

BMO_{loc}-QC MAPPINGS BETWEEN KLEIN SURFACES

CABIRIA ANDREIAN CAZACU and VICTORIA STANCIU

To Professor Corneliu Constantinescu

We generalize normality and compactness criteria obtained in [AC S1] for BMO_{loc} - QC and BMO - QC from Riemann to Klein surfaces.

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1. Introduction

In this paper we understand by a *Klein surface* a non-orientable surface without border X endowed with a dianalytic structure given by a dianalytic atlas [AG, 1, §2] and [SS, 3.2]. Such a surface X is also called a non-orientable Riemann surface, e.g. by Schiffer and Spencer [Sch S, 2, §1].

Our method is based on covering theory [N, 2, §4], [L, 4, §2], [S, 2], namely on the double orientable covering $(\tilde{X}, \tilde{\Pi}, X)$ and Klein Theorem [AG, 1, §6], [SS, 3.4], and the universal coverings: (\tilde{X}, Π, X) and $(\tilde{X}, \tilde{\Pi}, \tilde{X})$ with $\Pi = \tilde{\Pi} \circ \tilde{\Pi}$, [SS, 3.6].

The orientable surfaces \tilde{X} as well as \tilde{X} are Riemann surfaces with analytic atlases such that Π and $\tilde{\Pi}$ are dianalytic projections, and $\tilde{\Pi}$ is an analytic one. We take as \tilde{X} the Riemann surfaces $\hat{\mathbb{C}}, \mathbb{C}$ or D - the unit disk, endowed with the spherical, euclidean or hyperbolic metric respectively. This metric is transported by the projections $\tilde{\Pi}$ and Π to \tilde{X} and to X [SS, 3.8].

In this paper we deal with two homeomorphic Klein surfaces X and X' , their double orientable coverings and their universal coverings. In order to prove normality and compactness properties for certain families of homeomorphisms $f : X \rightarrow X'$ we shall lift the homeomorphisms to these coverings which are realized by Riemann surfaces and then apply the results we established in [AC S1].

First we lift f to the double coverings $(\tilde{X}, \tilde{\Pi}, X)$ and $(\tilde{X}', \tilde{\Pi}', X')$ obtaining a homeomorphism $\tilde{f} : \tilde{X} \rightarrow \tilde{X}'$, such that

$$\tilde{\Pi}' \circ \tilde{f} = f \circ \tilde{\Pi} \text{ and } \sigma' \circ \tilde{f} = \tilde{f} \circ \sigma. \quad (1.1)$$

Here σ and σ' are the antianalytic involutions $\sigma : \tilde{X} \rightarrow \tilde{X}$ and $\sigma' : \tilde{X}' \rightarrow \tilde{X}'$ respectively, which are also cover mappings: $\tilde{\Pi} \circ \sigma = \tilde{\Pi}$, $\tilde{\Pi}' \circ \sigma' = \tilde{\Pi}'$, and generate the corresponding covering groups of order 2. The existence of \tilde{f} is given by a Theorem of I. Bernstein [B], [St], as well as by Proposition 1.6.2 in [AG], and \tilde{f} is uniquely determined by fixing a pair of corresponding points. Remember that in general there are two liftings for every f , namely \tilde{f} and $\sigma' \circ \tilde{f}$.

Further we lift such a lifting, e.g. \tilde{f} , to $(\hat{X}, \hat{\Pi}, \hat{X})$ and $(\hat{X}', \hat{\Pi}', \hat{X}')$ and obtain a homeomorphism $\hat{f} : \hat{X} \rightarrow \hat{X}'$, which also represents a lifting of f to (\hat{X}, Π, X) and (\hat{X}', Π', X') .

In [AC S1] we proved the following auxiliary Propositions 1.1 and 1.2, which we formulate for Klein surfaces and for both coverings:

PROPOSITION 1.1. *Let $\{f_n\}$ be a sequence of homeomorphisms $f_n : X \rightarrow X'$ and for each n , let $\tilde{f}_n : \tilde{X} \rightarrow \tilde{X}'$ ($\tilde{f}_n : \tilde{X} \rightarrow \tilde{X}'$) be a lifting of f_n to the universal (double orientable) covering. If the sequence $\{\tilde{f}_n\}$ ($\{\tilde{f}_n\}$) locally uniformly (l.u.) converges to a mapping \tilde{f}_0 (\tilde{f}_0 respectively), then*

- (i) \tilde{f}_0 (\tilde{f}_0) is the lifting of the same continuous mapping $f_0 : X \rightarrow X'$, and
- (ii) the sequence $\{f_n\}$ l.u. converges to f_0 .

If in addition \tilde{f}_0 (\tilde{f}_0) is 1° a l. homeomorphism, 2° an embedding (i.e. homeomorphism into) or 3° a homeomorphism, then f_0 has the same property.

PROPOSITION 1.2. *Let \mathcal{F} be a family of homeomorphisms $f : X \rightarrow X'$ and $\hat{\mathcal{F}}$ ($\tilde{\mathcal{F}}$) a family consisting of at least one lifting $\hat{f} : \hat{X} \rightarrow \hat{X}'$ ($\tilde{f} : \tilde{X} \rightarrow \tilde{X}'$) for every $f \in \mathcal{F}$.*

- (i) *If $\hat{\mathcal{F}}$ ($\tilde{\mathcal{F}}$) is normal, then \mathcal{F} is normal.*
- (ii) *If $\hat{\mathcal{F}}$ ($\tilde{\mathcal{F}}$) is normal and closed, then \mathcal{F} is also normal and closed.*

The proofs of these propositions are similar to those in [AC S1].

PROPOSITION 1.3. *Let $\{f_n\}$ be a sequence of homeomorphisms $f_n : X \rightarrow X'$, which l.u. converges to f_0 , and consider the corresponding liftings $\tilde{f}_n : \tilde{X} \rightarrow \tilde{X}'$ and $\tilde{f}_0 : \tilde{X} \rightarrow \tilde{X}'$. If there is a point $\tilde{p} \in \tilde{X}$ such that $\tilde{f}_n(\tilde{p}) \rightarrow \tilde{f}_0(\tilde{p})$, then $\{\tilde{f}_n\}$ l.u. converges to \tilde{f}_0 .*

Proof. 1° Let \tilde{f}_0 be a lifting of f_0 . The condition $\tilde{f}_n(\tilde{p}) \rightarrow \tilde{f}_0(\tilde{p}) = \tilde{p}'$ for a point $\tilde{p} \in \tilde{X}$ can be always fulfilled for convenient liftings \tilde{f}_n . Indeed, we take for $p = \tilde{\Pi}(\tilde{p})$ a distinguished neighborhood v' of $p' = f_0(p)$ with respect to $(\tilde{X}', \tilde{\Pi}', X')$, i.e. a neighborhood such that $\tilde{\Pi}'^{-1}(v') = \tilde{v}' \cup \sigma'(\tilde{v}')$, where \tilde{v}' is a neighborhood of \tilde{p}' , $\tilde{v}' \cap \sigma'(\tilde{v}') = \emptyset$, $\tilde{\Pi}'|_{\tilde{v}'} : \tilde{v}' \rightarrow v'$ is an isometry in the metrics induced from \tilde{X}' to \tilde{X}' and X' ; moreover, we suppose, that v' is a relatively compact domain in X' , namely the disk $D(p', r) = \{q' \in X'; d'(q', p') < r\}$ in the metric d' on X' . For great n , $f_n(p) \in v'$ and we choose \tilde{f}_n such that $\tilde{f}_n(\tilde{p}) \in \tilde{v}'$.

2°. We prove now that there exists a neighborhood \tilde{v} of \tilde{p} such that $\tilde{f}_n \rightarrow \tilde{f}_0$ u. in \tilde{v} .

Let $u' = D(p', r')$, $0 < r' < r$, and v be a distinguished neighborhood of p with respect to $(\tilde{X}, \tilde{\Pi}, X)$ such that $f_0(v) \subset u'$. Let \tilde{v} and \tilde{u}' be the liftings to the double coverings of v and u' , such that $\tilde{p} \in \tilde{v}$ and $\tilde{p}' \in \tilde{u}' \subset \tilde{v}'$. Then $\tilde{f}_0(\tilde{v}) \subset \tilde{u}'$. As $f_n(v) \subset v'$ for great n , it follows that $\tilde{f}_n(\tilde{v}) \subset \tilde{v}'$ and the isometry $\tilde{\Pi}'|_{\tilde{v}'} : \tilde{v}' \rightarrow v'$ implies that $\tilde{f}_n \rightarrow \tilde{f}_0$ u. in \tilde{v} .

3°. Finally we prove that $\tilde{f}_n \rightarrow \tilde{f}_0$ l.u. in \tilde{X} .

Let $\tilde{X}_1 = \{\tilde{q} \in \tilde{X}; \tilde{f}_n(\tilde{q}) \rightarrow \tilde{f}_0(\tilde{q})\}$ and $\tilde{X}_2 = \tilde{X} \setminus \tilde{X}_1$. For any $\tilde{q} \in \tilde{X}_2$ there is a subsequence $\tilde{f}_{n_k}(\tilde{q}) \rightarrow \sigma' \circ \tilde{f}_0(\tilde{q})$, since the limit points of $\{\tilde{f}_n(\tilde{q})\}$ belong to $\{\tilde{f}_0(\tilde{q}), \sigma' \circ \tilde{f}_0(\tilde{q})\}$. Step 2° applied to the sequence $\{\tilde{f}_{n_k}\}$ shows that there is a neighborhood of \tilde{q} where $\{\tilde{f}_{n_k}\}$ u. converges to $\sigma' \circ \tilde{f}_0$ and thus belongs to \tilde{X}_2 . Both sets \tilde{X}_1 and \tilde{X}_2 being open and

\tilde{X} connected, we proved that $\tilde{X} = \tilde{X}_1$ and again from step 2° the l.u. convergence in \tilde{X} of $\{\tilde{f}_n\}$ to \tilde{f}_0 .

COROLLARY 1.1. *Under the hypotheses of Proposition 1.2, if $\tilde{\mathcal{F}}$ is closed, then \mathcal{F} is closed.*

Our aim is to treat BMO_{loc} -qc mappings between Klein surfaces, so that we extend the definitions in [AC S1]:

A function $Q : X \rightarrow \mathbb{R}$ belongs to $BMO_{loc}(X)$ (or $BMO(X)$) if $\tilde{Q} = Q \circ \Pi \in BMO_{loc}(\tilde{X})$ (or $BMO(\tilde{X})$) respectively and this is equivalent to $\tilde{Q} = Q \circ \tilde{\Pi} \in BMO_{loc}(\tilde{X})$ (or $BMO(\tilde{X})$).

A homeomorphism (an embedding) $f : X \rightarrow X'$ is Q -qc if there exists a lifting $\hat{f} : \hat{X} \rightarrow \hat{X}'$ of f , which is a \hat{Q} -qc homeomorphism (embedding) with $\hat{Q} = Q \circ \Pi$. We write $f \in BMO_{loc}$ -QC (or BMO -QC). This condition is equivalent to the existence of a lifting $\tilde{f} : \tilde{X} \rightarrow \tilde{X}'$, which is $\tilde{Q} = Q \circ \tilde{\Pi}$ -qc homeomorphism (embedding).

The mapping families we shall deal with will be normalized by a compacity condition. As in [AC S1] we choose the normalization by pairs of corresponding points: e.g. $p_0 \in X, p'_0 \in X'; \tilde{p}_0 \in \tilde{\Pi}^{-1}(p_0), \tilde{p}'_0 \in \tilde{\Pi}'^{-1}(p'_0); \hat{p}_0 \in \hat{\Pi}^{-1}(\tilde{p}_0) \subset \Pi^{-1}(p_0), \hat{p}'_0 \in \hat{\Pi}'^{-1}(\tilde{p}'_0) \subset \Pi'^{-1}(p'_0)$.

Let $Q \in BMO_{loc}(X)$. We consider the following families:

$$\mathcal{F} = \{f : X \rightarrow X'; f \text{ } Q\text{-qc homeomorphism, } f(p_0) = p'_0\},$$

$$\tilde{\mathcal{F}} = \{\tilde{f} : \tilde{X} \rightarrow \tilde{X}'; \tilde{f} \text{ lifting of } f \in \mathcal{F}, \tilde{f}(\tilde{p}_0) = \tilde{p}'_0\} \text{ and}$$

$$\hat{\mathcal{F}} = \{\hat{f} : \hat{X} \rightarrow \hat{X}'; \hat{f} \text{ lifting of } \tilde{f} \in \tilde{\mathcal{F}}, \hat{f}(\hat{p}_0) = \hat{p}'_0\}.$$

As the analytic structure of \tilde{X} and \tilde{X}' is fixed such that $\tilde{\Pi}$ and $\tilde{\Pi}'$ be analytic, \tilde{f} is either sense-preserving i.e. a \tilde{Q} -qc homeomorphism or sense-reversing. In this case $\sigma' \circ \tilde{f}$ is \tilde{Q} -qc homeomorphism and we say that \tilde{f} is an \tilde{Q} -antiqc homeomorphism.

Example: $X = X' = \mathbb{P}\mathbb{R}^2, \tilde{X} = \tilde{X}' = \hat{\mathbb{C}}; p_0 = p'_0 = (0, \infty), \tilde{p}_0 = \tilde{p}'_0 = 0; \sigma(z) = -\frac{1}{\bar{z}}$ for $z \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}, \sigma(0) = \infty, \sigma(\infty) = 0; \sigma' = \sigma; Q = 1$. If $f = 1_{\mathbb{P}\mathbb{R}^2}$, then $\tilde{f} = 1_{\hat{\mathbb{C}}}$ (while $\sigma' \circ \tilde{f} = \sigma$). If $f(p) = p'$, where $p = (z, \sigma(z))$ and $p' = (\bar{z}, \sigma'(\bar{z}))$, then $\tilde{f}(z) = \bar{z}$ (and $\sigma' \circ \tilde{f}(z) = -\frac{1}{z}, z \in \hat{\mathbb{C}}$).

From this view-point \mathcal{F} decomposes into two disjoint subfamilies: $\mathcal{F}_1 = \{f \in \mathcal{F}; \tilde{f} \in \tilde{\mathcal{F}} \text{ is sense-preserving}\}$ and $\mathcal{F}_2 = \mathcal{F} \setminus \mathcal{F}_1$. Evidently, this decomposition can be also derived by working with $\hat{\mathcal{F}}$. Thereafter $\tilde{\mathcal{F}} = \tilde{\mathcal{F}}_1 \cup \tilde{\mathcal{F}}_2$ and $\hat{\mathcal{F}} = \hat{\mathcal{F}}_1 \cup \hat{\mathcal{F}}_2$, where $\tilde{\mathcal{F}}_j = \{\tilde{f} \in \tilde{\mathcal{F}}; f \in \mathcal{F}_j\}$ and $\hat{\mathcal{F}}_j = \{\hat{f} \in \hat{\mathcal{F}}; f \in \mathcal{F}_j\}, j = 1, 2$.

The decomposition $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2$ depends on the choice of the points \tilde{p}_0 and \tilde{p}'_0 . If we take the pair \tilde{p}_0 and $\sigma'(\tilde{p}'_0)$, then the lifted family of \mathcal{F} is $\sigma'(\tilde{\mathcal{F}}) = \{\sigma' \circ \tilde{f}; \tilde{f} \in \tilde{\mathcal{F}}\}$ and thus the previous \mathcal{F}_2 becomes the new \mathcal{F}_1 .

For this new choice of points \tilde{p}_0 and $\sigma'(\tilde{p}'_0), \sigma'(\tilde{\mathcal{F}}_2) = \{\sigma' \circ \tilde{f}; \tilde{f} \in \tilde{\mathcal{F}}_2\}$ plays the role of $\tilde{\mathcal{F}}_1$, i.e. $\sigma'(\tilde{\mathcal{F}}_2) = (\sigma'(\tilde{\mathcal{F}}))_1$. Similarly, if $\hat{\sigma}'$ is a lifting of σ' to $(\hat{X}', \hat{\Pi}', \hat{X}')$, the family $\hat{\sigma}'(\hat{\mathcal{F}}_2) = \{\hat{\sigma}' \circ \hat{f}; \hat{f} \in \hat{\mathcal{F}}_2\}$ corresponds to $\hat{\mathcal{F}}_1$.

In the following, we give results for general families $\mathcal{F}, \tilde{\mathcal{F}}, \hat{\mathcal{F}}$. However, by the above remarks, we prove them only in the case $\mathcal{F} = \mathcal{F}_1, \tilde{\mathcal{F}} = \tilde{\mathcal{F}}_1$ and $\hat{\mathcal{F}} = \hat{\mathcal{F}}_1$. They will be also

valid for $\mathcal{F}_2, \sigma'(\tilde{\mathcal{F}}_2)$ and $\hat{\sigma}'(\hat{\mathcal{F}}_2)$ and, by composition with σ' and $\hat{\sigma}'^{-1}$, for $\tilde{\mathcal{F}}_2$ and $\hat{\mathcal{F}}_2$ as well. Thus they will be established for arbitrary $\mathcal{F}, \tilde{\mathcal{F}}$ and $\hat{\mathcal{F}}$.

2. Normality

THEOREM 2.0. *Let X and X' be homeomorphic Klein surfaces, $p_0 \in X$ and $p'_0 \in X'$ and $Q \in BMO_{loc}(X)$. The family \mathcal{F} of the Q -qc homeomorphisms $f : X \rightarrow X'$ with $f(p_0) = p'_0$ is normal.*

Proof based on the double orientable covering, $X' \neq \mathbb{P}\mathbb{R}^2$.

Let $(\tilde{X}, \tilde{\Pi}, X)$ and $(\tilde{X}', \tilde{\Pi}', X')$ be the double orientable coverings for X and X' respectively, and $\tilde{\mathcal{F}}$ the lifting family $\tilde{f} : \tilde{X} \rightarrow \tilde{X}'$ of the homeomorphisms $f \in \mathcal{F}$ normalized by $\tilde{f}(\tilde{p}_0) = \tilde{p}'_0$, where $\tilde{p}_0 \in \tilde{\Pi}^{-1}(p_0)$ and $\tilde{p}'_0 \in \tilde{\Pi}'^{-1}(p'_0)$. According to Theorem 2.0 in [AC S1] the family

$$\tilde{\mathcal{G}} = \{\tilde{g} : \tilde{X} \rightarrow \tilde{X}'; \tilde{g} \tilde{Q} - qc \text{ homeomorphism, } \tilde{Q} = Q \circ \tilde{\Pi} \in BMO_{loc}(\tilde{X}), \quad (2.1)$$

$$\tilde{g}(\tilde{p}_0) = \tilde{p}'_0\},$$

except for the case $\tilde{X}' = \hat{\mathbb{C}}$, is normal. As $\tilde{\mathcal{F}} \subset \tilde{\mathcal{G}}$, $\tilde{\mathcal{F}}$ is a normal family too. By Proposition 1.2 (i) the family \mathcal{F} is then normal.

Proof for Theorem 2.0 in the case $X' = \mathbb{P}\mathbb{R}^2$.

The orienting double of $\mathbb{P}\mathbb{R}^2$ is $\hat{\mathbb{C}}$, which also gives the universal covering surface of $\mathbb{P}\mathbb{R}^2$. We consider the covering $(\hat{\mathbb{C}}, \Pi, \mathbb{P}\mathbb{R}^2)$ and $\sigma : \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$, $\sigma(z) = -(\bar{z})^{-1}$ for $z \in \mathbb{C}^*$, $\sigma(0) = \infty$ and $\sigma(\infty) = 0$ the corresponding antianalytic involution. Each lifting \hat{f} of any $f \in \mathcal{F}$ will verify the relation

$$\hat{f} \circ \sigma = \sigma \circ \hat{f}. \quad (2.2)$$

As usually, we denote by $\hat{\mathcal{F}}$ the family of these $\hat{Q} = Q \circ \Pi$ -qc homeomorphisms and prove in two steps that $\hat{\mathcal{F}}$ is 1. normal, hence normal.

1°. Let Δ be an open hemisphere of $\hat{\mathbb{C}}$ such that $\infty \notin \bar{\Delta}$, $\Gamma = \partial\Delta$ and $\Delta^c = \hat{\mathbb{C}} \setminus \bar{\Delta}$. The Jordan curve $\hat{f}(\Gamma)$ consists of pairs of diametrically opposite points, since if $z \in \Gamma$ then by (2.2), $\hat{f}(z)$ and $\sigma\hat{f}(z) \in \hat{f}(\Gamma)$. As the spherical distance between two diametrically opposite points in $\hat{\mathbb{C}}$ is equal to $\pi/2$ and $\hat{f}(\Delta) \cap \hat{f}(\Gamma) = \emptyset$, every $\hat{f} : \Delta \rightarrow \hat{\mathbb{C}}$ omits $\hat{f}(\Gamma)$, hence pairs of points with spherical distance $\pi/2$. Since $\Delta \subset \mathbb{C}$, Theorem 5.1 in [RSY 1] or Theorem 3B in [T] implies that the family $\hat{\mathcal{F}}|_{\Delta}$ is normal.

2°. Let now $\infty \in \Delta$. According to step 1°, $\hat{\mathcal{F}}|_{\Delta^c}$ is normal and thus the family $\{\sigma \circ \hat{f} \circ \sigma|_{\Delta}; \hat{f} \in \hat{\mathcal{F}}\}$ is also normal, but this family coincides by (2.2) with $\hat{\mathcal{F}}|_{\Delta}$.

It follows from Proposition 1.2 (i) that \mathcal{F} is also normal.

Remark 2.1. This proof shows that Theorem 2.0 for $X' = \mathbb{P}\mathbb{R}^2$ remains valid without the condition $f(p_0) = p'_0$. The same is true for any X' compact.

THEOREM 2.1. *Let X, X', Q and \mathcal{F} be as in Theorem 2.0, (\hat{X}, Π, X) and (\hat{X}', Π', X') universal coverings, $\hat{p}_0 \in \Pi^{-1}(p_0)$ and $\hat{p}'_0 \in \Pi'^{-1}(p'_0)$, $\hat{Q} = Q \circ \Pi$ and $\hat{\mathcal{F}}$ the family of \hat{Q} -qc and \hat{Q} -antiqc homeomorphisms $\hat{f} : \hat{X} \rightarrow \hat{X}'$, where \hat{f} is the lifting of a homeomorphism $f \in \mathcal{F}$ normalized by the condition $\hat{f}(\hat{p}_0) = \hat{p}'_0$. Then $\hat{\mathcal{F}}$ is normal.*

Proof. In the case $X' = \mathbb{P}\mathbb{R}^2$, hence $\widehat{X}' = \widehat{C}$, the normality of $\widehat{\mathcal{F}}$ was established in the above proof of Theorem 2.0. Otherwise, let us remark that the family $\widehat{\mathcal{F}}$ is obtained by lifting of the family $\widetilde{\mathcal{F}}$ with respect to $(\widetilde{X}, \widetilde{\Pi}, \widetilde{X})$ and $(\widetilde{X}', \widetilde{\Pi}', \widetilde{X}')$ if we chose $\widetilde{p}_0 = \widetilde{\Pi}(\widehat{p}_0)$ and $\widetilde{p}'_0 = \widetilde{\Pi}'(\widehat{p}'_0)$. Hence $\widehat{\mathcal{F}}$ is a subfamily of the family $\widehat{\mathcal{G}}$ consisting in the liftings \widehat{g} of $\widetilde{g} \in \widetilde{\mathcal{G}}$, (2.1), with the condition $\widehat{g}(\widehat{p}_0) = \widehat{p}'_0$. Theorem 2.1 in [AC S1] implies that $\widehat{\mathcal{G}}$ is normal and thereafter $\widehat{\mathcal{F}}$ is normal as well.

Remark 2.2. From Theorem 2.1 and Proposition 1.2 (i) we obtain another proof of Theorem 2.0.

3. Limit mappings

THEOREM 3.0. *Under the hypotheses of Theorem 2.0, the limit f_0 of a l.u. convergent sequence $\{f_n\} \subset \mathcal{F}$ is a Q -qc embedding.*

Proof. We lift f_n and f_0 respectively to the double coverings $(\widetilde{X}, \widetilde{\Pi}, X), (\widetilde{X}', \widetilde{\Pi}', X')$. By Proposition 1.3, the sequence $\{\widetilde{f}_n\}$ l.u. converges to \widetilde{f}_0 and we deduce from Theorem 3.0 in [AC S1] that \widetilde{f}_0 is a \widetilde{Q} -qc embedding. Now we apply Proposition 1.1 2° to derive that f_0 is also a Q -qc embedding.

THEOREM 3.1. *Let $\widehat{\mathcal{F}}$ be the lifting of the family \mathcal{F} above to the universal coverings. If $\{\widehat{f}_n\} \subset \widehat{\mathcal{F}}$ l.u. converges to \widehat{f}_0 , then f_0 is a \widehat{Q} -qc or \widehat{Q} -antiqc embedding.*

Proof. By working with a family $\widetilde{\mathcal{F}}$ as in the proof of Theorem 2.1, $\{\widehat{f}_n\}$ is the lifting of the sequence $\{\widetilde{f}_n\}$ and we derive from Theorem 3.1 in [AC S1] that \widehat{f}_0 is a \widehat{Q} -qc embedding. If $X' = \mathbb{P}\mathbb{R}^2$, the assertion is Theorem 3.0 in [AC S1] for \widehat{C} .

Remark 3.1. Once again Proposition 1.1 2° gives that f_0 is a Q -qc embedding, hence another proof of Theorem 3.0.

4. Compactness

In the BMO_{loc} - QC class there are l.u. convergent sequences $\{f_n\} \subset \mathcal{F}$ so that $f_0 = \lim f_n$ is only a Q -qc embedding and not a homeomorphism. Hence $f_0 \notin \mathcal{F}$, what shows that \mathcal{F} is not always closed.

Example. We use Example 1° in 4. [AC S1] to obtain first for a $Q \in BMO_{loc}(D^*)$, $D^* = \{w; 0 < |w| < 1\}$, a sequence $\{F_n\}$ of Q -qc homeomorphisms $F_n : D^* \rightarrow D^*$, $F_n(w) = w'$, which l.u. converges to $F_0 : D^* \rightarrow A = \{w'; e^{-\frac{1}{2}} < |w'| < 1\}$, a Q -qc embedding: $D^* \rightarrow D^*$.

To this end we consider $f_n| : \{z; x < 0\} \rightarrow \{z'; x' < 0\}$ defined by (4.1) and (4.2) in [AC S1]:

$$z' = f_0(z) = \arctan x + i(x + y), \quad \arctan 0 = 0, \quad (4.1)$$

and

$$z' = f_n(z) = \begin{cases} f_0(z) & \text{if } |x| \leq x_n \\ \frac{\arctan x_n}{x_n} x + i[\pm x_n + (1 + x_n^2) \frac{\arctan x_n}{x_n} (x \mp x_n) + y] & \\ \text{with the upper sign if } x > x_n \text{ and the lower if } x < -x_n. \end{cases} \quad (4.2)$$

where $x_n > 0, x_n \rightarrow \infty$, and the universal coverings $w = e^z : \{z; x < 0\} \rightarrow D^*$ and $w' = e^{z'} : \{z'; x' < 0\} \rightarrow D^*$. The homeomorphisms F_n are obtained from $f_n|$ by factorization

with respect to these coverings. Similarly, $f_0| : \{z; x < 0\} \rightarrow \{z'; -\frac{\pi}{2} < x' < 0\}$, the l.u. limit of the sequence $\{f_n|\}$, factorizes to F_0 . Let us remark that F_n and F_0 can be homeomorphically continued to $|w| = 1$ by the identical mapping.

Let now Y be a Klein surface, $p_0 \in Y, W$ a Jordan domain on Y with $p_0 \in W$ and $W \subset\subset U$, where (U, κ) is a chart from the dianalytic atlas of Y , $\kappa(p) = w$ with $p \in U$, $\kappa(p_0) = 0$, $\kappa(W) = D \subset \kappa(U)$. We take $Y' = Y$, $p'_0 = p_0, W' = W, (U', \kappa') = (U, \kappa), X = Y \setminus \{p_0\}$ and $X' = Y' \setminus \{p'_0\}$. We transport by means of the charts (U, κ) and (U', κ') the mappings F_n and F_0 to the Klein surfaces X and X' : we define $g_n = \kappa^{-1} \circ F_n \circ \kappa : W \setminus \{p_0\} \rightarrow W' \setminus \{p'_0\}$ and $g_0 = \kappa^{-1} \circ F_0 \circ \kappa : W \setminus \{p_0\} \rightarrow \kappa^{-1}(A)$ and continue them by the identity: $X \setminus W \rightarrow X' \setminus W'$, preserving the notations g_n and g_0 respectively. Thus we obtained the sequence $\{g_n\}$ of Q -qc homeomorphisms, $g_n : X \rightarrow X'$, which l.u. converges to $g_0 : X \rightarrow X'$, a Q -qc embedding.

Remark 4.1. The example is also valid for Y , hence X , Riemann surface.

Nevertheless, there are cases when the family \mathcal{F} is closed:

THEOREM 4.0. *Under the hypotheses of Theorem 2.0, if X' is compact then \mathcal{F} is closed. Similarly, the family of all Q -qc homeomorphisms $f : X \rightarrow X'$ is closed.*

Proof. Let $\{f_n\} \subset \mathcal{F}$ be a sequence which l.u. converges to f_0 . By Theorem 3.0, f_0 is a Q -qc embedding. Since X is compact, $f_0(X)$ is open and closed in X' , whereas $f_0(X) = X'$ and $f_0 \in \mathcal{F}$.

THEOREM 4.0*. *Let \mathcal{F} be a family as in Theorem 2.0 with the property: there exists a function $C \in BMO_{loc}(X')$ such that for every $f \in \mathcal{F}, f^{-1}$ is a C -qc homeomorphism. Then \mathcal{F} is closed.*

Proof. If $\{f_n\} \subset \mathcal{F}$ l.u. converges to f_0 , then by Theorem 3.0 f_0 is a Q -qc embedding. The family $\{f_n^{-1}\}$ being also normal, cf. Theorem 2.0, there is a subsequence $\{f_{n_k}^{-1}\}$ l.u. convergent to a mapping φ_0 which is a C -qc embedding again by Theorem 3.0. By passing to the limit it follows that $\varphi_0 = f_0^{-1}$, i.e. $f_0 \in \mathcal{F}$.

5. Compactness in BMO case

THEOREM 5.0. *If $Q \in BMO(X)$, then the family \mathcal{F} in Theorem 2.0 is closed.*

Proof. Suppose that $\{f_n\} \subset \mathcal{F}$ l.u. converges to f_0 . By Proposition 1.3 we lift $\{f_n\}$ to the double coverings, obtaining a sequence $\{\tilde{f}_n\}$ which l.u. converges to \tilde{f}_0 . Then Theorem 5.0 in [AC S1] implies $\tilde{f}_0 \in \tilde{\mathcal{F}}$ and Proposition 1.1 3^o yields $f_0 \in \mathcal{F}$.

THEOREM 5.1*. *Let $Q \in BMO(X)$ and \mathcal{F} a family as in Theorem 2.0, then $\tilde{\mathcal{F}}$, the lifting of \mathcal{F} to the double orientable coverings, is closed.*

Proof. Let $\{\tilde{f}_n\} \subset \tilde{\mathcal{F}}$ be a l.u. convergent sequence and \tilde{f}_0 its limit. Since the family $\tilde{\mathcal{G}}$ (2.1), which includes $\tilde{\mathcal{F}}$, is closed by Theorem 5.0 in [AC S1], $\tilde{f}_0 \in \tilde{\mathcal{G}}$ and thus \tilde{f}_0 is a \tilde{Q} -qc homeomorphism. Proposition 1.1 implies that $\{f_n\}$ l.u. converges to f_0 , where $f_n \in \mathcal{F}$ has the lifting \tilde{f}_n and f_0 the lifting \tilde{f}_0 . By Theorem 5.0, $f_0 \in \mathcal{F}$ and thus $\tilde{f}_0 \in \tilde{\mathcal{F}}$, whence $\tilde{\mathcal{F}}$ is closed.

THEOREM 5.1. *Under the hypotheses of Theorem 5.1* $\hat{\mathcal{F}}$, the lifting of \mathcal{F} to the universal coverings, is closed.*

Proof. Let $\{\hat{f}_n\} \subset \hat{\mathcal{F}}$ be l.u. convergent to \hat{f}_0 . Proposition 1.1 applied to $\tilde{\mathcal{F}}$ and $\hat{\mathcal{F}}$ shows that $\{\tilde{f}_n\}$ l.u. converges to \tilde{f}_0 , where \hat{f}_n and \hat{f}_0 are liftings of \tilde{f}_n and \tilde{f}_0 respectively. By Theorem 5.1*, $\tilde{f}_0 \in \tilde{\mathcal{F}}$ and then, by Theorem 5.1 in [AC S1] $\hat{f}_0 \in \hat{\mathcal{F}}$, i.e. $\hat{\mathcal{F}}$ is closed.

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