LIBERTAS MATHEMATICA, VOL. XVII (1997)

THE CIRCULANT GRAPHS. APPLICATIONS

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1. Introduction

In this paper, we consider only simple graphs, i.e. undirected, without multiple edges and loops. For a graph G, we denote by V(G) the vertex set and E(G) the edge set and we write G=(V(G), E(G)) or briefly G=(V, E). For $(x, y) \in E(G)$, we also use $x \sim y$; if not, we use $x \not\sim y$. $\overline{G}$ is the complement of G. A *cycle subgraph* is defined as an induced subgraph isomorphic to a cycle graph C_n. O_n and K_n represent, as usually, the null graph with n vertices and the complete graph with n vertices, respectively.

Two graphs G_1 and G_2 are said to be *disjoint* if they have no vertex in common. The *union* $G_1 \cup G_2$ (*intersection* $G_1 \cap G_2$, respectively) of two graphs G_1 and G_2 is the graph having the vertex set $V(G_1) \cup V(G_2)$ ($V(G_1) \cap V(G_2)$) and the edge set $E(G_1) \cup E(G_2)$ ($E(G_1) \cap E(G_2)$).

Let $G_i = (V_i, E_i)$, i=1, 2,..., n be n graphs. We denote by $G_1 \otimes G_2 \otimes \ldots \otimes G_n$ or $\bigotimes_{i=1}^{n} G_i$ the graph defined by:

$$V(\bigotimes_{i=1}^{n} G_i) = \mathop{\mathrm{X}}_{i=1}^{n} V(G_i) = \{(x_1, x_2, \ldots, x_n): x_i \in V(G_i)\}$$

and

$$(x_1, x_2, \ldots, x_n) \sim (y_1, y_2, \ldots, y_n) \Leftrightarrow x_i \sim y_i,\ i = 1, 2, \ldots, n.$$

This product is called *categorical* or *cardinal* or *tensorial*.

The class of quasi-Cayley graphs was introduced in [1,2]. If Γ is a multiplicative group (with identity 1) and S a subset of Γ with the properties: $1 \notin S$; $x \in S \Rightarrow x^{-1} \in S$, then the graph $G(\Gamma, S)$, with $V(G(\Gamma, S)) = \Gamma$, $E(G(\Gamma, S)) = \{(x, y): (x, y) \in \Gamma \times \Gamma \text{ and } xy^{-1} \in S\}$, is called the *quasi-Cayley graph* of Γ with respect to S. In [7] it is defined the circulant graph which is a particular quasi-Cayley graph. In this paper, we present the structure of the circulant graphs, the operations with these graphs, and their latticeal properties. Also, we give the properties of the categorical product of the circulant graphs, the structure of a categorical product of atom circulant graphs and their applications.

2. The Circulant Graphs. Properties

We consider the aditive group $Z_n = \{\bar{0}, \bar{1}, \ldots, \overline{n-1}\}$, $\bar{a} = \{a + nh: h \in Z\}$, a = 0, 1, ..., n-1. The group Z_n is cyclic, $Z_n = <\bar{1}>$. Now we define the circulant graph [7].

Definition 2.1

Let $S \subseteq Z_n$ be such that $\bar{0} \notin S$ and if $\bar{s} \in S$, then $-\bar{s} \in S$. The *circulant graph* $G(Z_n, S)$ is the simple graph having the vertex set Z_n, and edge set

$$E(G(Z_n, S)) = \{(\bar{a}, \bar{b}): \bar{b} - \bar{a} \in S\}$$

The set S is called the *symbol* of the circulant graph $G(Z_n, S)$.

Remarks:

1) Every circulant graph is a quasi-Cayley graph [1, 2];
2) A circulant graph $G(Z_n, S)$ is a regular graph of degree $|S|$, because every vertex $\bar{h}$ is adjacent with the vertices $\bar{h} + \bar{s}$, $\bar{s} \in S$;
3) The null graph and the complete graph are circulant graphs; $O_n = G(Z_n, \phi)$, $K_n = G(Z_n, Z_n^{\bullet})$, $Z_n^{\bullet} = Z_n \backslash \{\bar{0}\}$;
4) If we denote by $S^{-1} = \{-\bar{s} : \bar{s} \in S\}$, then $\bar{s} \in S$ implies $-\bar{s} \in S \Leftrightarrow S = S^{-1}$.

Theorem (2.2)

If $H \subsetneq Z_n$ is a subgroup of order m, then $H^* = H - \{\bar{0}\}$ is a symbol and the circulant graph $G(Z_n, H^*)$ is the disjoint union of n/m complete graphs K_m.

Proof:

Indeed, if we denote as usually $c + H = \{\bar{a} + \bar{h} : \bar{h} \in H\}$, then:

$$(\bar{a}, \bar{b}) \in E(G(Z_n, H^*)) \Leftrightarrow \bar{a} - \bar{b} \in H^* \Leftrightarrow \bar{b} \in \bar{a} + H,$$

that is $\bar{a} + H$ generates the complete graph K_m and

$$Z_n = \bigcup_{\bar{a} \in Z_n} (\bar{a} + H) \qquad (2.1)$$

Let us prove that there are no edges which join vertices of different complete graphs. Indeed, if $\bar{a} + \bar{h}_1 \in \bar{a} + H$, $\bar{b} + \bar{h}_2 \in \bar{b} + H$, and $\bar{a} + \bar{h}_1 \sim \bar{b} + \bar{h}_2$, then $(\bar{a} + \bar{h}_1) - (\bar{b} + \bar{h}_2) \in H$ implies $\bar{a} - \bar{b} \in H$ and $\bar{a} + H = \bar{b} + H$.

In the following theorem it is presented the structure of a circulant graph $G(Z_n, S)$, with $|S| \leq 2$, which is called the *atom graph*.

Theorem (2.3)

Let $G(Z_n, S)$ be a circulant graph with $|S| \leq 2$. The following statements are true:

1) If $\bar{a}$ is a generator for the group $(Z_n, +)$, then $G(Z_n, \{\bar{a}, -\bar{a}\})$, is the cycle graph C_n.
2) If $n = pq$, $p \neq 2$, $q \neq 2$, then the graph $G(Z_n, \{\bar{p}, -\bar{p}\})$ is a union of p disjoint cycle graphs C_q.
3) If $n = 2q$, then the graph $G(Z_{2q}, \{\bar{q}\})$ is a union of q graphs K_2.

Proof:

1) If $\bar{a}$ is a generator for the group $(Z_n, +)$, then n is the least positive integer such that $n\bar{a} = \bar{0}$ and $Z_n = \{\bar{0}, \bar{a}, 2\bar{a}, \ldots, (n-1)\bar{a}\}$. For $i\bar{a}, k\bar{a} \in Z_n$, we have:

$$i\bar{a} \sim k\bar{a} \Leftrightarrow i\bar{a} - k\bar{a} \in \{\bar{a}, -\bar{a}\} \Leftrightarrow (i-k+1)\bar{a} = \bar{0} \text{ or}$$
$$(i-k+1)\bar{a} = \bar{0} \Leftrightarrow i-k-1 = 0 (\text{mod } n) \text{ or}$$
$$i-k+1 = 0 (\text{mod } n) \Leftrightarrow |i-k| = 1 (\text{mod } n),$$

that is $(k-1)\bar{a} \sim k\bar{a}$, $k = 0, 1, \ldots, n-1$. The graph $G(Z_n, \{\bar{a}, -\bar{a}\})$, is regular of degree two hence it is a cycle graph.

2) From $n = 2q$, we deduce that ord $\bar{p} = q$ and the subgroup generated by $\bar{p}$ is

$$<\bar{p}> = \{\bar{0}, \bar{p}, 2\bar{p}, \ldots, (q-1)\bar{p}\}.$$

In the graph $G(Z_n, \{\bar{p}, -\bar{p}\})$, the elements of the subgroup $<\bar{p}>$ generate the cycle subgraph C_q.

The subgroup $<\bar{p}>$ defines the partition:

$$Z_n = \bigcup_{t=0}^{p-1} (\bar{t} + <\bar{p}>) \tag{2.2}$$

where $\bar{t} + <\bar{p}> = \{\bar{t} + h\bar{p} : h = 0, 1, \ldots, p-1\}$ is the equivalence class modulo $<\bar{p}>$. In the same manner as above we have

$$\bar{t} + i\bar{p} \sim \bar{t} + k\bar{p} \Leftrightarrow (\bar{t} + i\bar{p}) - (\bar{t} + k\bar{p}) \in \{\bar{p}, -\bar{p}\} \Leftrightarrow |i - k| = 1 (\mathrm{mod}\ q),$$

i.e. the set $\bar{t} + <\bar{p}>$ generates a cycle graph C_q.

The subgraphs generated by $\bar{t}_1 + <\bar{p}>$ and $\bar{t}_2 + <\bar{p}>$, $t_1 \neq t_2$, have no common edge. Indeed, if $\bar{t}_1 + i\bar{p} \sim \bar{t}_2 + j\bar{p}$, $t_1 \neq t_2$, then $(i\bar{p} + t_1) - (j\bar{p} + t_2) \in \{\bar{p}, -\bar{p}\}$, hence $(i - j \pm 1)$ $\bar{p} + (\bar{t}_1 - \bar{t}_2) = \bar{0}$, that is $p/t_1 - t_2$, contradicting the fact that $t_1, t_2 \leq p - 1$.

3) If $n = 2q$, then $-\bar{q} = \bar{q}$, hence $\{\bar{q}, -\bar{q}\} = \{\bar{q}\}$ and $G(Z_{2q}, \{\bar{q}\}) = q K_2$.

3. The Boole Algebra of the Circulant Graphs. Decomposition

A symbol of a circulant graph $G(Z_n, S)$ is called the *symbol* of the group $(Z_n, +)$.

Theorem (3.1)

The set of symbols

$$\mathcal{S}(Z_n) = \{S : S \subseteq Z_n, \bar{0} \notin S, S = S^{-1}\}$$

of a group Z_n is a Boole subalgebra of the Boole algebra $(\mathcal{P}(Z_n^*), \cap, \cup, ')$.

Proof:

Indeed, if $S_1, S_2 \in \mathcal{S}(Z_n)$, we have:

$$(S_1 \cap S_2)^{-1} \in S_1^{-1} \cap S_2^{-1}, (S_1 \cup S_2)^{-1} \in S_1^{-1} \cup S_2^{-1}$$

hence $S_1 \cap S_2, S_1 \cup S_2 \in \mathcal{S}(Z_n)$ and $\bar{0} \notin S_1 \cap S_2$, $\bar{0} \notin S_1 \cup S_2$. It results that $\mathcal{S}((Z_n), \cap, \cup)$ is a distributive lattice.

We remark that the empty set ϕ is a symbol group of Z_n hence ϕ is a zero element and Z_n^* is a unit element of $\mathcal{S}(Z_n)$.

Obviously, if $S \in \mathcal{S}(Z_n)$, then $S' = Z_n^* \setminus S \in \mathcal{S}(Z_n)$, hence $\mathcal{S}(Z_n)$ is a Boole subalgebra of $\mathcal{P}(Z_n^*, \cap, \cup, ')$.

If we denote

$$\mathcal{C}(Z_n) = \{G(Z_n, S) : S \in \mathcal{S}(Z_n)\},$$

we have:

Theorem (3.2)

The set $\mathcal{C}(Z_n)$ of the circulant graphs is a Boole algebra with respect to graphs' union, graphs' intersection and complementarity.

Proof: If $G(Z_n,S_1)$, $G(Z_n,S_2) \in \mathcal{G}(Z_n)$ it is easy to check that the following equalities hold:

$$G(Z_n, S_1) \cap G(Z_n, S_2) = G(Z_n, S_1 \cap S_2) \quad (3.1)$$

$$G(Z_n, S_1) \cup G(Z_n, S_2) = G(Z_n, S_1 \cup S_2) \quad (3.2)$$

Consequently, we may deduce that $(\mathcal{G}(Z_n), \cap, \cup)$ is a lattice.

In the lattice $\mathcal{G}(Z_n)$, the graph $G(Z_n, \phi) = O_n$ is a zero element and the graph $G(Z_n, Z_n^*)$ is a unit element. If $G(Z_n, S) \in \mathcal{G}(Z_n)$, the circulant graph $G(Z_n, S')$ verifies:

$$G(Z_n, S) \cap G(Z_n, S') = G(Z_n, S \cap S') = G(Z_n, \phi) = O_n$$

$$G(Z_n, S) \cup G(Z_n, S') = G(Z_n, S \cup S') = G(Z_n, Z_n^*) = K_n,$$

that is $G(Z_n, S')$ is the complement of $G(Z_n, S)$.

Obviously

$$G(Z_n, S') = \overline{G(Z_n, S)} \quad (3.3)$$

Corollary (3.3)

If $S_i \in \mathcal{S}(Z_n)$, $i = 1, 2, \ldots, t$, then:

$$G(Z_n, \bigcap_{i=1}^{t} S_i) = \bigcap_{i=1}^{t} G(Z_n, S_i) \quad (3.4)$$

$$G(Z_n, \bigcup_{i=1}^{t} S_i) = \bigcup_{i=1}^{t} G(Z_n, S_i) \quad (3.5)$$

Theorem (3.4)

The Boole algebras $\mathcal{S}(Z_n)$ and $\mathcal{G}(Z_n)$ are isomorphic.

Proof:

We consider the map $\varphi: \mathcal{S}(Z_n) \to \mathcal{G}(Z_n)$, $\varphi(S) = G(Z_n, S)$. According to (3.1), (3.2), and (3.3) it results that φ is a morphism of Boole algebras.

From definition it results that φ is surjective. If $S_1 \neq S_2$, then there is an element $\bar{k} \in S_1 - S_2$. Hence $\bar{0} \sim \bar{k}$ in the graph $G(Z_n, S_1)$ and $\bar{0} \nsim \bar{k}$ in $G(Z_n, S_2)$. So $\varphi(S_1) \neq \varphi(S_2)$, i.e. φ is injective.

Remark (3.5)

It is possible that $S_1 \neq S_2$, the graphs $\varphi(S_1)$ and $\varphi(S_2)$ are isomorphic but not equal. For example the graphs $G(Z_5, \{\bar{1}, \bar{4}\})$ and $G(Z_5, \{\bar{2}, \bar{3}\})$ are isomorphic but not equal.

From theorem (3.4) it follows the following corollary.

Corollary (3.6)

If $S_1, S_2 \in \mathcal{S}(Z_n)$ then:

1) $S_1 \subseteq S_2$ implies $G(Z_n, S_1) \subseteq G(Z_n, S_2)$;
2) $S_1 \cap S_2 = \phi$ implies $G(Z_n, S_1) \cap G(Z_n, S_2) = O_n$.

Theorem (3.7)

Let $G(Z_n, S)$ be a circulant graph. If the symbol set S contains an element of order n, then $G(Z_n, S)$ is connected.

Proof:

Let $Z_n = <\bar{a}>$ with $\bar{a} \in S$. We can write $S=\{\bar{a},-\bar{a}\} \cup T$, where $T \in \mathcal{S}(Z_n)$. We have

$$G(Z_n, S) = G(Z_n, \{\bar{a},-\bar{a}\}) \cup G(Z_n, T)$$

In that case $G(Z_n, \{\bar{a},-\bar{a}\})$ is a cycle graph C_n (Cf. 2.3) and it results that $G(Z_n, S)$ is connected.

Remarks:

1) The condition from theorem (3.7) is sufficient but it is not necessary; for example the graph $G(Z_6, \{\bar{2},\bar{3},\bar{4}\})$ is connected, but order $\bar{2} = 3$, order $\bar{3} = 2$, order $\bar{4} = 3$;
2) If the set S contains an element of order n, then the graph $G(Z_n, S)$ is hamiltonian;
3) If the graph $G(Z_n, S)$ is not connected then each element from S has the order less than n.

Theorem (3.8)

Let $G(Z_n, S)$ be a circulant graph. The following statements are true:

(i) If $n = 2q + 1$ or $n = 2q$ and $\bar{q} \notin S$, then the graph $G(Z_n, S)$ can be uniquely written as a union of cycle graphs with no common edge;

(ii) If $n = 2q$ and $\bar{q} \in S$, then the graph $G(Z_n, S)$ can be written as a union of a cycle graph and q subgraphs K_2.

Proof:

(i) In this condition, S can be written as

$$S = \bigcup_{1 \le p \le (q-1)} \{\bar{p},-\bar{p}\} \tag{3.6}$$

and according to (3.5), we have

$$G(Z_n, S) = \bigcup_{1 \le p \le (q-1)} G(Z_n,\{\bar{p},-\bar{p}\}) \tag{3.7}$$

(ii) If $n = 2q$ and $\bar{q} \in S$, the symbol S of the circulant graph $G(Z_n, S)$ is of the form

$$S = \{\bar{q}\} \cup \left(\bigcup_{1 \le p \le (q-1)} \{\bar{p},-\bar{p}\} \right) \tag{3.8}$$

hence

$$G(Z_n, S) = G(Z_n, \{\bar{q}\}) \cup \left(\bigcup_{1 \le p \le (q-1)} G(Z_n,\{\bar{p},-\bar{p}\}) \right) \tag{3.9}$$

Remarks:

1) If ord $\bar{q} = 2$, then $\{\bar{q}, -\bar{q}\} = \{\bar{q}\}$ and (3.8) implies (3.7).
2) If a circulant graph $G(Z_n, S)$ is in the shape of (3.7), then its complement has the form:

$$\overline{G(Z_n,S)} = \bigcup_{p \in Z_n^* - S} G(Z_n,\{\bar{p},-\bar{p}\})$$

Theorem (3.9)

For a complete graph the following statements are true:

(i) The graph K_{2q} can be written as a union of cycle graphs with no common edge and of q graphs K_2;

(ii) The graph K_{2q+1} can be written as a union of cycle graphs with no common edge.

Proof:

(i) Indeed, $K_{2q} = G(Z_{2q}, Z_{2q}^{\bullet})$ and

$$Z_{2q}^{\bullet} = \{\bar{1}, \overline{2q-1}\} \cup \{\bar{2}, \overline{2q-2}\} \cup \ldots \cup \{\overline{q-1}, \overline{q+1}\} \cup \{\bar{q}\}$$

Using (3.5) we obtain

$$K_{2q} = G(Z_{2q}, \{\bar{1}, \overline{2q-1}\}) \cup G(Z_{2q}, \{\bar{2}, \overline{2q-2}\}) \cup \ldots \cup G(Z_{2q}, \{\overline{q-1}, \overline{q+1}\}) \cup G(Z_{2q}, \{\bar{q}\}) \quad (3.10)$$

(ii) Similarly, we obtain

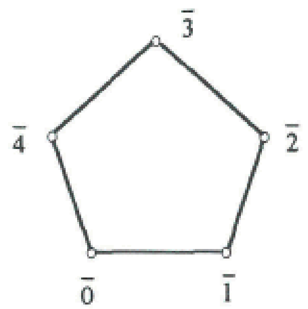
$$K_{2q+1} = G(Z_{2q+1}, \{\bar{1}, \overline{2q}\}) \cup G(Z_{2q+1}, \{\bar{2}, \overline{2q-1}\}) \cup \ldots \cup G(Z_{2q+1}, \{\bar{q}, \overline{q+1}\}) \quad (3.11)$$

Remark (3.10)

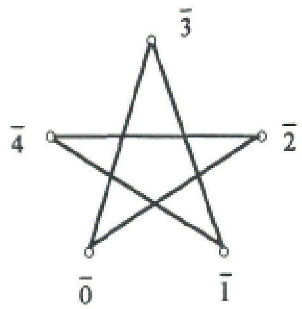
If n is an odd prime number, then K_n could be written as a union of (n-1)/2 cycle graphs C_n; Indeed, n has the form $n = 2q+1$ and each atom graph from (3.7) is a cycle graph C_{2q+1}.

Example (3.11)

For the group $(Z_5,+)$ the atom circulant graphs are $G(Z_5, \{\bar{1}, \bar{4}\})$ and $G(Z_5, \{\bar{2}, \bar{3}\})$.



$G(Z_5, \{\bar{1}, \bar{4}\}) = C_5$



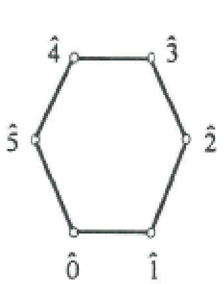
$G(Z_5, \{\bar{2}, \bar{3}\}) = C_5$

Hence,

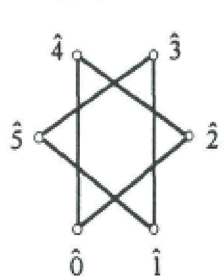
$$G(Z_5, \{\bar{1}, \bar{4}\}) \cup G(Z_5, \{\bar{2}, \bar{3}\}) = G(Z_5, Z_5^{\bullet}) = K_5$$

Example (3.12)

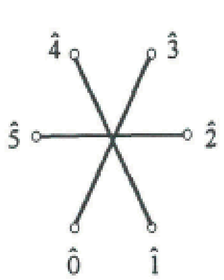
The lattice $\mathcal{C}(Z_6)$ of the circulant graphs has the following atom graphs:



$G(Z_6, \{\hat{1}, \hat{5}\}) = C_6$



$G(Z_6, \{\hat{2}, \hat{4}\}) = 2C_3$



$G(Z_6, \{\hat{3}\}) = 3K_2$

We obtain:

$$G(Z_6,\{\hat{1},\hat{5}\}) \cup G(Z_6,\{\hat{2},\hat{4}\}) \cup G(Z_6,\{\hat{3}\}) = G(Z_6, Z_6^*) = K_6$$

4. The categorical product of circulant graphs. Applications

Theorem (4.1)

The categorical product of two circulant graphs is a circulant graph.

Proof:

Let $G_1 = G(Z_m, S)$, $G_2 = G(Z_n, T)$ be two circulant graphs. It is well known that the set $Z_m \times Z_n$ is a group with respect to the operation

$$(\bar{x}_1, \hat{x}_2) \oplus (\bar{y}_1, \hat{y}_2) = (\overline{x_1 + y_2}, x_1 \hat{+} y_2) \tag{4.1}$$

where $\bar{x}_1, \bar{y}_1 \in Z_m$, $\hat{x}_2, \hat{y}_2 \in Z_n$ and $(\bar{0}, \hat{0})$ is the identity element. It is easy to check that $S \times T = \{(\bar{s}, \hat{t}) : \bar{s} \in S, \hat{t} \in T\}$ is a symbol set into the group $(Z_m \times Z_n, \oplus)$. We show now that

$$G(Z_m, S) \otimes G(Z_n, T) = G(Z_m \times Z_n, S \times T) \tag{4.2}$$

These graphs have the same vertex set. The equivalencies

$$((\bar{x}_1, \hat{x}_2), (\bar{y}_1, \hat{y}_2)) \in E(G_1 \otimes G_2) \Leftrightarrow (\bar{x}_1, \hat{y}_1) \in E(G_1) \wedge (\bar{x}_2, \hat{y}_2) \in E(G_2) \Leftrightarrow$$

$$\Leftrightarrow (\bar{x}_1 - \bar{y}_1 \in S) \wedge (\hat{x}_2 - \hat{y}_2) \in T \Leftrightarrow (\bar{x}_1, \hat{x}_2) - (\bar{y}_1, \hat{y}_2) \in S \times T \Leftrightarrow$$

$$\Leftrightarrow ((\bar{x}_1, \hat{x}_2), (\bar{y}_1, \hat{y}_2)) \in E(G(Z_m \times Z_n, S \times T))$$

prove the equality of the edge sets.

In a similar way we may prove the following result.

Theorem (4.2)

If $G(Z_{n_i}, S_i) \in \mathcal{C}(Z_{n_i})$ and $S_i \in \mathcal{S}(Z_{n_i})$, $i = 1, 2, \ldots, t$, then $\mathop{X}\limits_{i=1}^{t} S_i \in \mathcal{S}(\mathop{X}\limits_{i=1}^{t} Z_{n_i})$ and

$$\mathop{\otimes}_{i=1}^{t} G(Z_{n_i}, S_i) = G(\mathop{X}_{i=1}^{t} Z_{n_i}, \mathop{X}_{i=1}^{t} S_i) \tag{4.3}$$

Theorem (4.3)

Let $G(Z_m, S_i)$, $i = 1, 2, \ldots, s$, and $G(Z_n, T_j)$, $j = 1, 2, \ldots, t$ be circulant graphs. Then

$$G(Z_m, \bigcup_{i=1}^{s} S_i) \otimes G(Z_n, \bigcup_{j=1}^{t} T_j) = \bigcup_{i=1}^{s} \bigcup_{j=1}^{t} [G(Z_m, S_i) \otimes G(Z_n, T_j)] \tag{4.4}$$

and

$$(\bigcup_{i=1}^{s} G(Z_m, S_i)) \otimes (\bigcup_{j=1}^{t} G(Z_n, T_j)) = \bigcup_{i=1}^{s} \bigcup_{j=1}^{t} [G(Z_m, S_i) \otimes G(Z_n, T_j)] \tag{4.5}$$

Proof:

Indeed $\bigcup_{i=1}^{s} S_i \in \mathcal{S}(Z_m)$, $\bigcup_{j=1}^{t} T_j \in \mathcal{S}(Z_n)$ and $S_i \times T_j \in \mathcal{S}(Z_m \times Z_n)$. According to (4.2), the properties of the cartesian product of sets and corollary 3.3. we have

$$G(Z_m, \bigcup_{i=1}^{s} S_i) \otimes G(Z_n, \bigcup_{j=1}^{t} T_j) = G(Z_m \times Z_n, (\bigcup_{i=1}^{s} S_i) \times \bigcup_{j=1}^{t} T_j) =$$

$$G(Z_m \times Z_n, \bigcup_{i,j=1}^{s,t} (S_i \times T_j)) = \bigcup_{i,j=1}^{s,t} (G(Z_m, S_i) \otimes G(Z_n, T_j))$$

and

$$G(Z_m, \bigcup_{i=1}^{s} S_i) \otimes G(Z_n, \bigcup_{j=1}^{t} T_j) = (\bigcup_{i=1}^{s} G(Z_m, S_i)) \otimes (\bigcup_{j=1}^{t} G(Z_n, T_j))$$

Remark.

In a similar way, we obtain the corresponding result for intersection.

The next theorem presents the structure of the categorical product of atom circulant graphs.

Theorem (4.4)

Let $G(Z_m, S)$, $G(Z_n, T)$ be circulant graphs with $|S| \leq 2$, $|T| \leq 2$. The following statements are true:

1) If $|S| = |T| = 1$ and $S=\{\bar{p}\}$, $T=\{\hat{q}\}$ then $m = 2p$, $n = 2q$ and

$$G(Z_{2p}, \{\bar{p}\}) \otimes G(Z_{2q}, \{\hat{q}\}) = 2pqK_2 \tag{4.6}$$

2) If $|S| = 1$, $|T| = 2$, $m = 2p$, $T = \{\hat{b}, -\hat{b}\}$, order $\hat{b} = \beta \neq 2$, then we have:

For $\beta = 2s$,

$$G(Z_{2p}, \{\bar{p}\}) \otimes G(Z_n, \{\hat{q}\}) = p\frac{n}{s}C_{2s} \tag{4.7}$$

For $\beta = 2s + 1$,

$$G(Z_{2p}, \{\bar{p}\}) \otimes G(Z_n, \{\hat{q}\}) = p\frac{n}{2s+1}C_{2s+1} \tag{4.8}$$

3) If $|S| = |T| = 2$, $S = \{\bar{a}, -\bar{a}\}$, $T = \{\hat{b}, -\hat{b}\}$, order $\bar{a} = \alpha \neq 2$, order $\hat{b} = \beta \neq 2$, then

$$G(Z_m, S) \otimes G(Z_n, T) = 2\frac{mn}{[\alpha, \beta]}C_{[\alpha,\beta]} \tag{4.9}$$

where $[\alpha,\beta]$ is the least common multiple of α and β.

Proof:

1) According to (4.2), we have

$$G(Z_{2p}, \{\bar{p}\}) \otimes G(Z_{2q}, \{\hat{q}\}) = G(Z_{2p} \times Z_{2q}, \{(\bar{p}, \hat{q})\}).$$

From $(\bar{p}, \hat{q}) + (\bar{p}, \hat{q}) = (\bar{0}, \hat{0})$ we deduce that order $(\bar{p}, \hat{q}) = 2$ in the group $Z_{2p} \times Z_{2q}$ whose order is $4pq$. It results from theorem 2.3, 3) that

$$G(Z_{2p} \times Z_{2q}, \{(\bar{p}, \hat{q})\}) = 2pqK_2.$$

2) The set $S \times T = \{(\bar{p}, \hat{b}), (\bar{p}, -\hat{b})\}$ and $|Z_{2p} \times Z_n| = 2pn$. In the group $Z_{2p} \times Z_n$, for $\beta = 2s$, order $(\bar{p}, \hat{b}) = 2s$. According to theorem 2.3, 2), it results that

$$G(Z_{2p} \times Z_n, (\bar{p}, \hat{b})) = \frac{2pn}{2s}C_{2s}$$

i.e. (4.7).

For $\beta = 2s + 1$, order $(\bar{p}, \hat{b}) = 2(2s + 1)$, hence

$$G(Z_{2p} \times Z_n, (\bar{p}, \hat{b})) = \frac{2pn}{2(2s+1)} C_{2(2s+1)} = p \frac{n}{2s+1} C_{2(2s+1)}$$

3) We have

$$S \times T = \{(\bar{a}, \hat{b}), (-\bar{a}, -\hat{b})\} \cup \{(-\bar{a}, \hat{b}), (\bar{a}, -\hat{b})\}$$

and $\{(\bar{a}, \hat{b}), (-\bar{a}, -\hat{b})\}, \{(-\bar{a}, \hat{b}), (\bar{a}, -\hat{b})\} \in \mathcal{S}(Z_{2p} \times Z_n)$. Each element of $S \times T$ has the order $[\alpha, \beta]$ in the group $Z_m \times Z_n$ and $|Z_m \times Z_n| = mn$. According to (4.2) and the theorem 2.3, 2), we have

$$G(Z_m, S) \otimes G(Z_n, T) = G(Z_m \times Z_n, S \times T) = G(Z_m \times Z_n, \{(\bar{a}, \hat{b}), (-\bar{a}, -\hat{b})\}) \cup$$
$$G(Z_m \times Z_n, \{(-\bar{a}, \hat{b}), (\bar{a}, -\hat{b})\}) = \frac{mn}{[\alpha,\beta]} C_{[\alpha,\beta]} \cup \frac{mn}{[\alpha,\beta]} C_{[\alpha,\beta]} = 2\frac{mn}{[\alpha,\beta]} C_{[\alpha,\beta]}$$

5. The applications

Theorem (5.1)

Let m, n, p, and q be positive numbers. Then:

$$p K_2 \otimes q K_2 = 2pqK_2 \quad (5.1)$$
$$K_2 \otimes C_{2q} = 2C_{2q} \quad (5.2)$$
$$K_2 \otimes C_{2q+1} = 2C_{2(2q+1)} \quad (5.3)$$
$$C_m \otimes C_n = 2(m, n)C_{[m,n]} \quad (5.4)$$

where (m, n) is the greatest common factor of m and n; [m, n] is the least common multiple of m and n.

Proof:

(5.1) results from theorem 4.4, 1). If in (4.7) we consider $p = 1$, $n = \beta = 2q$, $s = q$, then (5.2) follows. From (4.8), for $p = 1$, $n = 2q + 1$, $s = q$, we obtain (5.3). We can write $C_m = G(Z_m, \{\bar{1}, \overline{m-1}\})$, $C_n = G(Z_n, \{\hat{1}, \widehat{n-1}\})$ hence $\alpha = m$ and $\beta = n$. Since $(m,n)[m,n] = mn$ (5.4) results from (4.4).

Corollary (5.2)

([2]) Let C_m and C_n be cycle graphs with $m, n > 2$. If $(m, n) = 1$, then

$$C_m \otimes C_n = 2C_{mn} \quad (5.5)$$

Problem (5.3)

Decompose the graph $K_m \otimes K_n$ as a union of cycle graphs.

Solution:

It is enough to study the following cases: a) $K_{2p} \otimes K_{2q}$; b) $K_{2p} \otimes K_{2q+1}$; c) $K_{2p+1} \otimes K_{2q+1}$.

a) We consider $K_{2p} = G(Z_{2p}, Z^*_{2p})$, $K_{2q} = G(Z_{2q}, Z^*_{2q})$ and

$$Z^*_{2p} = \{\bar{1}, \bar{2}, \ldots, \overline{2p-1}\} = \{\bar{1}, \overline{2p-1}\} \cup \{\bar{2}, \overline{2p-2}\} \cup \ldots \cup \{\bar{p}\},$$

$$Z^*_{2q} = \{\hat{1}, \hat{2}, \ldots, \widehat{2q-1}\} = \{\bar{1}, \widehat{2q-1}\} \cup \{\hat{2}, \widehat{2q-2}\} \cup \ldots \cup \{\hat{q}\}.$$

Using (4.2) and (4.5), we have:

$$K_{2p} \otimes K_{2q} = G(Z_{2p} \times Z_{2q}, Z^*_{2p} \times Z^*_{2q}) = G(Z_{2p}, \{\bar{1}, \overline{2p-1}\}) \otimes G(Z_{2q}, \{\hat{1}, \widehat{2q-1}\}) \cup$$

$$G(Z_{2q}, \{\bar{1}, \overline{2p-1}\}) \otimes G(Z_{2q}, \{\hat{2}, \widehat{2q-2}\}) \cup \ldots \cup G(Z_{2p}, \{\bar{p}\}) \otimes G(Z_{2q}, \{\hat{q}\}).$$

Now, we use the relations (5.1), (5.2), (5.3), and (5.4).

Cases b) and c) can be approached in the same manner.

In particular, for m = 4, n = 5, we have

$$K_4 = G(Z_4, Z^*_4) = G(Z_4, \{\bar{1}, \bar{3}\}) \cup G(Z_4, \{\bar{2}\}) = C_4 \cup 2K_2,$$

$$K_5 = G(Z_5, Z^*_5) = G(Z_4, \{\hat{1}, \hat{4}\}) \cup G(Z_5, \{\hat{2}, \hat{3}\}) = C_5 \cup C_5 = 2C_5$$

Using (4.4), (5.5), and (5.3), we obtain

$$K_5 \otimes K_5 = (C_4 \cup 2K_2) \otimes 2C_5 = 2C_4 \otimes C_5 \cup 4K_2 \otimes C_5 = 2C_{20} \cup 4C_{10}.$$

References

1. St. Antohe, **Quasi-Cayley Graphs**, Analele Univ. of Galati, fasc. II, pp. 28-32, 1989.
2. St. Antohe and E. Olaru, **On the Structure of Quasi-Cayley Graphs. Decomposition Theorems**, Libertas Mathematica, vol. XIV, Arlington, Texas, pp. 137-149, 1994.
3. N. L. Biggs, **Algebraic Graph Theory**, Cambridge Univ. Press, 1974.
4. M. Farzan and D. Waller, **Kronecker Products and Local Joins of Graphs**, Can. J. Math., vol. XXX, No. 2, pp. 255-269, 1977.
5. C. Nastasescu, **Teoria dimensiunii in algebra necomutativa**, Ed. Acad. R. S. Romania, Bucuresti, 1983.
6. D. D. Popescu and C. Vraciu, **Elemente de teoria grupurilor finite**, Editura Stiintifica si Enciclopedica, Bucuresti, 1986.
7. H. P. Yap, **Total Colourings of Graphs**, Springer, 1996.