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ON A SEPARABLE NONLINEAR VOLTERRA INTEGRAL EQUATION WITH POWER NONLINEARITY

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Abstract. The nonlinear Volterra integral equation

$$\varphi^m(x) = a(x)\int_0^x b(t)\varphi(t)dt + f(x)$$

with separable kernel of rank 1 with real functions $a(x)$, $b(x)$ and $f(x)$ is studied. Closed form solutions are given for special cases. Inequalities are used to estimate the solution $\varphi(x)$. Existence and uniqueness results are given for a separable kernel of rank n.

1. Introduction. Consider the Volterra nonlinear convolution integral equation

(1) $\varphi^m(x) = a(x)\int_0^x k(x-t)\varphi(t)dt + f(x)$, with $m \in \mathbb{R}$.

Equation (1) arises in nonlinear theory of wave propagation J. J. Keller [5] and water perlocation Okransinski [7]. Equation (1) is studied in Karapetyants, Kilbas and Saigo [4] where a list of related articles are given. We study a similar integral equation with a separable kernel of rank 1.

(2) $\varphi^m(x) = a(x)\int_0^x b(t)\varphi(t)dt + f(x),\ m \in \mathbb{R}.$

First we note that if we let $m = -1$, equations (1) and (2) are related to Chandrasekhar H-function given by (3) in the theory of radiative transfer [2].

(3) $H(\mu) = 1 + \mu H(\mu)\int_0^1 \frac{\psi(\mu')}{\mu+\mu'} H(\mu')d\mu'.$

Divide both sides of (3) by $H(\mu)$ to obtain

(4) $(H(\mu))^{-1} = 1 - \mu\int_0^1 \frac{\psi(\mu')}{\mu+\mu'} H(\mu')d\mu'.$

For a general reference to the theory of integral equations we recommend Corduneanu [3]. In (3) if we let $\eta(x) = \varphi^m(x)$ and $p = \frac{1}{m},\ m \neq 0$, we obtain

(5) $\eta(x) = a(x)\int_0^x b(t)\eta^p(t)dt + f(x).$

A generalization of (5) is treated in Bushell [1]. Special case of (2) with $m = -1$ is discussed in Nestell and Ghandehari [6].

In Section 2, we obtain a differential equation by differentiating (2). Under various assumptions, we discuss existence, uniqueness results. In Section 3 we consider special cases of (2) and obtain closed form solutions. Inequality related to (2) are discussed in section 4.

2 Existence and Uniqueness.

In equation (2) assume a, b, φ, f are differentiable for $x \geq 0$. Differentiate both sides of (2) with respect to x to obtain

(6) $m\varphi_{(x)}^{m-1}\varphi'_{(x)} = a'(x)\int_0^x b(t)\varphi(t)dt + a(x)b(x)\varphi(x) + f'(x).$

use (2) to obtain (7) and substitute on (6) to obtain (8).

(7) $$\int_0^x b(t)\varphi(t)dt = \frac{\varphi^m_{(x)} - f(x)}{a(x)},$$

(8) $$m\varphi^{m-1}_{(x)}\varphi'_{(x)} = a'(x)\frac{\varphi^m_{(x)} - f(x)}{a(x)} + a(x)b(x)\varphi(x) + f'(x).$$

Simplify (8) to obtain (9).

(9) $$\varphi'(x) = \frac{1}{m}\left[a'(x)\varphi(x) + a'(x)b(x)\varphi(x)^{2-m} + (a(x)f'(x) - f(x)b'(x))\varphi^{m-1}_{(x)}\right].$$

Now we consider cases and use (5) to provide existence and uniqueness results.

Case 1. If $m = 0$ the integral equations (2) reduces to the theory of linear integral equations.

Case 2. If $m = -1$, $f(x) = 1$ this case is treated in Nestell and Ghandehari [6].

Case 3. The general power with $m < 0$ is discussed below.

Let

$$g(x,\varphi) = \frac{1}{m}\left[a'(x)\varphi(x) + a'(x)b(x)\varphi^{2-m} + (a(x)f'(x) - f(x)b'(x))\varphi^{m-1}_{(x)}\right].$$

The right handside is continuous as a function of φ. Thus, there is a solution of (9) provided a and f are differentiable and c is continuous in a rectangle containing the initial condition. For uniqueness we need $\dfrac{\partial g}{\partial \varphi}$ to be continuous. For $m < 0$ this is satisfied.

Case 4. If $0 < m < 1$ there is a solution. Uniqueness theorem does not apply.

Case 5. If $m \geq 1$, existence, uniqueness theorem does not apply.

We now discuss the nonlinear Volterra equation given below with separable kernel of rank n.

(10) $\varphi^m(x) = \int_0^x (\sum_{i=1}^{n} A_i(x)B_i(y))\varphi(y)\, dy.$

Let

(11) $\phi_i(x) = \int_0^x B_i(y)\varphi(y)\, dy.$

Then interchanging summation and integral signs in (10) and using (11) we obtain

(12) $\varphi^m(x) = \sum_{i=1}^{n} A_i(x)\phi_i(x)\,.$

Also, by differentiating (11) and using (12) we get

(13) $\phi_i'(x) = B_i(x)(\sum_{i=1}^{n} A_i(x)\phi_i(x))^{\frac{1}{m}}\,.$

Using a standard existence and uniqueness theorem for systems, we conclude that if $0 < m < 1$ the system (13) has a unique solution as long as $A_i(x)$ and $B_i(x)$ are continuous for all $i,\ x \geq 0$.

We conclude this section by considering a special case before discussing closed form solutions. In (9) if we let $m = -1$ we obtain Abel's differential equation

(14) $\varphi'(x) = -\,[a'(x)\varphi(x) + a'(x)b(x)\varphi^3(x)]$ where we have used $f(x) = 0$.

3. Closed Form Solutions.

Let $a(x)b(t) = e^{x-t}$, $f(x) = 1$. Then equation (9) reduces to

$$\varphi'(x) = \frac{1}{m}e^x\left[\varphi + \varphi^{2-m} - \varphi^{1-m}\right]$$

which is separable, to obtain

(15) $$\frac{1}{m}e^{x} = \int \frac{d\varphi}{\varphi(1+\varphi^{1-m}-\varphi^{-m})}$$

In this case if m is a negative integer using the fundamental theorem of algebra the right hand side of (15) can be integrated by partial fractions as sums of logarithms.

Now consider the homogeneous case

(16) $$\varphi^{m}(x) = \int_0^x A(x)B(y)\varphi(y)\,dy.$$

Equation (16) can be reduced to Bernaulli's ordinary differential equation as follows. Differentiate both sides of (16) to get

(17) $$m\varphi(x)^{m-1}\varphi'(x) = A'(x)\int_0^x B(y)\varphi(y)\,dy + A(x)B(x)\varphi(x).$$

Use (16) to solve for $\int_0^x B(y)\varphi(y)\,dy$ to obtain

(18) $$\int_0^x B(y)\varphi(y)\,dy = \frac{\varphi(x)^m}{A(x)}\ .$$

Substitute (18) in (17) to find

(19) $$m\varphi(x)^{m-1}\varphi'(x) = A'(x)\cdot\frac{\varphi(x)^m}{A(x)} + A(x)B(x)\varphi(x).$$

simplify (19) to obtain (20) which is a Bernoulli equation

(20) $$\varphi'(x) - \frac{1}{m}\frac{A'(x)}{A(x)}\varphi(x) = \frac{1}{m}A(x)B(x)\varphi(x)^{2-m}.$$

The solution to (20) can be obtained by letting $\eta(x) = \varphi^{1-(2-m)} = = \varphi^{m-1}$. Then we obtain a first order linear differential equation for η. After solving for η we find φ.

4. <u>Inequalities.</u>

We prove the following theorem.

<u>Theorem.</u> Let $m, \alpha, \mu \in R$ such that $m > 1$, $0 < \alpha \leq 1$, $m + \alpha + \mu > 1$. Assume $a(n)$, $b(n)$ satisfy

$$a(x) \geq ax^{\mu},\ b(x) \geq bx^{\alpha-1},\ x \in (0, \infty)$$

then $\varphi(x) \geq \left(\frac{\alpha b(m-1)}{\mu+\alpha+m-1}\right)^{\frac{1}{m-1}} x^{(\mu+\alpha)/(m-1)}$.

<u>Proof.</u>

The proof is similar to [4]. However the convolution kernel is used there. Assume φ is a positive solution

$$\varphi^m \geq ax^{\mu} \int_0^x by^{\alpha-1}\varphi(y)dy$$

Since $y < x$ and $\alpha - 1 \leq 0$ then $y^{\alpha-1} \geq x^{\alpha-1}$. Thus $\varphi^m \geq abx^{\mu} \cdot x^{\alpha-1} \int_0^x \varphi(y)dy$

(21) $$\varphi^m \geq ab\, x^{\alpha+\mu-1} \int_0^x \varphi(y)dy.$$

Let $y = \int_0^x \varphi(t)dt$, then

$$y'^m \geq ab\, x^{\alpha+\mu-1} y,$$

$$y' y^{-1/m} \geq (ab)^{1/m} x^{(\alpha+\mu-1)/m}$$

integrating the latter inequality on (a, x) and using $m + \alpha + \mu > 1$, $m > 1$, and $y = \int_0^x \varphi(t)dt$, we arrive at

(22) $$y(x) \geq (ab)^{1/m-1} \left(\frac{m-1}{\mu+\alpha+m-1}\right)^{m/m-1} x^{\{(\mu+\alpha)/(m-1)\}+1}.$$

Substitute (2) into (1) we obtain

$$\varphi(x) \geq \left(\frac{ab(m-1)}{\mu+\alpha+m-1}\right)^{1/m-1} x^{(\mu+\alpha)/(m-1)}$$

which is the desired result.

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