LIBERTAS MATHEMATICA, VOL.XVII (1997)

RECESSION METHODS FOR NONCOERCIVE VARIATIONAL INEQUALITIES

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The variational inequalities invented by G.Stampacchia and G.Fichera in the 60's in potential theory and mechanics and developed afterwards by the French and Italian schools,turned out to be an effective tool for a concurrent investigation of wide class of linear and nonlinear problems in applied mathematics. Upon some coerciveness conditions,the existence principles of solutions to variational inequalities are well known. However, the variational model of many mechanical problems leads, generally, to variational inequalities, which are noncoercive. The lack of coerciveness appears when boundary conditions are underdetermined or in the presence of a destabilizing term depending on a parameter as it is the case for the unilateral buckling in elasticity. In the past decades, the noncoercive inequalities have been approached by using recession analysis,Leray-Schauder degree and critical point theory. Finally,in a remarkably short time, the frame of variational inequalities has seen considerable developement both in functional analysis and mechanics by the introduction of the concept of hemivariational inequality due to P.D.Panagiotopoulos [12] as a variational formulation of unilateral problems.

We look for a solution $u \in C$ of the hemivariational inequality

(P) $\qquad (Au - f, v) \geq 0 \quad$ for all $\; v \in T_C(u)$,

where the set C is assumed to be closed and star-shaped with respect to a certain ball in a real reflexive Banach space X, $T_C(u)$ denotes Clarke's tangent cone of C at $u_0 \in C$, A is a pseudomonotone operator and f is given in the dual space X^*. As usual, $(\cdot,\cdot)$ stands for the duality pairing (X^*,X) while "$\longrightarrow$"

and " ⇀ " denote strong and weak convergence, respectively. The first existence theorem concerning problem (P), with a coerciveness condition on A, is due to Z.Naniewicz [10]. In this paper, we present recent refinements of this result when A is not coercive and study some unilateral problems where the boundary conditions are insufficiently blocked up. Most of them are basically due to D.Goeloven and co-workers [6].

§ 1. PREREQUISITES

The behavior at infinity of the functional $G:X \longrightarrow \mathbb{R}\cup\{+\infty\}$ is given by its *recession function*

$$G_\infty(x) := \liminf_{v \to x,\ t \to \infty} \frac{G(tv)}{t} = \inf\left\{ \liminf_{n \to \infty} \frac{G(t_n v_n)}{t_n} \;\middle|\; t_n \to \infty,\ v_n \longrightarrow x \right\}.$$

We observe that if $G, H: X \longrightarrow \mathbb{R}\cup\{+\infty\}$ then $(G + H)_\infty \geq G_\infty + H_\infty$.

For $u_o \in X$ we define the *recession function associated to a general operator* $A:X \longrightarrow X^*$ *with respect to* u_o by the formula

$$r_{u_o,A}(x) := \liminf_{v \to x,\ t \to \infty} \frac{(A(tv), tv - u_o)}{t}.$$

If we set $G(x) = (Ax, x - u_o)$ then clearly $r_{u_o,A}(x) = G_\infty(x)$. If $u_o = 0$, then $r_{o,A}$ reduces to the recession function used by H.Brezis and L.Nirenberg [2] for the characterization of the range of some nonlinear operators.

Let K be a set of X. The *recession cone* of K is the closed cone

$$K_\infty := \operatorname{dom}(\psi_K)_\infty = \{ x \in X \mid (\psi_K)_\infty < +\infty \},$$

where $\psi_K(x) = \begin{cases} 0 & \text{if } x \in K \\ \infty & \text{if } x \notin K \end{cases}$ is the indicator function of K. Equivalently, x belongs to K_∞ if and only if there exist sequences $\{t_n\} \subset \mathbb{R}$, $\{x_n\} \subset K$ such that $t_n \to \infty$ and $\frac{1}{t} x_n \longrightarrow x$. In particular, if K is closed and convex, then

$$K_\infty := \bigcap_{t>0} \left[\frac{K - x_o}{t} \right]$$

where x_o is an arbitrary element of K. The notion of recession cone has been used to sharpen many results of the convex analysis.

With a nonempty closed C in X, we associate

$$T_C(u) = \{k \in X \quad \forall \{u_n\} \subset C, \{\lambda_n\} \subset \mathbb{R}_+, \quad u_n \longrightarrow u, \lambda_n \to 0,$$
$$\exists \{k_n\} \subset X, \quad k_n \longrightarrow k, \quad u + \lambda_n k_n \in C\}$$

Clarke's tangent cone of C at u, and

$$N_C(u) = \{ u^* \in X^* \mid (u^*,k) \leq 0 \quad \forall k \in T_C(u) \}$$

Clarke's normal cone to C at u.

Let $d_C(u) = \inf \|w - u\|$ be the *distance function* of C. We set

$$d^\circ_C(u,v) := \lim_{y \to u} \sup_{t > 0} \frac{d_C(y + tv) - d_C(v)}{t}$$

the *generalized directional derivative in direction* v *of* d_C *at* v, and

$$\partial d_C(u) = \{ w \in X^* \mid d^\circ_C(u,v) \geq (w,v) \quad \forall v \in X \}$$

the *generalized gradient of Clarke of* d_C *at* u.

Let $B(u_o,\rho)$ be a closed ball in X with center u_o and radius ρ. We say that C is *star-shaped* with respect to $B(u_o,\rho)$ if

$$v \in C \iff \lambda v + (1 - \lambda) w \in C, \quad \forall \lambda \in [0,1], \quad \forall w \in B(u_o,\rho).$$

The distance function of a star-shaped C has the following properties:

LEMMA 1.1 ([10]). *Let* C *be a nonempty closed set in a real Banach space* X. *If* C *is star-shaped with respect to* $B(u_o,\rho)$ *then*

$$d^\circ_C(u,u_o - u) \leq d_C(u) - \rho, \quad \forall u \notin C$$

and

$$d^\circ_C(u,u_o - u) = 0, \quad \forall u \in C.$$

For an operator A, generally nonlinear, from X to X^* we denote by $D(A)$ its domain, by $R(A)$ its range and by $G(A)$ its graph. The operator A is *monotone* if

$$(Au - Av, u - v) \geq 0 \quad \forall u,v \in D(A),$$

and *maximal monotone* if its graph is maximal with respect to inclusion among all monotone operators. The operator A is *hemicontinuous* if $D(A)$ is convex and for all $y \in D(A)$ the map $t \longmapsto A((1-t)x + ty)$ is continuous from $[0,1]$ into X^*, endowed with the weak topology.

The operator A satisfies *condition* (S) if each sequence $\{u_n\}$ in X with $u_n \rightharpoonup u$ for which

$$\limsup\, (Au_n, u_n - u) \leq 0 \tag{1}$$

is in fact strongly convergent.

A single-valued operator A from X into X^* is said to be *pseudomonotone* if for any sequence $\{u_n\}$ in X converging weakly to some $u \in X$ such that (1) holds, it follows that

$$\liminf \langle Au_n, u_n - v\rangle \geq \langle Au, u - v\rangle \quad \text{for all } v \in X.$$

As we readily realize, the pseudomonotonicity is linked to the investigation of variational inequalities. A degree-theoretic approach of variational inequalities involving pseudomonotone mappings was recently carried out in [8]. Sometimes, in applications, the following variant acts effectively:

A single-valued operator A from X into X^* is said to be *generalized pseudomonotone* if for any sequence $\{u_n\}$ in X with $u_n \rightharpoonup u$ for which (1.1) holds, it follows that $\langle Au_n, u_n\rangle \to \langle Au, u\rangle$ and $Au_n \rightharpoonup Au$ in X^*.

We can prove that any pseudomonotone mapping is generalized pseudomonotone while a bounded generalized pseudomonotone operator is pseudomonotone [3].

LEMMA 1.2 ([9]). Let X be a real reflexive Banach space and $f_i : X \longrightarrow \mathbb{R}$ *a finite collection of locally Lipschitzian convex functions defined on* X. *Define* $f : X \longrightarrow \overline{\mathbb{R}}$ *as*

$$f(u) = \min\{f_i(u) \mid i = 1, \ldots, N\}, \; u \in X.$$

If $A : X \longrightarrow X^*$ *is a monotone maximal operator with* $D(A) = X$ *and satisfying condition* (S), *then the sum* $A + \partial f$ *is pseudomonotone.*

As an immediate consequence of this result, we have

PROPOSITION 1.3. *Let* A *be as in the previous Lemma and let* C *be a set of* X *which can be represented as the union of a finite collection of nonempty closed convex set* C_j, $j = 1, \ldots, N$, *of* X, *i.e.* $C = \bigcup_{j=1}^{N} C_j$. *We assume that* $\bigcap_{j=1}^{N} C_j \neq \emptyset$. *Then*

(*i*) C *is star-shaped with respect to a certain ball;*

(*ii*) $A + \lambda \partial d_C$ *is pseudomonotone for each* $\lambda \geq 0$.

Let us now introduce the set of *resting directions* [1]:

$$\mathcal{R}(A, f, C, u_o) = \left\{ w \in X \mid \exists\, \{u_n\} \subset C, \|u_n\| \to \infty,\; w_n = \frac{u_n}{\|u_n\|} \rightharpoonup w,\; \langle Au_n, u_n - u_o\rangle \leq \langle f, u_n - u_o\rangle \right\}.$$

On this set, we introduce a compactness condition:

The set $\mathcal{R}(A,f,C,u_o)$ is *asymptotically compact* (*a-compact*) if for each $w \in \mathcal{R}(A,f,C,u_o)$ the sequences $\{w_n\}$ which appear in the definition of this set are strongly convergent to w.

Now we point out some properties of the recession set.

PROPOSITION 1.4 [4]. *Let* u_o *be given in* X *and* f *in* X^*. *If*

(j) A *satisfies condition* (S);

(jj) $(Au,u) \geq o, \ \forall\ u \in X$;

(jjj) A *is weakly continuous,i.e.* $u_n \rightharpoonup u \ \Rightarrow\ Au_n \rightharpoonup Au$;

(jv) A *is positively homogeneous;*

(v) B *is monotone on* X.

Then $\mathcal{R}(A + B,f,C,u_o)$ *is a-compact and*

$$\mathcal{R}(A + B,f,C,u_o) \subset \{\ w \in C_\infty \setminus \{0\}\ (Aw,w) = 0\ \}.$$

REMARK 1.5. If $A:X \longrightarrow X^*$ is bounded linear and coercive, i.e, there is $x > o$ such that $(Au,u) \geq x\|u\|^2$, $\forall\ u \in X$, then A satisfies hypotheses (j)-(jv). Moreover,when B is monotone, we derive directly that $\mathcal{R}(A + B,f,C,u_o) \neq \emptyset$.

§ 2. NONCOERCIVE HEMIVARIATIONAL INEQUALITIES

In this section we will investigate constrained problems on real reflexive Banach spaces, in which the set C of all admissible elements is not convex in general but star-shaped. Due to the nonconvexity of C, the variational formulations lead to the hemivariational inequalities of the form (P) stated above. However, in the particular case when C is convex, the problem (P) reduces to the following variational inequality: find $u \in C$ such that

(P') $$(Au - f,v - u) \geq 0, \quad \forall\ v \in C,$$

and we can derive some new existence results concerning problem (P').

PROPOSITION 2.1 [5]. *Let* u_o *be given in* X *and* f *in* X^*. *If*

(i) $\mathcal{R}(A,f,C,u_o)$ *is a-compact;*

(ii) *there is a nonempty set* W *of* $X\setminus\{0\}$ *such that* $\mathcal{R}(A,f,C,u_o) \subset W$ *and*

(2) $$r_{u_o,A}(w) > (f,w) \quad \forall\ w \in W$$

then $\mathcal{R}(A,f,C,u_o)$ *is empty.*

We make further the following assumptions:

(H_1) X is a real reflexive Banach space and C is a nonempty closed set of X with is star-shaped with respect to a ball $B(u_o,\rho)$;

(H_2) $A + \lambda\partial d_C$ is pseudomonotone for each $\lambda > 0$;

(H_3) A is bounded.

THEOREM 2.2. *Suppose that assumptions* (H_1)-(H_3) *are fulfilled. If*

$$R(A,f,C,u_o) = \emptyset$$

then problem (P) *has at least one solution.*

§3. AN EXISTENCE THEOREM FOR SEMICOERCIVE HEMIVARIATIONAL INEQUALITIES

This section emphasizes an existence result of solutions for nonlinear perturbations of linear semicoercive hemivariational inequalities.

We suppose that C is a "piecewise convex" set in a real Hilbert space X; this means that the following hypothesis is fulfilled:

(H) C is nonempty, closed and can be represented as the union of a finite collection of nonempty closed convex sets C_j, $j=1,\ldots,N$, of X such that

$$\operatorname{int}\left(\bigcap_{j=1}^{N} C_j\right) \neq \emptyset.$$

Under this hypothesis, C is star-shaped with respect to a certain ball.

The operator $A:X \longrightarrow X$ is *semicoercive* if there is $\varkappa > 0$ such that

$$(Au,u) \geq \varkappa \|Pu\|^2 \quad \forall\, u \in X,$$

with $P = I - Q$, where I denotes the identity mapping and Q the orthogonal projector of X onto $\operatorname{Ker}(A + A^*)$, A^* is the adjoint operator.

THEOREM 3.1. *Let* X *be a real Hilbert space and suppose that hypothesis* (H) *is satisfied. Let* g *be given in* X. *Assume that*

1° $A:X \longrightarrow X$ *is bounded linear and semicoercive;*

2° $\dim\{\operatorname{Ker}(A + A^*)\} < \infty$;

3° $u_o \in C \cap \operatorname{Ker}(A)$;

4° $(g,w) < 0$, $\forall\, w \in C_\infty \cap \operatorname{Ker}(A + A^*) \setminus \{0\}$;

5° $B:X \longrightarrow X$ *is bounded hemicontinuous monotone and* $u_o \in \operatorname{Ker} B$;

or

$5.^{\circ}$ *$B:X \longrightarrow X$ is bounded pseudomonotone and*

$$(Bu, u - u_o) \geq 0 \quad \forall\, u \in X.$$

Then there exists a solution $u \in C$ of the hemivariational inequality

$$(Au + Bu - g, v) \geq 0 \quad \forall\, v \in T_C(u).$$

§4. AN EXISTENCE THEOREM FOR SEMICOERCIVE VARIATIONAL - - HEMIVARIATIONAL INEQUALITIES

We will present the result of previous section to so called variational-hemivariational inequalities.

Let $(X, \|\cdot\|)$ be a real Hilbert subspace of $L^2(\Omega)$, $X \subset L^2(\Omega) \subset X^*$, where Ω is a regular bounded domain of $\mathbb{R}^N$. Suppose that the injection $X \hookrightarrow L^2(\Omega)$ is dense and compact.

With a given a continuous bilinear form $a: X \times X \longmapsto \mathbb{R}$ we associate the operator $A \in L(X, X^*)$ by

$$a(u,v) = (Au, v) \quad \forall\, u,v \in X,$$

and assume that a is *semicoercive*, i.e., there is constant $c > 0$ such that

$$a(u,u) \geq c\|u\|^2 \quad \forall\, u \in \operatorname{Ker}(A + A^*)^{\perp}.$$

Later on, we suppose the following hypotheses are fulfilled:

(A_1) $a: X \times X \longmapsto \mathbb{R}$ is bilinear, continuous and semicoercive;

(A_2) $\dim \ker (A + A) < \gamma$, $\gamma \geq 0$;

(A_3) $\phi: X \longmapsto (-\gamma, \gamma]$ is a convex l.s.c. functional with $\Phi(0) = 0$;

(A_4) $j: \mathbb{R} \longmapsto \mathbb{R}$ is Lipschitz continuous;

(A_5) $j^{\circ}(u,v) \in L^1(\Omega)$ for $u,v \in X$.

A variational-hemivariational inequality is the problem of finding an element $u \in X$ such that

$$\text{(VHI)} \quad a(u, v-u) + \int_\Omega j^{\circ}(u, v-u)\, d\Omega + \Phi(v) - \Phi(u) \geq (f, v-u) \quad \forall\, v \in X.$$

Define the recession function

$$(J^{\circ})_{\infty}(x) = \lim_{v \to x} \inf_{t \to \infty} \int_\Omega j^{\circ}(u, v-u)\, d\Omega.$$

THEOREM 4.1 [7]. *Assume that hypotheses* (A_1)-(A_5) *are satisfied. If*

(3) $$(J^{\circ})_{\infty}(w) + \Phi_{\infty}(w) > (f,w) \quad \forall\, w \in \mathrm{Ker}\,(A + A^{*}) \setminus \{0\},$$

then Problem (HVI) *has at least one solution.*

The assumptions (2) and (3) provide conditions of Landesman-Lazer type.

§ 5. AN APPLICATION IN NONCONVEX MECHANICS

As example, we show why the constrained echilibrium of a material point is governed by problem (P).

Let us consider a material point with mass m which is constrained to remain in a closed set C of $\mathbb{R}^3$. When m is in contact without friction with the boundary of C, then there is a reaction force R which is normal to the boundary

$$R \in -N_C(x).$$

Therefore, if $f(x)$ is an external force acting on m, then to obtain equilibrium it is necessary and suficient that $x \in C$ and $f = -R$, namely

$$x \in C \quad \text{and} \quad (-f(x),v) \geq 0, \quad \forall\, v \in T_C(x).$$

Moreover, we suppose that $f(x) = mg - Ax$ where $A \in \mathbb{R}^{3\times 3}$ is a positive semidefinite matrix and mg is the gravity force. Finally, we have

$$(Ax - mg,v) \geq 0 \quad \forall\, v \in T_C(x),$$

here $(x,y) = \sum_{i=1}^{3} x_i y_i$. This example shows that problem (P) is nothing else that a general expression of the classical principle of virtual work.

When C is a nonempty star-shaped set, we can prove [4] that the map

$$x \longrightarrow \partial d_C(x)$$

is pseudomonotone as well as the sum $A + \lambda\partial d_C(x)$ for all $\lambda > 0$. By virtue of Theorem 3.1, the condition

$$(g,v) < 0 \quad \forall\, v \in C_{\infty} \cap \ker(A + A^{T}) \setminus \{0\}$$

is sufficient for the existence of an equilibrium.

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