LOCALLY CONFORMAL KAEHLER STRUCTURES ON TANGENT MANIFOLD OF A SPACE FORM

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Dedicated to Professor Radu Rosca on his 90th birthday anniversary

Abstract. A set of locally conformal KaeHLer structures on tangent manifold $TM$ of a space form $M$ is pointed out. This is found in a study of a type of Sasaki metric whose second term is a special deformation of the first one. Introducing an adequate almost complex structure we find at first a large class of locally conformal almost KaeHLer structures on $TM$ for $M$ a (pseudo)-Riemannian manifold. When $M$ is a space form, a subset of it is made of locally conformal KaeHLer structures. One of them was found by R. Miron in [3].

1. INTRODUCTION

Let $(M,g)$ be a (pseudo)-Riemannian manifold and $\nabla$ its Levi-Civita connection. In a local chart $(U_i, (x^i))$ we set $g_{ij} = g(\partial_i, \partial_j)$, where $\partial_k : \mathbb{R}^n \to \mathbb{R}^n$ and we denote by $\gamma^i_{jk}(x)$ the Christoffel symbols giving $\nabla$. Let $(x^i, y^j) \equiv (x, y)$ be the local coordinates on the manifold $TM$ projected on $M$ by $\tau$. The indices $i, j, k, \ldots$ will run from 1 to $n = \dim M$.

The functions $N^i_j(x, y) := \gamma^i_{jk}(x)y^k$ are the local coefficients of a nonlinear connection, that is, the local vector fields $\xi_i = \partial_i - N^i_j(x, y)\partial_j$, where $\partial_j : \mathbb{R}^n \to \mathbb{R}^n$ span a distribution on $TM$ called horizontal which is supplementary to the vertical distribution $u \to V_uTM = \ker\tau \otimes \mathbb{R}u \subset TM$. Let us denote by $u \mapsto H_uTM$ the horizontal distribution and let $\{\partial_i, \partial_k\}$ be the basis adapted to the decomposition $T_uTM = H_uTM \oplus V_uTM, u \in TM$. The bi-symmetric of it is $(dx^i, dy^j)$ with $dy^j = dy^j + N^j_i(x, y)dx^k$.

The Sasaki metric on $TM$ is as follows

$$G_S = g_{ij}(x)dx^i \otimes dx^j + g_{ij}(x)dy^i \otimes dy^j.$$  

(1.1)

If in the second term of $G_S$ one replaces $g_{ij}(x)$ with the components $h_{ij}(x, y)$ of a generalized Lagrange metric (see Ch. X in [4]) one gets a type of Sasaki metric

$$G(x, y) = g_{ij}(x)dx^i \otimes dx^j + h_{ij}(x, y)dy^i \otimes dy^j.$$  

(1.2)

In particular, $h_{ij}(x, y)$ could be a deformation of $g_{ij}(x)$, a case studied by the present author and H. Shimada in [4].

In this paper we are concerning with the metrical structure (1.2) in the case when $h_{ij}(x, y)$ is the following special deformation of $g_{ij}(x)$
(1.3) \[ h_{ij}(x, y) = a(L^2)g_{ij}(x) + b(L^2)y_i y_j, \]
where \( L^2 = g_{ij}(x)y^i y^j, y_i = g_{ij}(x)y^j \) and \( a, b : \text{Im}(L^2) \subseteq \mathbb{R}_+ \rightarrow \mathbb{R}_+ \), with \( a > 0, b \geq 0 \).

For \( b = 0 \) and \( a = \frac{g_{ii}^2}{L^2} \), for any constant \( c \), the metrical structure (1.2), (1.3) was studied by R. Miron in [3] as an homogeneous lift of \( g_{ij}(x) \) to \( TM \).

In the following Section we introduce an almost complex structure which paired with \( G \) given by (1.2), (1.3) provides a large set of almost Hermitian structures on \( TM \). Then, in Section 3 we show that all these structures are locally conformal almost Kaehler structures. Finally, we find in Section 4 that, when \( (M, g) \) is of constant curvature, a part of them are locally conformal Kaehler structures.

2. SOME ALMOST HERMITIAN STRUCTURES ON TM

Let \( F_\theta \) be the almost complex structure on \( TM \) given in the adapted basis \( (e_i, \bar{e}_i) \) by

\[ F_\theta(e_i) = -\bar{e}_i, F_\theta(\bar{e}_i) = e_i. \]

It is well known that the pair \( (G_\theta, F_\theta) \) is an almost Kaehler structure on \( TM \), that is \( G_\theta(F_\theta X, F_\theta Y) = G_\theta(X, Y) \) and the 2-form\( \Omega(X, Y) = G_{\theta}( F_\theta(X), Y ) \) is closed, \( X, Y \in \chi(M) \).

The pair \( (G, F_\theta) \) with \( G \) given by (1.2), (1.3) is no longer an almost Hermitian structure.

We look for a new almost complex structure which paired with \( G \) to provide an almost Hermitian structure. We modify \( F_\theta \) to a linear map \( F \) given in the basis \( (e_i, \bar{e}_i) \) as follows

\[ F(e_i) = (\alpha \delta^j + \beta y^j y^k) \delta_k, F(\bar{e}_i) = (\gamma \delta^j + \delta y^j y^h) \delta_h, \]

where \( \alpha, \beta, \gamma, \delta \) are functions on \( TM \) to be determined. The condition \( F^2 = -I \) (identity) leads to

\[ \alpha \gamma = -1, \alpha \delta + \beta \gamma + \beta \delta L^2 = 0. \]

Then the condition \( G(F(X), F(Y)) = G(X, Y) \) gives

\[ ax^2 = 1, \gamma^2 = \alpha, 2 \alpha \beta + \delta^2 L^2 = b, (2 \alpha \beta + \beta^2 L^2)(\alpha + b L^2) + b a^2 = 0. \]

The solution of the system of equations (2.3), (2.4) is

\[ \alpha = \frac{-1}{\sqrt{a}}, \beta = \frac{\sqrt{a} + \sqrt{a + b L^2}}{L^2 \sqrt{a + b L^2}}, \gamma = \sqrt{a}, \delta = -\frac{\sqrt{a} + \sqrt{a + b L^2}}{L^2}. \]

We notice that for \( b = 0 \), besides the solution provided by (2.5), that is

\[ \alpha = -\frac{1}{\sqrt{a}}, \gamma = \sqrt{a}, \beta = 0, \delta = 0, \]

there exists also the solution

\[ \alpha = -\frac{1}{\sqrt{a}}, \gamma = \sqrt{a}, \beta = 0, \delta = 0. \]

Let us make the substitution \( a \rightarrow \frac{g_{ii}^2}{L^2}, b \rightarrow \frac{b \sqrt{a}}{L^2}. \)
Then (2.5) and (2.6) are unified to

\[ \alpha = \frac{-L}{a}, \beta = \frac{a+b}{abL}, \gamma = \frac{a}{L}, \delta = \frac{a+b}{L^2}, b \geq a > 0 \]

and (2.7) modifies to

\[ \alpha = \frac{-L}{a}, \gamma = \frac{a}{L}, \beta = \delta = 0. \]

The metric \( G \) takes the form

\[ (2.10) \quad G_{a,b}(x,y) = g_{ij}(x)dx^i \otimes dx^j + \frac{a^2}{L^2} g_{ij}(x) + \frac{\partial^2 - a^2}{L^2} y_i y_j \delta y^i \otimes \delta y^j, b \geq a > 0. \]

Let \( F_{a,b} \) be the almost complex structures given by (2.2), (2.8) and \( F_a \) those given by (2.2), (2.9). Then the pairs \( (G_{a,b}, F_{a,b}) \) and \( (G_{a,a}, F_a) \) are almost Hermitian structures on \( TM \).

For \( a^2 = \frac{L^2}{1+t^2}, b = L^2 \), the metric \( G_{a,b}(x,y) \) is the Cheeger-Gromoll metric \([5],[6]\]

\[ (2.11) \quad G_{CC}(x,y) = g_{ij}(x)dx^i \otimes dx^j + \frac{1}{1+L^2} (g_{ij}(x) + y_i y_j \delta y^i \otimes \delta y^j). \]

If \( a^2 = \varphi^2, b^2 = L^2(\varphi^2 + 2\varphi''L^2) \) for \( \varphi : \mathbb{R}_+ \rightarrow \mathbb{R}_+ \) with \( \varphi'(t) \neq 0, t \in \text{Im}(L^2) \), one obtains the Antonelli-Hrimiuc metrical structure \([2]\)

\[ (2.12) \quad G_{AH}(x,y) = g_{ij}(x)dx^i \otimes dx^j + (\varphi y_{ij}(x) + 2\varphi''y_i y_j) \delta y^i \otimes \delta y^j. \]

3. LOCALLY CONFORMAL ALMOST KAHLER STRUCTURES ON \( TM \)

Let \( \Omega(X,Y) = G_{a,b}(F_{a,b}X,Y), X,Y \in \chi(TM) \) be the 2-form associated to the almost Hermitian structure \( (G_{a,b}, F_{a,b}) \).

**Theorem 3.1** The almost Hermitian structures \( (G_{a,b}, F_{a,b}) \) are locally conformal almost Kaehlerian structures, that is

\[ (3.1) \quad d\Omega = \Omega \wedge \theta, \theta = \frac{2a^2L + b}{a^2L} dl. \]

**Proof.** We shall check (3.1) on the basis \((\partial_i, \partial_\theta)\). If we rewrite (2.2) in the form

\[ (3.2) \quad \mu(\partial_i, \partial_\theta) = A^i_\theta \partial_\theta, \mu(\partial_\theta, \partial_i) = H^i_\theta \partial_i, \]

we easily get

\[ (3.3) \quad \Omega(\delta_i, \delta_j) = 0, \Omega(\delta_i, \partial_\theta) = A^i_\theta \delta_i, \Omega(\delta_j, \partial_\theta) = H^j_\theta \delta_i, \Omega(\partial_\theta, \partial_i) = 0. \]

Thus \( \Omega \) is completely determined by

\[ (3.4) \quad \Omega_{ij} := H^j_\theta \delta_i = \partial_i y_j + \delta_i y_j; \Omega = \partial_i y^j \wedge dx^j. \]
Next we have the following essential components of \( d\Omega \):
\[
d\Omega(\delta_i, \delta_j, \delta_k) = \delta_i \Omega_{jk} - \gamma^a_{ij} \Omega_{jk} - \delta_i \Omega_{jk} - \gamma^a_{ij} \Omega_{jk},
\]
\[
d\Omega(\delta_i, \delta_j, \delta_k) = \partial_{\delta_i} \Omega_{jk} - \partial_{\delta_k} \Omega_{ij}.
\]

Now we consider the Berwald connection \( (\mathcal{V}^j = \gamma^j_{ij} x^j, \gamma^j_{ij}(x), 0) \) on \( TM \) (see Ch.8 in [4]) and denote by \( \delta_k \) its \( k \)-covariant derivative. Thus because of \( \Omega_{ijk} = \delta_i \Omega_{jk} - \gamma^a_{ij} \Omega_{ak} - \gamma^a_{ij} \Omega_{aj} \), we have \( d\Omega(\delta_i, \delta_j, \delta_k) = \Omega_{ijk} - \Omega_{jik} \).

The following formulæ are verified by a direct calculation.
\[
(3.5) \quad g_{ijlk} = 0, y_{ijl} = 0, y_{ijlk} = 0, \delta_k L^2 = 0, \delta_k \psi(L^2) = 0,
\]
\[
\delta_k y_{ik} = g_{ik}, \delta_k L^2 = 2y_k, \delta_k \psi(L^2) = 2y_k \psi(L^2),
\]
for any \( \psi : \text{Im}(L^2) \subseteq \mathbb{R}_+ \to \mathbb{R}_+ \).

Using (3.5) it immediately results \( \Omega_{ijk} = 0 \) and so \( d\Omega(\delta_i, \delta_j, \delta_k) = 0 \). Consequently, \( d\Omega \) is completely determined by \( d\Omega(\delta_i, \delta_j, \delta_k) = (\delta_{\theta} \gamma)(y_{ik} - (\delta_k \gamma) y_{ik} + (\delta_j \delta) y_{ik} y_k - (\delta_k \delta) y_{ij} y_k + (\delta_k \gamma) y_{ij} \).

Inserting here \( \delta_{\theta} \gamma, \delta_{\theta} \delta \) with \( \gamma, \delta \) from (2.8) one arrives to
\[
(3.6) \quad d\Omega(\delta_i, \delta_j, \delta_k) = (2\gamma - \delta)(g_{ik} y_{ij} - g_{ij} y_k) = \frac{2a' L^2 + b}{L^3} (g_{ik} y_{ij} - g_{ij} y_k).
\]

Let be \( \theta_0 = dL^2 = 2y_k \delta y^k \). Thus \( \theta_0(\delta_i) = 0 \) and \( \theta_0(\delta_j) = 2y_j \). Evaluating \( \Omega \wedge \theta_0 \) on the basis \( (\delta_i, \delta_k) \) one finds the essential component
\[
(3.7) \quad \Omega \wedge \theta_0(\delta_i, \delta_j, \delta_k) = 2(\Omega_{ik} y_{jk} - \Omega_{ij} y_k) = \frac{2a}{L} (g_{ik} y_{ij} - g_{ij} y_k).
\]

Comparing (3.6) with (3.7) one obtains \( d\Omega = \frac{2a' L^2 + b}{a L^3} \Omega \wedge \theta_0 \) which is just (3.1).\( \blacksquare \)

Obviously \( \theta \) is globally defined. Moreover, \( \theta \) is closed. This fact can be directly verified using (3.5) or by differentiating (3.1).

Looking at (3.6) we notice that contracting \( g_{ik} y_{ij} - g_{ij} y_k = 0 \) with \( y^k \) one gets \( (n - 1)y_j = 0 \) which is a contradiction. Thus we have

**Theorem 3.2** The almost Hermitian structures \((G_{a,b}, F_{a,b})\) are almost Kaehler structures if and only if
\[
(3.8) \quad 2a' L^2 + b = 0,
\]
holds good.

We put \( t = L^2 \) and think (3.8) as a first order differential equation: \( 2a'(t) + b(t) = 0 \). Its general solutions is \( a(t) = c - \frac{t}{2} \int b(t) dt \) for a constant \( c \). Thus for various functions \( b \) we find a set of pairs \( (a, b) \) for which (3.8) holds. Choosing among these pairs those which verify \( b \geq a > 0 \) we find a set of almost Kaehler structures on \( TM \). For instance, if we take \( b(t) = 2t \) it results \( a(t) = c - t \) and \( b \geq a > 0 \) holds if \( \frac{c}{2} \leq L^2(x, y) < c \), for \( c > 0 \). When \( a = b \), the equation (3.8) has the general solution \( u(t) = \frac{c}{\sqrt{t}} \). It follows

**Corollary 3.1** The almost Hermitian structures \((G_{a,a}, F_{a,a})\) are almost Kaehler structures if and only if \( a(L^2) = \frac{c}{\sqrt{L^2}}, c > 0 \).
The almost Hermitian structures \((G_{a}, a, F_{a})\) have to be separately considered. Repeating for them the proof of Theorem 3.1 one obtains

**Theorem 3.3** The almost Hermitian structures \((G_{a}, a, F_{a})\) are locally conformal almost Kaehler structures, that is

\[
d\Omega = \Omega \wedge \theta, \theta = \frac{2a'}{aL} - \frac{a}{aL}dL.
\]

The following corresponds to Theorem 3.2.

**Theorem 3.4** The almost Hermitian structures \((G_{a}, a, F_{a})\) are almost Kaehler structures if and only if \(a = c\sqrt{L^2}, c > 0\).

**Proof:** The almost Kaehler condition is now \(2a' L^2 - a = 0\). Integrating the equation \(2a' L^2 - a = 0\) one gets \(a = c\sqrt{L}\).

**Remark 3.1** For \(a = c\sqrt{L^2}, c > 0, G_{a, a}\) is very close to \(G_{a}\) which is obtained for \(c = 1\).

4. **LOCALLY CONFORMAL KAHLER STRUCTURES ON TM**

In order to find conditions that \((G_{a}, F_{a})\) be a locally conformal Kaehler structure we have to put zero for the Nijenhuis tensor field of \(F := F_{a, b}\).

\[
\]

As the evaluation of \(N_F\) on the basis \((\delta, \hat{\delta})\) is in general very complicated we confine ourselves to the structures \((G_{a, a}, F_{a})\). In this case, the conditions

\[
N_F(\delta_i, \delta_j) = 0, N_F(\delta_i, \hat{\delta}_j) = 0, N_F(\hat{\delta}_i, \hat{\delta}_j) = 0,
\]

are equivalent with six equations. Three of them are identities because of \(\delta_i \partial_i = \delta_i \gamma = 0\) and the other three are each one equivalent with

\[
R_{ij}^{k} = \frac{2a' L^2 - a}{\mu^3} (y_j \delta_i^k - y_i \delta_j^k),
\]

where \(R_{ij}^{k} = R_{ij}^{k}(x) y^s\) and \(R_{ij}^{k}\) is the curvature tensor of \(\nabla\).

By a contraction with \(g_{ik}\) the Eq. (4.3) reduces to

\[
R_{\alpha ij}(x) y^\alpha = \frac{2a' L^2 - a}{\mu^3} (y_j \delta_i - y_i \delta_j) y^\alpha.
\]

The Eq. (4.4) remember us the condition that \((M, g)\) is of constant curvature (space form). It suggests us to look for functions \(a\) such that \(2a' L^2 - a = k\), where \(k\) is a constant.

For \(t = L^2\), solving the Bernoulli equation \(a' = \frac{1}{2} \mu + k \mu^3\), one gets \(a(L^2) = \sqrt{\frac{L^2}{c L^2}}\), for \(c = k L^2 > 0\), where \(c\) is a constant of integration. For these functions \(a\), the Eq. (4.4) becomes

\[
R_{\alpha ij}(x) y^\alpha = -k (y_j g_{i\alpha} - y_i g_{j\alpha}) y^\alpha,
\]

which says that \((M, g)\) is of constant curvature \(-k\). Thus we have proved
Theorem 4.1 If the (pseudo)-Riemannian manifold \((M, g)\) is of constant curvature \(k \in \mathbb{R}\), for \(a(L^2) = \sqrt{\frac{L^2}{c + kL^2}}\) with \(c\) a constant such that \(c+kL^2 > 0\), the structures \((G_{a,\alpha}, F_a)\) are locally conformal Kähler structures on \(TM\).

The explicit form of these structures is as follows.

\[
G_{a,\alpha}(x,y) = g_{ij}(x)dx^i \otimes dx^j + \frac{1}{c + kL^2} (g_{ij}(x))\delta y^i \otimes \delta y^j.
\]

(4.6)

\[
F_a(\delta_i) = -\sqrt{c + kL^2} \hat{\partial}_i, \quad F_a(\hat{\partial}_i) = \frac{1}{\sqrt{c + kL^2}} \delta_i.
\]

(4.7)

The 1-form \(\theta\) is

\[
\theta = -\frac{kL}{c + kL^2} \eta.
\]

(4.8)

Corollary 4.1 For \(a(L^2) = c_0 \sqrt{L^2}\), with \(c_0\) a strict positive constant, the pairs \((G_{a,\alpha}, F_a)\) are Kähler structures on \(TM\) if and only if \((M, g)\) is flat.

Proof. If \((M, g)\) is flat, by the Theorem 4.1 for \(a(L^2) = c_0 \sqrt{L^2}, c_0 = \frac{1}{\sqrt{c}}\), the pair \((G_{a,\alpha}, F_a)\) is a locally conformal Kähler structure and by the Theorem 3.4 this is almost Kähler. Thus \((G_{a,\alpha}, F_a)\) is a Kähler structure on \(TM\). Conversely, if the pair \((G_{a,\alpha}, F_a)\) with \(a(L^2) = c_0 \sqrt{L^2}\) is a Kähler structure, the Eq. (4.3) gives \(R_{ij}^k = 0\), equivalently \(R_{xij}(x) = 0\), that is \((M, g)\) is flat. ■

Looking at (4.6) and (4.7) we see that the structures \((G_{a,\alpha}, F_a)\) from Corollary 4.1 are very close to \((G_{3}, F_5)\) which is obtained for \(c = 1\). Thus the Corollary 4.1 covers a well-known result: \((G_{3}, F_5)\) is a Kählerian structure if and only if \((M, g)\) is flat.

Finally, we notice that for \(c = 0\) and \(k \to \frac{1}{\sqrt{c}}\) in (4.6) and (4.7) one obtains the locally conformal Kähler structure found by R.Miron in [3].

REFERENCES