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SOME COMMENTS ON THE THEORY OF RIEMANNIAN SUBMANIFOLDS

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This paper is dedicated to Radu Rosca on the occasion of his 90th birthday. It is a small token of my great respect for him as an inspiring mathematician and as a wise guide to life in general.

In 1971, at the Mathematical Institute of the Romanian Academy at Bucharest, Radu Rosca introduced me into the modern differential geometry of submanifolds. My doctoral work was concerned mainly with one of the notions introduced by Radu Rosca in this field, namely that of the so called pseudo–nul or pseudo–isotropic submanifolds (i.e. submanifolds with ametrical Gauss maps). The purpose of this text is to give a somewhat loose formulation of some personal opinions on the scope at present of the field of submanifold theory, or, more precisely, on the theory of Riemannian submanifolds. These opinions are based on some results of the research done since 1993 by Bang–yen Chen, who directed my post–doctoral work in submanifold theory at Michigan in 1975–1976. They concern essentially the introduction for any Riemannian manifold of two series of new scalar valued Riemannian curvature invariants, called δ–curvatures, and the discovery of optimal inequalities relating each individual δ–curvature with the mean curvature of any isometric immersion of this Riemannian manifold in a Euclidean space. These results of Bang–yen Chen started off by answering a crucial question of Shiing–shen Chern concerning Riemannian obstructions to minimal immersibility in Euclidean spaces, a question in perfect accordance with the criteria to be satisfied by "good mathematical problems" in the sense of David Hilbert.

The theorema egregium of C.F. Gauss (1827) amongst others brought about a deep novel understanding of the differential geometrical properties of the surfaces M^2 which live in the Euclidean world $\mathbb{E}^3$. It allowed to make distinction between the intrinsic and the extrinsic features of surfaces M^2 in $\mathbb{E}^3$, and made the investigations of the interactions between their intrinsic and extrinsic geometries an important field of study.

In the vast extension of the intrinsic geometry of surfaces M^2 in $\mathbb{E}^3$ which constitutes the universe of "modern differential geometry" as the mathematics of differential manifolds M^n endowed with some geometrical structure, the world of Riemannian geometry might at this stage be considered as the earth. In Riemannian geometry (1854), differential manifolds M^n are endowed with a Riemannian metric g, which is probably the simplest geometrical structure to study in accordance with the natural geometry of our human experiences, (simplest at least from the technical point of view and at this moment, versus the maybe still more plausible geometrical structures, of which Finsler geometry may be the prototype version, which focuss directly on abstractly defined measures of lengths on M^n, rather than on scalar products g first, which then, in their turn, automatically determine nice length measures). During one and half century, the study of abstract Riemannian spaces has been in the center of important researches in pure and applied mathematics.

The imbedding theorem of J. Nash (1954) made clear, even for less sophisticated observers of mathematical life, that the species which originated in B. Riemann's fertile brain from the seed of the theorema egregium, is not at all extraterestrial, but that its members, up to dimensional size, are qualitatively most clearly of the same family as the surfaces M^2 with which also these observers are well aquanted since their youth, when growing up in their neighborhoud $\mathbb{E}^3$. So, for about half a century, it was known that Riemannian manifolds (M^n, g) are living as submanifolds in Euclidean spaces $\mathbb{E}^m$. However, there was no systematic quantitative information available on their life styles there.

The question of S.S. Chern, mentioned before, was crucial towards a better understanding of the behaviour of the members (M^n, g) of the Riemannian family with respect to the tension they can not avoid to experience in the environment $\mathbb{E}^m$ in which they live. The answers to this question given recently by B.-y. Chen disclosed the DNA structure itself of the Riemannian beings (M^n, g), with their δ-curvatures as main genetic characteristics. Moreover, the links between these intrinsic and the most important extrinsic quantities of Riemannian submanifolds as given by the Chen inequalities, in my view, "close a circle of mathematical evolution", in the following sense. Rudimentally stated, the purpose of the differential geometry of surface M^2 in $\mathbb{E}^3$ is to describe their shape. In the course of its development, various kinds of curvatures turned out to be of greater value than others, and the principal curvatures, the mean curvature and the Gauss curvature rightfully gained a high status of merit in this respect. C. F. Gauss succeeded to prove that the Gauss curvature equals the product of the principal curvatures and that it remains unchanged by those deformations of a surface M^2 in $\mathbb{E}^3$ which preserve the distance between any two points on a surface such as this distance is measured along connecting paths fully lying within this surface. The intrinsic geometry of surfaces M^2 in $\mathbb{E}^3$ as the geometry of such surfaces M^2 with as group of transformations the one consisting of such isometries of surfaces, was generalized to abstract Riemannian geometry. The theorem of J. Nash proved the concrete realizability of all Riemannian manifolds (M^n, g) as submanifolds in Euclidean spaces $\mathbb{E}^m$. The δ-curvatures of B.-y. Chen, while giving on the one hand the intrinsic blue print of any Riemannian manifold (M^n, g), via B.-y. Chen's inequalities also show on the other hand to have a strong impact on the possible shapes that (M^n, g) can assume as a submanifold in $\mathbb{E}^m$, i.e. on its extrinsic geometry.

Thus, the Riemannian geometry can now largely be seen as the theory of Riemannian submanifolds (M^n, g) of Euclidean spaces $\mathbb{E}^m$, which conceptually is indeed nothing but the differential geometry of surfaces M^2 in $\mathbb{E}^3$, practiced now of course for arbitrary dimensions and codimensions.

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