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ON THE INTEGRABILITY OF SOME DISTRIBUTIONS DEFINED BY A PSEUDO f-STRUCTURE ON A VECTOR BUNDLE OF ALMOST PRODUCT MANIFOLDS

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Dedicated to Professor Radu Rosca on his 90th birthday anniversary.

Abstract. *This paper introduce a pseudo f-structure on a certain vector bundle on an almost product manifold. And we mainly study integrable conditions of certain distributions defined from this pseudo f-structure.*

Introduction. On 1976, one of the authors introduced the notion of pseudo f-structure ([M]) which is closely related with an almost product structure ([Y]) and an almost paracontact structure ([Y]).

In this paper, we define a vector bundle over an almost product manifold. Then we introduce a pseudo f-structure on this vector bundle.

And we consider integrability conditions of certain distributions on the vector bundle which admits this structure.

1. PSEUDO f-STRUCTURE ON A VECTOR BUNDLE OVER AN ALMOST PRODUCT MANIFOLD.

Let M^n be an n-dimensional differentiable manifold with an almost product structure H ;

$$H^2 = I, \qquad (H \neq I) \tag{1.1}$$

where I denotes the identity matrix of order n. Then we know that the eigenvalues of H are 1 and -1.

Let P (resp. Q) be the projection operator of H with respect to the eigenvalue 1 (resp. -1). Then we have

$$P + Q = I, \quad PQ = QP = 0, \quad P^2 = P, \quad Q^2 = Q$$

We put

$$D^1 = \mathrm{Im}P, \quad \dim D^1 = p \quad \text{and} \quad D^2 = \mathrm{Im}Q, \quad \dim D^2 = n - p = q,$$

where p (resp q) is the multiplicity of 1 (resp. -1).

Now, we define a vector bundle $\eta = (D_1(M^n), \pi^1, M^n)$ which π^1 denotes the bundle projection.

Let U be a local coordinate neighbouhood with a coordinate system $(x^1, x^2, ..., x^n)$ in M^n. Then we can represent

$$P = p_i{}^j \partial_j \otimes dx^i, \quad Q = q_i{}^j \partial_j \otimes dx^i, \tag{1.2}$$

where $(p_i{}^j)$ (resp. $q_i{}^j$) is a matrix of odrer n which the rank is p (resp. q) and $\partial_i = \frac{\partial}{\partial x^i}$.

Hereafter, we assume that the first minor matrix $(p_a{}^b)$ of order p of P and the last minor matrix $q_u{}^v$ of order q of Q are non-singular, where the indices $a, b, ..., c$ (resp, $u, v, ..., w$) run over the range 1,2,...,p (resp.p+1,p+2,...,n).

For each $x \in U$, we define a n vectors $\{X_1, X_2, ..., X_n\}$ as follows;

$$X_a = p_a{}^i \partial_i, \quad X_u = q_u{}^i \partial_i. \tag{1.3}$$

Since $\{X_a\}$ (resp. $\{X_u\}$) is a basis of $D_x{}^1$ (resp. $D_x{}^2$), the vector $\{X_i\}$ is a basis of the tangent vector space $T_x M^n$ at x of M^n.

Next, let $\{\theta^i\} = \{\gamma_j{}^i dx^j\}$ be the dual basis of $\{X_i\}$. Then the matrix $(\gamma_j{}^i)$ is the inverse one of $(p_a{}^i, q_u{}^i)$, that is,

$$\gamma_j{}^i p_a{}^j = \delta_a{}^i, \quad \gamma_j{}^i q_u{}^j = \delta_u{}^i, \quad \gamma_j{}^a p_a{}^i + \gamma_j{}^u q_u{}^i = \delta_j{}^i. \tag{1.4}$$

Now, for $\sigma \in (\pi^1)^{-1}(U) = U \times \boldsymbol{R}^p$ ($\boldsymbol{R}^p$ being the p-dimensional number space), we can put $\sigma = \xi^a X_a$. Then, $(x^1, x^2, ..., x^n, \xi^1, \xi^2, ..., \xi^p)$ is a local coordinate system on $(\pi^1)^{-1}(U)$. Thus $D^1(M^n)$ is an $(n+p)$-dimensional differentiable manifold. The coordinate transformation of $\{U \times \boldsymbol{R}^p\} \cap \{U' \times \boldsymbol{R}^p\}$ with local coordinates system $(y^1, y^2, ..., y^{n+p}) = (x^1, x^2, ...x^n, \xi^1, \xi^2, ..., \xi^p)$ and $(y'^1, y'^2, ..., y'^{n+p}) = (x'^1, x'^2, ..., x'^n, \xi'^1, \xi'^2, ..., \xi'^p)$ is given by

$$y^i = y^i(y') \quad (x^i = x^i(x')), \quad y^a = y'^b p'_b{}^j \frac{\partial x^i}{\partial x'^j} \gamma_i{}^a \quad (\xi^a = \xi'^b p'^j_b \frac{\partial x^i}{\partial x'^j} \gamma_i{}^a).$$

So, the Jacobian matrix of this transformation is given by

$$\left(\frac{\partial y^\alpha}{\partial y'^\beta}\right) = \begin{pmatrix} \frac{\partial y^i}{\partial y'^j} & \frac{\partial y^a}{\partial y'^j} \\ \frac{\partial y^b}{\partial y'^i} & \frac{\partial y^a}{\partial y'^b} \end{pmatrix},$$

where

$$\begin{cases} \dfrac{\partial y^i}{\partial y'^j} = \dfrac{\partial x^i}{\partial x'^j}, \\ \dfrac{\partial y^a}{\partial y'^j} = \dfrac{\partial p'^k_c}{\partial x'^j} \dfrac{\partial x^i}{\partial x'^k} \gamma_i{}^a \xi'^c + p'_c{}^k \dfrac{\partial^2 x^i}{\partial x'^j \partial x'^k} \gamma_i{}^a \xi'^c + p'_c{}^k \dfrac{\partial x^i}{\partial x'^k} \dfrac{\partial \gamma^a_i}{\partial x^l} \dfrac{\partial x^l}{\partial x'^j} \xi'^c + p'_c{}^k \dfrac{\partial x^i}{\partial x'^k} \gamma_i{}^a \dfrac{\partial \xi'^c}{\partial x'^j}, \\ \dfrac{\partial y^b}{\partial y'^i} = 0, \\ \dfrac{\partial y^a}{\partial x'^b} = p'^j_b \dfrac{\partial x^i}{\partial x'^j} \gamma_i{}^a \end{cases}$$

The tangent space $T_\sigma(D^1(M^n))$ at a point σ of $D^1(M^n)$ is expressed as

$$T_\sigma(D^1(M^n)) = T_x(M^n) \oplus F_x = D^1_x \oplus D^2_x \oplus F_x, \tag{1.5}$$

where $T_x(M^n)$, $D_x{}^1$ (resp. $D_x{}^2$) and F_x are respectively the tangent vector space at x of M^n, the tangent plane at x belong to D^1 (resp. D^2) and $F_x = (\pi^1)^{-1}(x)$ which is the fibre of $x \in M$.

Let $j : D_x{}^1 \to F_x$ be the natural projection. Then, for $X = \mu^a X_a$, we have $j(X) = \mu^a \partial_a$, where $\partial_a = \partial/\partial\xi^a$.

Now, let ω^* be a linear connection with local components $\Gamma_j{}^a{}_b$ with respect to the local coordinate system $\{x^i, \xi^a\}$ on η^1.

For $X \in T_x(M^n)$, let $X^{\mathcal{H}}$ be the horizontal lift of X with respect to ω^*. Then, we define a linear map F_σ of $T_\sigma(D^1(M^n))$ as follows;

$$\begin{cases} F_\sigma(X^{\mathcal{H}}) = 0, & \text{for X} \in D_x^2 \\ F_\sigma(X^{\mathcal{H}}) = j(X), & \text{for X} \in D_x^2 \\ F_\sigma(X^{\mathcal{H}}) = (j^{-1}(X))^{\mathcal{H}}, & \text{for X} \in F_x \end{cases} \tag{1.6}$$

Then we can easily prove that

Proposition 1.1. *The linear map F_σ defined by (1.6) is a pseudo-f-structure on $D^1(M^n)$, that is, it satisfies $F_\sigma{}^3 - F_\sigma = 0$.*

Remark. Hereafer, we neglect the index σ in F_σ.

Since

$$\partial_i^{\mathcal{H}} = \partial_i - \Gamma_i{}^a{}_b \xi^b \partial_a,$$

we have

$$X^{\mathcal{H}} = X - \Gamma_i{}^a{}_b \xi^b X^i \partial_a$$

for any $X \in T_x M$, where $\partial_a = \frac{\partial}{\partial \xi^a}$.

Especially, for $X = \mu^a X_a = \mu^a p_a{}^j \partial_j \in D_x{}^1$, we have

$$X^{\mathcal{H}} = \mu^a p_a{}^j \partial_j - \Gamma_i{}^a{}_b \xi^b \mu^c p_c{}^i \partial_a, \tag{1.7}$$

and for $X = \mu^u X_u = \mu^u q_u{}^i \partial_i \in D_x^2$, we get

$$X^{\mathcal{H}} = \mu^u q_u{}^i \partial_i - \Gamma_i{}^a{}_b \xi^b \mu^u q_u{}^i \partial_a. \tag{1.8}$$

Now, we put

$$F = (F_\beta{}^\alpha) = \begin{pmatrix} F_j{}^i F_j{}^a \\ F_b{}^i F_b{}^a \end{pmatrix}, \tag{1.9}$$

where $\alpha, \beta, ..., \gamma$ run over the range $\{1, 2, ..., n+p\}$.

Then, by the straightfoward calculation, we obtain

$$(F_j{}^i - F_b{}^i \Gamma_j{}^b) q_u{}^j = 0, \tag{1.10}$$

$$(F_j{}^a - F_b{}^a \Gamma_j{}^b) q_u{}^j = 0, \tag{1.11}$$

$$(F_j{}^i - F_b{}^i \Gamma_j{}^b) p_a{}^j = 0, \tag{1.12}$$

$$(F_j{}^a - F_c{}^a \Gamma_j{}^c) p_b{}^j = \delta_b{}^a \tag{1.13}$$

$$F_b{}^i = p_a{}^i, \quad F_b{}^a = -\Gamma_j{}^a p_b{}^j, \tag{1.14}$$

where we put $\Gamma_j{}^a = \Gamma_j{}^a{}_b \xi^b$.

From (1.4) and the above equations, we easily obtain

$$F_j{}^i = F_b{}^i\Gamma_j{}^b = p_b{}^i\Gamma_j{}^b, \quad F_j{}^a = F_b{}^a\Gamma_j{}^b = \gamma_j{}^a - \Gamma_l{}^a p_c{}^l\Gamma_j{}^c.$$

Thus, we can sum up these as follows;

$$(F_\beta{}^\alpha) = \begin{pmatrix} F_j{}^i & F_j{}^a \\ F_b{}^i & F_b{}^a \end{pmatrix} = \begin{pmatrix} p_b{}^i\Gamma_j{}^b & \gamma_j{}^a - \Gamma_k{}^a\Gamma_j{}^c p_c{}^k \\ p_b{}^i & -\Gamma_j{}^a p_b{}^j \end{pmatrix}.$$

From (1.15), we can easily see that $F^3 - F = 0$. This means that F is a pseudo-f-structure. Thus we have

Theorem 1.2. *If an n-differentiable manifold M^n admits an almost product structure H, then there exists a pseudo-f-structure F in the bundle space of the $(n+p)$-dimensional vector bundle $D^1(M^n)$ over M^n and if a connection ω' with components $\Gamma_j{}^a{}_b$ in $D^1(M^n)$ is given, then the pseudo-f-structure $F = (F_\beta{}^\alpha)$ should be determined by* (1.15).

2. NIJENHUIS TENSOR OF $F_\beta{}^\alpha$.

In this section, we caluculate the Nijenhuis tensor of the pseudo-f-structure F which satisfies (1.15)

Let $N_\gamma{}^\alpha{}_\beta$ of F is defined by

$$N_\gamma{}^\alpha{}_\beta = F_\gamma{}^\delta\partial_\delta F_\beta{}^\alpha - F_\beta{}^\delta\partial_\delta F_\gamma{}^\alpha - (\partial_\gamma F_\beta{}^\delta - \partial_\beta F_\gamma{}^\delta)F_\delta{}^\alpha. \tag{2.1}$$

Using (1.15), we do straightfoward calculation. Then we get

$$\left\{\begin{array}{l} N_j{}^h{}_i = -S_j{}^h{}_i + p_b{}^k p_c{}^h(\Gamma_j{}^b R_{ki}{}^c - \Gamma_i{}^b R_{kj}{}^c) + \Gamma_j{}^c\Gamma_i{}^b S_c{}^h{}_b, \\ N_j{}^a{}_i = -R_{ji}{}^a + \Gamma_k{}^a S_j{}^k{}_i - \Gamma_j{}^c\gamma_i{}^b p_c{}^k p_b{}^l R_{kl}{}^a - \Gamma_l{}^a p_c{}^k p_b{}^l(\Gamma_j{}^c R_{ki}{}^b - \Gamma_i{}^c R_{kj}{}^b) \\ \qquad +p_c{}^k(\Gamma_j{}^c S_k{}^a{}_i - \Gamma_i{}^c S_k{}^a{}_j) - \Gamma_l{}^a\Gamma_j{}^c\Gamma_i{}^b S_c{}^l{}_b, \\ N_j{}^h{}_b = \Gamma_j{}^c S_c{}^h{}_b - p_b{}^k p_c{}^h R_{kj}{}^c, \\ N_j{}^a{}_b = -p_b{}^k S_k{}^a{}_j + p_b{}^k\Gamma_l{}^a p_c{}^l R_{kj}{}^c - \Gamma_j{}^c\Gamma_l{}^a S_c{}^l{}_b - p_c{}^k\Gamma_j{}^c p_b{}^l R_{kl}{}^a, \\ N_c{}^h{}_b = S_c{}^h{}_b, \\ N_c{}^a{}_b = -\Gamma_j{}^a S_c{}^j{}_b - p_c{}^k p_b{}^j R_{kj}{}^a, \end{array}\right. \tag{2.2}$$

where we put

$$\left\{\begin{array}{l} S_j{}^h{}_i = p_d{}^h\{\partial_j\gamma_i{}^d - \partial_i\gamma_j{}^d - (\gamma_j{}^a\Gamma_i{}^d{}_a - \gamma_i{}^a\Gamma_j{}^d{}_a),\} \\ S_c{}^h{}_b = p_c{}^k\partial_k p_b{}^h - p_b{}^k\partial_k p_c{}^h - (p_c{}^k\Gamma_k{}^d{}_b - p_b{}^k\Gamma_k{}^d{}_c), \\ S_j{}^a{}_i = \partial_j\gamma_i{}^a - \partial_i\gamma_j{}^a - (\gamma_j{}^b\Gamma_i{}^a{}_b - \gamma_i{}^b\Gamma_j{}^a{}_b), \\ R_{jib}{}^a = \partial_j\Gamma_i{}^a{}_b - \partial_i\Gamma_j{}^a{}_b + \Gamma_j{}^a{}_d\Gamma_i{}^d{}_b - \Gamma_i{}^a{}_d\Gamma_j{}^d{}_b, \\ R_{ji}{}^a = R_{jib}{}^a\xi^b. \end{array}\right. \tag{2.3}$$

Remark. In general, for a (1,1) tensor field F, the Nijenhuis tensor N_F with respect to F is defined as

$$N_F(Y,X)=[FY,FX]-F[FY,X]-F[Y,FX]+F^2[X,Y]$$

for any vector fields X and Y, where [,] denotes the Lie bracket.

From (2.1) and (2.1), we obtain

Theorem 2.1. *A necessary and sufficient condition for the pseudo-f-structure induced from an almost product structure be integreble with respect to* ω^*, *that is,* $N_\gamma{}^\alpha{}_\beta=0$, *is that the tensor* $S_k{}^i{}_j$ *(or equivalently* $S_k{}^a{}_j$*),* $S_c{}^h{}_b$ *vanish identically and the connection* ω^* *is of zero curvature.*

3. INTEGRABILITY CONDITIONS

For the pseudo-f-structure F on $D^1(M^n)$, we put

$$L=F^2,\quad M=-L+I=-F^2+I. \tag{3.1}$$

Then we have easily seen that

$$L+M=I,\quad LM=ML=0. \tag{3.2}$$

Definition 3.1. (i) The pseudo-f-structure F is said to be *partially integrable* if its Nijenhuis tensor $N_\gamma{}^\alpha{}_\beta$ satisfies $L_\gamma{}^\delta L_\beta{}^\omega N_\delta{}^\alpha{}_\omega=0$. (ii) L (resp. M) is *integrable* if the Nijenhuis tensor $N_\gamma{}^\alpha{}_\beta$ satisfies $M_\delta{}^\alpha N_\gamma{}^\delta{}_\beta=0$ (resp. $M_\gamma{}^\delta M_\beta{}^\omega N_\delta{}^\alpha{}_\omega=0$).

On $D^1(M^n)$, we define an almost product structure K as

$$K=M-L=-2F^2+I. \tag{3.3}$$

Next, we define $\bar P$ and $\bar Q$ from K as follows;

$$\bar P=\frac{1}{2}(I-K),\quad \bar Q=\frac{1}{2}(I+K). \tag{3.4}$$

Definition 3.2. $\bar P$ (resp. $\bar Q$) is *completely integrable* if the Nijenhuis tensor $\tilde N$ of K satisfies $\tilde N_\gamma{}^\alpha{}_\beta+\tilde N_\gamma{}^\delta{}_\beta K_\delta{}^\alpha=0$ (resp. $\tilde N_\gamma{}^\alpha{}_\beta-\tilde N_\gamma{}^\delta{}_\beta K_\delta{}^\alpha=0$).

We can calculate the followings;

$$L \text{ is integrable} \Longleftrightarrow F_\omega{}^\delta F_\gamma{}^\beta(\partial_\delta M_\beta{}^\alpha-\partial_\beta M_\delta{}^\alpha)=0. \tag{3.5}$$

$$M \text{ is integrable} \Longleftrightarrow L_\beta{}^\alpha\{M_\gamma{}^\delta\partial_\delta M_\omega{}^\beta-M_\omega{}^\delta\partial_\delta M_\gamma{}^\beta\}=0. \tag{3.6}$$

Next, by the straightfoward calculation, we get

$$\begin{cases}\partial_j M_i{}^h=-\partial_j(p_a{}^h\gamma_i{}^a),\\ \partial_j M_i{}^a=-\partial_j\Gamma_i{}^a+\Gamma_k{}^a p_d{}^k\partial_j\gamma_i{}^d+\gamma_i{}^d(p_d{}^k\partial_j\Gamma_k{}^a+\Gamma_k{}^a\partial_j p_d{}^k),\\ \partial_d M_j{}^a=\gamma_j{}^b p_b{}^k\Gamma_k{}^a{}_d-\Gamma_j{}^a{}_k,\\ \text{the other cases}=0.\end{cases} \tag{3.7}$$

Moreover, we obtain

$$(L_\beta{}^\alpha) = \begin{pmatrix} L_j{}^i L_j{}^a \\ \\ L_b{}^i L_b{}^a \end{pmatrix} = \begin{pmatrix} \gamma_j{}^a p_a{}^i \Gamma_j{}^a - \gamma_j{}^b p_b{}^k \Gamma_k{}^a \\ \\ 0 \quad \delta_b{}^a \end{pmatrix}, \tag{3.8}$$

$$(M_\beta{}^\alpha) = \begin{pmatrix} M_j{}^i M_j{}^a \\ \\ M_b{}^i M_b{}^a \end{pmatrix} = \begin{pmatrix} \delta_j{}^i - \gamma_j{}^a p_a{}^i \gamma_j{}^b p_b{}^k \Gamma_k{}^a - \Gamma_j{}^a \\ \\ 0 \quad 0 \end{pmatrix}, \tag{3.9}$$

$$(K_\beta{}^\alpha) = \begin{pmatrix} K_j{}^i K_j{}^a \\ \\ K_b{}^i K_b{}^a) \end{pmatrix} = \begin{pmatrix} \delta_j{}^i - 2\gamma_j{}^a p_a{}^i 2(-\Gamma_j{}^a + \gamma_j{}^b p_b{}^k \Gamma_k{}^a) \\ \\ 0 \quad -\delta_b{}^a \end{pmatrix}. \tag{3.10}$$

Let the distribution L be integrable. Then we know $M_\delta{}^\alpha N_\gamma{}^\lambda{}_\beta = 0$ by Definition 3.1. This is equivalent to the followings;

$$\gamma_k{}^u S_j{}^k{}_i = 0, \quad \gamma_k{}^u S_c{}^k{}_b = 0. \tag{3.11}$$

Thus we have

Theorem 3.1. *The distribution L induced from the pseudo-f-structure F on the vector bundle $D^1(M^n)$ is integrable if and only if* (3.11) *is satisfied.*

Corollary 3.2. *If the tensor fields $S_j{}^h{}_i$ and $S_c{}^h{}_b$ vanish, then the distribution L is integrable.*

Next, we assume that the distribution M is integrable. Then we know (3.6). By virtue of (3.7),(3.8) and (3.9), we get

$$(3.6) \Longleftrightarrow \begin{cases} S_j{}^a{}_i + p_d{}^k(\gamma_j{}^d S_i{}^a{}_k - \gamma_i{}^d S_j{}^a{}_k) + p_d{}^k p_b{}^l \gamma_j{}^b \gamma_i{}^d S_l{}^a{}_k = 0, \\ \\ R_{ji}{}^a + p_c{}^k(\gamma_j{}^c R_{ik}{}^a - \gamma_i{}^c R_{jk}{}^a) + p_d{}^k p_b{}^l \gamma_j{}^b \gamma_i{}^d R_{lk}{}^a = 0. \end{cases} \tag{3.12}$$

Thus we have

Theorem 3.3. *The distribution M induced from the pseudo-f-structure F on the vector bundle $D^1(M^n)$ is integrable if and only if $S_j{}^a{}_i$ and $R_{ji}{}^a$ satisfy the equation* (3.12).

Corollary 3.4. If the tensor fields $S_j{}^a{}_i$ and $R_{jib}{}^a$ vanish, then the distribution M is integrable.

On the other hand, for the Nijenhuis tensor of the almost product structure $H = (h_j{}^i)$ on the base space, we have

$$N_j{}^h{}_i = h_j{}^k \partial_k h_i{}^h - h_i{}^k \partial_k h_j{}^h - (\partial_j h_i{}^k - \partial_i h_j{}^k) h_k{}^h.$$

Substituting $h_j{}^i = \delta_j{}^i - 2p_j{}^i$ into the above equation, we get

$$\frac{1}{4} N_j{}^h{}_i = p_j{}^k \partial_k p_i{}^h - p_i{}^k \partial_k p_j{}^k - (\partial_j p_i{}^k - \partial_i p_j{}^k) p_k{}^h. \tag{3.13}$$

Since we know

$$p_j{}^i + q_j{}^i = \delta_j{}^i, \quad p_j{}^k q_k{}^i = q_j{}^k p_k{}^i = 0, \quad p_j{}^k p_k{}^i = p_j{}^i, \quad q_j{}^k p_k{}^i = q_j{}^i, \tag{3.14}$$

these three equations mean

$$\gamma_j{}^u q_u{}^h = q_j{}^h, \quad \gamma_j{}^a p_a{}^i = p_j{}^i, \quad \gamma_j{}^a q_i{}^j = 0. \tag{3.15}$$

Now, by virtue of $(2.3)_3$ and (3.15), we get

$$q_j{}^l q_i{}^k S_l{}^a{}_k p_a{}^h = \{\partial_j p_i{}^k - \partial_i p_j{}^k - (p_j{}^l \partial_l p_i{}^k - p_i{}^l \partial_l p_j{}^k)\} p_k{}^h.$$

Futhermore, (3.13) implies

$$\frac{1}{4} N_j{}^k{}_i p_k{}^h = -\{(\partial_j p_i{}^k - \partial_i p_j{}^k) - (p_j{}^l \partial_l p_i{}^k - p_i{}^l \partial_l p_j{}^k)\} p_k{}^h.$$

Comparing the above two equations, we obtain

$$\frac{1}{4} N_j{}^k{}_i p_k{}^h = -q_j{}^l q_i{}^k S_l{}^a{}_k p_a{}^h. \tag{3.16}$$

Moreover, (3.12) and (3.14) teach us

$$q_l{}^j q_k{}^i S_j{}^a{}_i = 0, \quad q_l{}^j q_k{}^i R_{ji}{}^a = 0. \tag{3.17}$$

Conversely, if we assume $(3.17)_1$, we can easily see $(3.12)_1$.
Similarlly, if we assume that $(3.17)_2$, we have $(3.12)_2$. Thus we have

Theorem 3.5. *The distribution M induced from the pseudo-f-structure F on the vector bundle $D^1(M^n)$ is integrable if and only if $S_j{}^a{}_i$ and $R_{ji}{}^a$ satisfy* (3.17).

Next, we assume that the structure F is partially integrable. Then, by its definition, we know $L_\gamma{}^\delta L_\beta{}^\omega N_\delta{}^\alpha{}_\omega = 0$. The calculation of this equation by separating the indices gives us

$$\begin{cases} \gamma_j{}^a p_a{}^l p_b{}^k \gamma_i{}^b S_l{}^h{}_k + p_c{}^h p_a{}^l p_e{}^m (\gamma_j{}^a \Gamma_i{}^e R_{ml}{}^c - \gamma_i{}^a \Gamma_j{}^e R_{ml}{}^c) - \Gamma_j{}^e \Gamma_i{}^d S_e{}^h{}_d = 0, \\ \gamma_j{}^d p_d{}^l \gamma_i{}^b p_b{}^k (R_{lk}{}^a + \Gamma_m{}^a S_l{}^m{}_k) + \gamma_j{}^d p_d{}^l \Gamma_i{}^e p_e{}^m (\Gamma_n{}^a p_d{}^n R_{ml}{}^d - S_m{}^a{}_l) \\ \qquad + \gamma_i{}^c p_c{}^k \Gamma_j{}^e p_e{}^m (S_m{}^a{}_k - \Gamma_n{}^a p_d{}^n R_{mk}{}^d) \\ \qquad - \Gamma_j{}^e \Gamma_i{}^d (\Gamma_m{}^a S_c{}^m{}_d - p_e{}^m p_d{}^n R_{mn}{}^a) = 0, \\ \Gamma_j{}^e S_c{}^h{}_b - \gamma_j{}^d p_d{}^l p_b{}^k p_c{}^h R_{kl}{}^c = 0, \\ \gamma_j{}^d p_d{}^l (p_b{}^k S_k{}^a{}_l - p_b{}^k \Gamma_d{}^a R_{kl}{}^d) + \Gamma_j{}^e (\Gamma_l{}^a S_e{}^l{}_b + p_e{}^l p_b{}^m R_{lm}{}^a) = 0, \\ S_c{}^j{}_b = 0, \quad \Gamma_j{}^a S_c{}^j{}_b + p_c{}^k p_b{}^j R_{kj}{}^a = 0. \end{cases} \tag{3.18}$$

We can get that (3.18) equivalent with

$$S_c{}^h{}_b = 0, \quad p_c{}^k p_b{}^j R_{kjb}{}^a = 0, \quad p_d{}^l p_b{}^k S_k{}^a{}_l = 0. \tag{3.19}$$

Thus we have

Theorem 3.6. *The pseudo-f-structure F induced in the vector bundle $D^1(M^n)$ is partially integrable if and only if $S_c{}^h{}_b$, $R_{kjb}{}^a$ and $S_k{}^a{}_l$ satisfy* (3.19).

4. COMPLETELY INTEGRABLE CONDITION OF $\bar{\mathbf{P}}$ AND $\bar{\mathbf{Q}}$.

We first calculate the Nijenhuis tensor $\tilde{N}_\gamma{}^\alpha{}_\beta$ of $K = M - L$. Then it is written as

$$\begin{cases} \frac{1}{4}\tilde{N}_j{}^h{}_i = -p_c{}^h S_j{}^c{}_i + p_d{}^k p_c{}^h(\gamma_j{}^d S_k{}^c{}_i - \gamma_i{}^d S_k{}^c{}_j) + \gamma_j{}^d\gamma_i{}^c S_d{}^h{}_c, \\ \frac{1}{4}\tilde{N}_j{}^a{}_i = -R_{ji}{}^a + p_b{}^l(\gamma_j{}^b R_{li}{}^a - \gamma_i{}^b R_{lj}{}^a) + p_b{}^l\Gamma_l{}^a S_j{}^b{}_i \\ \qquad -p_d{}^k p_b{}^l \Gamma_l{}^a(\gamma_j{}^d S_k{}^b{}_i - \gamma_i{}^d S_k{}^b{}_j) - \gamma_j{}^d\gamma_i{}^b(p_k{}^d p_b{}^l R_{kl}{}^a + \Gamma_l{}^a S_d{}^l{}_b), \\ \text{the other components} = 0. \end{cases} \tag{4.1}$$

Now, let $\bar{P}$ be completely integrable. Then the Nijenhuis tensor $\tilde{N}_\gamma{}^\alpha{}_\beta$ satisfies $\tilde{N}_\gamma{}^\alpha{}_\beta + \tilde{N}_\gamma{}^\delta{}_\beta K_\delta{}^\alpha = 0$.

Using (3.10) and (4.1), the last equation means

$$S_d{}^h{}_c - p_b{}^h\gamma_k{}^b S_d{}^k{}_c = 0 \quad \text{or equivalently} \quad \gamma_k{}^a S_d{}^k{}_c = 0. \tag{4.2}$$

Thus we have

Theorem 4.1. *For the almost product structure K which is define from the pseudo-f-structure F on the vector bundle $D^1(M^n)$, the distribution $\bar{P}$ is completely integrable if and only if the tensor $S_d{}^h{}_c$ satisfies $(4.2)_1$ or equivalently $(4.2)_2$.*

By virtue of Theorem 3.1 and Theorem 4.1, we have

Corollary 4.2. *If the distribution L is integrable, then the distribution $\bar{P}$ is completely integrable.*

Next, we assume that the distribution $\bar{Q} = \frac{1}{2}(I + K)$ is completely integrable. Then the Nijenhuis tensor $\tilde{N}_\gamma{}^\alpha{}_\beta$ satisfies

$$\tilde{N}_\gamma{}^\alpha{}_\beta - \tilde{N}_\gamma{}^\delta{}_\beta K_\delta{}^\alpha = 0.$$

By virtue of (3.10) and (4.1), the above equation has the following case which are not identically zero;

$$\begin{cases} \tilde{N}_j{}^h{}_i - \tilde{N}_j{}^k{}_i K_k{}^h - \tilde{N}_j{}^d{}_i K_d{}^h = 0, \\ \tilde{N}_j{}^a{}_i - \tilde{N}_j{}^k{}_i K_k{}^a - \tilde{N}_j{}^d{}_i K_d{}^a = 0. \end{cases} \tag{4.3}$$

By the straightfoward calculation, we get that $(4.3)_1$ and $(4.3)_2$ respectively mean

$$S_j{}^a{}_i - p_d{}^k(\gamma_j{}^d S_k{}^a{}_i - \gamma_i{}^d S_k{}^a{}_j) - \gamma_j{}^d\gamma_i{}^c\gamma_k{}^a S_d{}^k{}_c = 0, \tag{4.4}$$

and

$$R_{jib}{}^a + p_c{}^l(\gamma_j{}^c R_{kib}{}^a - \gamma_i{}^c R_{ljb}{}^a - \gamma_j{}^d\gamma_i{}^c p_d{}^k p_c{}^l R_{klb}{}^a) = 0. \tag{4.5}$$

Thus we have

Theorem 4.3. *For the almost product structure K which is defined from the pseudo-f-structure F on the vector bundle $D^1(M^1)$, the distribution $\bar{Q}$ is completely integrable if and only if the equations (4.4) and (4.5) are satisfied.*

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