dual form of $\mathcal{T}$ ($Z \mapsto Z^{\flat} : TM \to T^*M$ means the *musical isomorphism* defined by g) we may express the equations (2.3) as

(2.4) $$\begin{cases} \nabla \xi_{2m+1} &= \omega \otimes \xi_{2m+2} - \eta^{2m+2} \otimes \mathcal{T} \\ \nabla \xi_{2m+2} &= -\omega \otimes \xi_{2m+1} + \eta^{2m+1} \otimes \mathcal{T} \end{cases} .$$

Let θ_r^A ($A = 1, \ldots, 2m$) be the *transversal connection* forms; then since ω is given by

(2.5) $$\omega = t_A \omega^A \quad , \quad t_A \in C^{\infty} M \quad ,$$

one gets from (2.4)

$$(2.6) \qquad \begin{cases} \theta^{A}_{2m+1} &= -t_A\eta^{2m+2} \\ \theta^{A}_{2m+2} &= t_A\eta^{2m+1} \end{cases} .$$

Next by the structure equations (1.3) one derives:

$$(2.7) \qquad \begin{cases} d\eta^{2m+1} &= 2\omega\wedge\eta^{2m+2} \\ d\eta^{2m+2} &= -2\omega\wedge\eta^{2m+1} \end{cases} ,$$

which shows that the Reeb covectors are *exterior recurrent* (D.K. Datta [D]) having $\pm 2\omega$ as recurrence forms.

On behalf of (2.7) one has $\eta^r\wedge d\eta^r \neq 0$ and $(d\eta^r)^2 = 0$, which proves (see (1.5)) that the Reeb covectors are of *class* 3. Consequently, this class being not maximal, we agree to say that the pairing $\{\xi_r\}$ defines a $\mathcal{T}-$ *Killing almost contact 2-structure* (abr. $\mathcal{T}.K.A.C.$ 2-structure) on M.

By (2.4) it is easily seen that one has

$$\nabla_{\xi_{2m+2}}\xi_{2m+1} + \nabla_{\xi_{2m+1}}\xi_{2m+2} = 0 \quad ,$$

which reveals the fact that $\{\xi_r\}$ define a *left invariant pairing.*

On the other hand exterior differentiation of (2.7) gives by a simple argument

$$(2.8) \qquad d\omega = \tau\eta^{2m+1}\wedge\eta^{2m+2} \quad , \quad \tau\in C^\infty M \quad .$$

Since in the above

$$(2.9) \qquad \varphi = \eta^{2m+1}\wedge\eta^{2m+2}$$

represents the simple unit form of the vertical distribution $D_p^\perp$, we agree to call φ (by abusing language) the *Killing 2-form* on M.

By (2.8) one has $D^\top \subset \ker\varphi\cap\ker(d\varphi)$, which shows that φ is an *integral invariant* of $D^\top$ and consequently $D^\top$ is involutive; we denote by $M^\top$ the leaf of $D^\top$.

Also, by the structure equations (1.2) we have $\theta^{2m+2}_{2m+1} = \omega$. Hence by (1.4) and (2.8) it follows that

$$(2.9) \qquad \Theta^{2m+2}_{2m+1} = (\tau + \|\mathcal{T}\|^2)\varphi \quad ,$$

where $\|\mathcal{T}\|^2 = g(\mathcal{T},\mathcal{T})$.

As a consequence, it is seen that the vertical curvature 2-form Θ^{2m+2}_{2m+1} is conformal to the Killing 2-form φ. Then setting $2t = \|\mathcal{T}\|^2$ and $s = \tau + 2t$ we may write

$$(2.10) \qquad d^{-d\lg s}\Theta^{2m+2}_{2m+1} = 0 \quad .$$

This is, in terms of d^ω-cohomology, Θ^{2m+2}_{2m+1} is a $d^{-d\lg s}$-exact form.

Finally if we consider the immersion $x : M^{\top} \to M$, then since ξ_r are normal sections associated with x, the second fundamental forms h_r associated to x are expressed by

$$h_r = - < dp^{\top}, \nabla\xi_r > \quad . \tag{2.11}$$

Since in the above, $dp^{\top} = dp|_{M^{\top}}$ means the restriction of dp on $M^{\top}$, it follows at once from (2.4) that $h_r = 0$, which proves that $M^{\top}$ is a *totally geodesic* submanifold of M.

Summarizing, we proved the following.

Theorem 1. Let $M(\xi_r, \eta^r, g)$ be a $(2m+2)$-dimensional manifold endowed with a $\mathcal{T}$-Killing almost contact 2-structure and let $D^{\top}$ the $2m$-dimensional horizontal distribution associated with this structure.

Then any such a manifold is foliated by $2m$-submanifolds $M^{\top}$ tangent to $D^{\top}$ and the immersion $x : M^{\top} \to M$ is totally geodesic. In addition:

i) the two skew symmetric Killing vector fields ξ_r on M $(r = 2m+1, 2m+2)$ define a left invariant pairing and the Killing 2-form

$$\varphi = \eta^{2m+1} \wedge \eta^{2m+2}$$

is closed;

ii) the vertical curvature 2-form Θ^{2m+2}_{2m+1} on M is conformal to φ.

3 f-structures on $M(\xi_r, \eta^r, g)$

We assume in this Section that the manifold $M(\xi_r, \eta^r, g)$ under discussion is equipped with a f-structure [YK2] (see also [M]) defined by ϕ. As is known one has the following equations

$$\phi^2 = -Id + \eta^r \otimes \xi_r \quad , \quad \phi\xi_r = 0 \quad , \quad \eta^r \circ \phi = 0 \quad ,$$

$$g(X,Y) = g(\phi X, \phi Y) + \Sigma\eta^r(X)\eta^r(Y) \quad , \quad g(X, \phi Y) + g(\phi X, Y) = 0 \quad , \tag{3.1}$$

$$\Omega(X,Y) = g(\phi X, Y) \quad ,$$

for all $X, Y \in \Gamma TM$, where Ω is the fundamental 2-form associated with the f-structure.

In our case one has in addition

$$\Omega^m \wedge \eta^{2m+1} \wedge \eta^{2m+2} \neq 0 \tag{3.2}$$

and setting

$$\mathcal{F} = \xi_{2m+1} \wedge \xi_{2m+2} \in A^1(M, TM) \quad , \tag{3.3}$$

one deduces on behalf of (3.1)

$$(\nabla\phi)X = g(\phi X, \mathcal{T})\mathcal{F} - (\mathcal{F}X)^{\flat} \otimes \phi\mathcal{T} \tag{3.4}$$

$(\mathcal{F}X = X^{2m+2}\xi_{2m+1} - X^{2m+1}\xi_{2m+2})$.

Next denoting by $\theta^A_B\,(A, B = 1, ..., 2m)$ the horizontal connection forms one infers from (3.1) the well-known Kählerian relations (see also [YK1])

$$\theta^a_b = \theta^{a^*}_{b^*} \quad , \quad \theta^a_{b^*} = \theta^b_{a^*} \quad ; a^* = a + m \,. \tag{3.5}$$

Setting

$$\Omega = \Sigma\omega^a \wedge \omega^{a^*} \quad , \tag{3.6}$$

then with the help of structure equation (1.3) and taking account of (3.6) and (2.6), we get

$$d\Omega = -2\bar{\omega} \wedge \varphi \quad , \tag{3.7}$$

where

$$\bar{\omega} = (\phi\mathcal{T})^\flat = \Sigma\,(t_a\omega^{a^*} - t_{a^*}\omega^a) \quad . \tag{3.8}$$

Since $d\varphi = 0$, then by a simple argument, it follows that $d\bar{\omega} = 0$.
On the other hand by (3.1) one has

$$< \phi dp, X >= \phi X \quad , \quad X \in \Gamma TM \tag{3.9}$$

and therefore any vector field V such that $\nabla V = f\phi dp$ for some scalar f, is a Killing vector field, i.e. $g(\nabla_X V, Y) + g(\nabla_Y V, X) = 0$.

Moreover let $M(\phi, dp, g)$ be a (pseudo-) Riemannian manifold with soldering form dp and equipped with a (1.1) tensor field ϕ such that

$$g(\phi X, Y) + g(X, \phi Y) = 0 \quad .$$

Since the wedge product $\wedge$ is skew symmetric with respect to g, then it is easily seen that any vector field V such that

$$\nabla V = f\phi dp + \lambda(V_1 \wedge V_2) \quad ; \quad f, \lambda \in C^\infty M \quad , \tag{3.10}$$

where V_1, V_2 are some vector fields, is a Killing vector field.

As a consequence of (3.10) we agree to define V as a *ϕ-soldering skew symmetric Killing* vector field (abr. $\phi - S.S.S.K.$).

Comming back to the case under discussion and setting $\mathcal{T}^\flat = \omega = t_A\omega^A$, one finds that the necessary and sufficient conditions for $\mathcal{T}$ to be $\phi - S.S.S.K.$ are expressed by the following equations

$$\begin{cases} dt_a - t_A\theta^A_a & = & -f\omega^{a^*} \\ dt_{a^*} - t_A\theta^A_{a^*} & = & f\omega^a \end{cases} ; \tag{3.11}$$

then the covariant differential of $\mathcal{T}$ satisfies

$$\nabla\mathcal{T} = f\phi dp + 2t\mathcal{F} \quad ; \quad 2t = g(\mathcal{T}, \mathcal{T}) \,. \tag{3.12}$$

From (3.11) one gets at once $dt = -f\bar{\omega}$, which shows that the scalar f is the *Euler multiplicator* of $\bar{\omega}$. By (3.1) we have $i_{\mathcal{T}}\Omega = \bar{\omega}$ and since $\mathcal{T}$ is a horizontal vector field it follows that $\mathcal{L}_{\mathcal{T}}\Omega = 0$. Hence $\mathcal{T}$ defines an *infinitesimal automorphism* of the fundamental 2-form Ω.

Further since by (3.1) one has $i_{\phi\mathcal{T}}\Omega = \omega$, then by (2.8), (2.9) and (3.7) we obtain

$$\mathcal{L}_{\phi\mathcal{T}}\Omega = -2(\tau + 2t)\varphi = -2s\varphi \quad , \tag{3.13}$$

that is on behalf of (2.10)

$$\mathcal{L}_{\phi\mathcal{T}}\Omega = -2\Theta^{2m+2}_{2m+1} \quad . \tag{3.14}$$

The above equation shows that the Lie derivative $\mathcal{L}_{\phi\mathcal{T}}\Omega$ is up to -2 equated by the vertical curvature 2-form of M.

Now by (3.12) and (3.4), one derives $(\nabla\phi)\mathcal{T} = 0$ and consequently

$$\nabla\phi\mathcal{T} = -f dp + f\eta^r \otimes \xi_r \quad . \tag{3.15}$$

Then since the *left inner product* $< \phi\mathcal{T}, \phi dp >= -\mathcal{T}$, we obtain from (3.12) and (3.15) that $\mathcal{T}$ and $\phi\mathcal{T}$ commute, i.e. $[\mathcal{T}, \phi\mathcal{T}] = 0$.

It also should be noticed that by (3.15) it follows that $\bar{\omega}$ is an *exact* form and using $dt = -f\bar{\omega}$, we get $f = f(t)$.

Recall now that there is an isomorphism between the Lie algebra of G and the space of Killing vector fields. Therefore if α is any 1-form and V_1, V_2 are two Killing vector fields one has

$$d\alpha(V_1, V_2) = 2\alpha([V_1, V_2]) \quad .$$

Then since in the case under discussion one finds

$$[\xi_{2m+1}, \xi_{2m+2}] = 2\mathcal{T} \quad , \quad [\mathcal{T}, \xi_{2m+1}] = 4t\xi_{2m+2} \quad , \quad [\mathcal{T}, \xi_{2m+2}] = -4t\xi_{2m+1} \quad ,$$

one has

$$\begin{cases} d\alpha([\xi_{2m+2}, \xi_{2m+1}]) = 4\alpha(\mathcal{T}) \\ d\alpha([\mathcal{T}, \xi_{2m+1}]) = 4t\alpha(\xi_{2m+2}) \\ d\alpha([\mathcal{T}, \xi_{2m+2}]) = -4t\alpha(\xi_{2m+1}) \end{cases} \quad , \tag{3.16}$$

It is worth to notice that the Killing vector fields $\xi_r, \mathcal{T}$ define a *perfect* Lie group.

On the other hand, $dt = -f\bar{\omega}$ implies

$$\nabla 2t = -2f\phi\mathcal{T} \quad \Rightarrow \quad \|\nabla 2t\|^2 = 8f^2 t \tag{3.17}$$

and by (3.15) one finds

$$div\, \phi\mathcal{T} = -2mf \quad . \tag{3.18}$$

Then since we have seen that $f = f(t)$, one has

$$div\,(\nabla 2t) = -4f'ft + 8mf^2 \quad ; \quad f' = \frac{df}{dt} \quad . \tag{3.19}$$

Since by (3.17) and (3.19) it follows that $\|\nabla\|\mathcal{T}\|^2\|^2$ and div $\nabla\|\mathcal{T}\|^2$ are both functions of $\|\mathcal{T}\|^2$, we conclude by reference to a known definition (see also [W]) that $\|\mathcal{T}\|^2 = 2t$ is an *isoparametric* function.

Making use of the structure equations (1.4) involving the curvature 2-forms, one finds by (2.6), (2.9) and (3.11) that the transversal curvature forms are expressed by

$$\begin{cases} \Theta_a^{2m+1} = -(t_a\omega \wedge \eta^{2m+1} + f\omega^{a^*} \wedge \eta^{2m+2})\,, \\ \Theta_{a^*}^{2m+1} = -(t_{a^*}\omega \wedge \eta^{2m+1} - f\omega^{a} \wedge \eta^{2m+2})\,, \\ \Theta_a^{2m+2} = -(t_a\omega \wedge \eta^{2m+2} - f\omega^{a^*} \wedge \eta^{2m+1})\,, \\ \Theta_{a^*}^{2m+2} = -(t_{a^*}\omega \wedge \eta^{2m+2} + f\omega^{a} \wedge \eta^{2m+1})\,. \end{cases} \tag{3.20}$$

From above we derive the two binomial 2-forms

$$\begin{cases} \psi_{2m+1} = \Sigma\, t_A\Omega_A^{2m+1} = -2t\omega \wedge \eta^{2m+1} - f\bar{\omega} \wedge \eta^{2m+2} \\ \psi_{2m+2} = \Sigma\, t_A\Omega_A^{2m+2} = -2t\omega \wedge \eta^{2m+2} + f\bar{\omega} \wedge \eta^{2m+1} \end{cases} \tag{3.21}$$

and we agree to denominate $\psi_r\,(r = 2m+1, 2m+2)$ the *distinguished transversal* 2-forms on $M(\phi, \Omega, \eta^r, \xi_r, g)$.

Next using (2.7) and (2.8), one finds by exterior differentiation

$$\begin{cases} d\psi_{2m+1} = -2u \wedge \psi_{2m+1} \\ d\psi_{2m+2} = 2u \wedge \psi_{2m+2} \end{cases} , \tag{3.22}$$

where we have set $u = d\lg t$. Consequently, u being an exact form the equations (3.22) may be expressed in terms of d^ω-cohomology as

$$d^{2u}\psi_{2m+1} = 0 \quad , \quad d^{-2u}\psi_{2m+2} = 0 \quad , \tag{3.23}$$

which shows that ψ_{2m+1} (resp. ψ_{2m+2}) is d^{2u}-exact (resp. d^{-2u}-exact).

Further making use of (3.4) one finds after some calculations that for any vector field X one has

$$\begin{aligned} \nabla^2 X + \phi\nabla^2\phi X = {} & -g(X,\mathcal{T})\omega \wedge dp^\perp - f(\phi X)^\flat \wedge \mathcal{F} + \\ & +\varphi \otimes \mathcal{F}X + \omega \wedge (X^\perp)^\flat \otimes \mathcal{T} + (\mathcal{F}X)^\flat \wedge \phi dp\,. \end{aligned} \tag{3.24}$$

In the above equation $dp^\perp = \eta^r \otimes \xi_r$ means the vertical component of the soldering form dp and

$$\mathcal{F} = \xi_{2m+1} \wedge \xi_{2m+2} \quad \Rightarrow \quad \mathcal{F}X = X^{2m+1}\xi_{2m+2} - X^{2m+2}\xi_{2m+1}\,. \tag{3.25}$$

If R denotes the curvature tensor we recall the general formula

$$R(Y,Z)X = \nabla^2 X(Y,Z) \quad , \quad Y, Z \in \Gamma TM \, .$$

Thus the relation (3.24) implies

$$R(Y,Z)X + \phi R(Y,Z)\phi X = g(X,\mathcal{T})(g(Z,\mathcal{T})Y^{\perp} - g(Y,\mathcal{T})Z^{\perp}) + \tag{3.26}$$

$$+ f\{g(Z,\phi X)\mathcal{F}Y - g(Y,\phi X)\mathcal{F}Z\} + \varphi(Y,Z)\mathcal{F}X + \varphi(X,Y)\phi Z - \varphi(X,Z)\phi Y \quad ,$$

where $Y^{\perp}$ (resp. $Z^{\perp}$) means the vertical component of Y (resp. Z).

It is seen by (3.4) that ϕ and ∇ commute on $M^{\top}$ and by (3.7) that the 2-form Ω is closed, i.e. Ω moves on $M^{\top}$ to a symplectic form (in order to simplify we denote the induced elements on $M^{\top}$ by the same letters).

Therefore we conclude that $M^{\top}$ is a Kähler submanifold of M.

It is also worth to remark that if Y, Z are any horizontal vector fields, i.e.

$$\eta^r(Y) = \eta^r(Z) = 0 \quad \Rightarrow \quad Y^{\perp} = Z^{\perp} = 0 \quad ,$$

then (3.26) moves to

$$R(Y,Z)X + \phi R(Y,Z)\phi X = 0 \quad .$$

On the other hand since ξ_r are two normal sections fot $M^{\top}$ the 2-fundamental quadratic forms associated with the immersion $x : M^{\top} \to M$ are given by $h_r = - < dp^{\top}, \nabla\xi_r >$ ($dp^{\top}$ is the soldering form of $M^{\top}$).

Then by (2.4) it is easily seen that $h_r = 0$, i.e. the immersion $x : M^{\top} \to M$ is *totally geodesic*.

We state the following.

Theorem 2. Assume that the manifold M discussed in Section 2 is equipped with a f-structure defined by the (1.1)-tensor field ϕ and by the fundamental 2-form Ω. Then any such a manifold $M(\phi, \Omega, \xi_r, \eta^r, g)$ is foliated by totally geodesic Kählerian submanifolds $M^{\top}$ tangent to the horizontal distribution $D^{\top}$ on M.

The following properties are also proved.

i) the dual form $\bar{\omega} = (\phi\mathcal{T})^{\flat}$ of $\phi\mathcal{T}$ (where $\mathcal{T}$ is the generator vector field of the $(\mathcal{T}.K.A.C.)$-structure of M) is closed;

ii) $\mathcal{T}$ is a ϕ-soldering skew symmetric Killing vector field, whose covariant differential satisfies

$$\nabla\mathcal{T} = f\phi dp + 2t\mathcal{F} \quad ,$$

where $2t = g(\mathcal{T},\mathcal{T})$ and $\mathcal{F} = \xi_{2m+1} \wedge \xi_{2m+2}$, and enjoys also the property to define an infinitesimal automorphism of Ω, i.e. $\mathcal{L}_{\mathcal{T}}\Omega = 0$;

iii) if Θ^{2m+2}_{2m+1} denotes the vertical curvature 2-form of M, then the following relation

$$\mathcal{L}_{\phi\mathcal{T}}\Omega + \Theta^{2m+2}_{2m+1} = 0$$

holds good;

iv) $\|\mathcal{T}\|^2 = 2t$ is an isoparametric function;

v) with the transversal curvature 2-forms are associated two binomial 2-forms ψ_r which are cohomologically exact;

vi) if X is any vector field on M, then the curvature tensor field R on M satisfies the following relation

$$R(Y,Z)X + \phi R(Y,Z)\phi X = g(X,\mathcal{T})(g(Z,\mathcal{T})Y^{\perp} - g(Y,\mathcal{T})Z^{\perp}) +$$

$$+ f\{g(Z,\phi X)\mathcal{F}Y - g(Y,\phi X)\mathcal{F}Z\} + \varphi(Y,Z)\mathcal{F}X + \varphi(X,Y)\phi Z - \varphi(X,Z)\phi Y\,,$$

where φ is the Killing 2-form, $Y^{\perp}, Z^{\perp}$ are the vertical components of $Y, Z \in \Gamma TM$ and $f \in C^{\infty}M$.

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