LIBERTAS MATHEMATICA, VOL XX (2000)

ON SOME CLASSES OF MEROMORPHICALLY p-VALENT STARLIKE FUNCTIONS WITH POSITIVE COEFFICIENTS

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Abstract

Let $\Sigma(p)$ denote the class of functions of the form

$$f(z) = \frac{a}{z^p} + \sum_{n=1}^{\infty} a_{p+n-1} z^{p+n-1}, \;\; a > 0, \;\; a_{p+n-1} \geq 0, \;\; p \in \mathbb{N} = \{1, 2, ...\},$$

which are regular and p-valent in the punctured disc $U^* = \{z : 0 < |z| < 1\}$. Let $\Sigma^*(p, \alpha)$ $(0 \leq \alpha < p)$ denote the subclasses of $\Sigma(p)$ satisfying

$$\operatorname{Re} \frac{zf'(z)}{f(z)} < -\alpha \quad \text{for } z \in U^*.$$

The functions in $\Sigma^*(p, \alpha)$ are called meromorphically p-valent starlike of order α functions with positive coefficients.

In this paper we obtain a characterisation theorem, coefficient estimates, a distortion theorem, closure theorems and a radius of convexity of order δ $(0 \leq \delta < p)$ for the class $\Sigma^*(p, \alpha)$.

1991 Mathematics Subject Classification. 30C45 and 30C50.

Key words - p-valent, regular, meromorphically

1. Introduction

Let $\Sigma(p)$ denote the class of functions of the form

$$f(z) = \frac{a}{z^p} + \sum_{n=1}^{\infty} a_{p+n-1} z^{p+n-1}, \;\; a > 0, \;\; a_{p+n-1} \geq 0, \;\; p \in \mathbb{N}, \tag{1}$$

which are regular and p-valent in the punctured disc $U^* = \{z : 0 < |z| < 1\}$. A function f in $\Sigma(p)$ is said to belong to $\Sigma^*(p, \alpha)$, the class of meromorphically p-valent starlike of order α functions with positive coefficients $(0 \leq \alpha < p)$ if and only if

$$\operatorname{Re} \frac{zf'(z)}{f(z)} < -\alpha \quad \text{for } z \in U^*. \tag{2}$$

A function f in $\Sigma(p)$ is said to be meromorphically p-valent convex of order α if

$$\mathrm{Re}\,\frac{zf''(z)}{f'(z)}+1<-\alpha \quad \text{for } z\in U^*. \tag{3}$$

In this paper we obtain a characterization theorem, coefficient estimates, a distortion theorem, closure theorems and a radius of convexity of order δ $(0\le\delta<p)$ for the class $\Sigma^*(p,\alpha)$, . The techniques used are similar to those of Silverman [5] , Uralegaddi and Ganigi [6] , Mogra [4] , Aouf [1] , [2] and Darwish, Aouf and Sălăgean [3] .

2. Characterization Theorem

Theorem 1. *Let the function f be defined by (1). Then f is in the class $\Sigma^*(p,\alpha)$ if and only if*

$$\sum_{n=1}^{\infty}(p+n-1+\alpha)a_{p+n-1}\le a\,(p-\alpha). \tag{4}$$

The result is sharp for the functions

$$f_{p+n-1}(z)=\frac{a}{z^p}+\frac{a(p-\alpha)}{p+n-1+\alpha}\,z^{p+n-1},\quad n\in\mathbb{N}. \tag{5}$$

Proof. We suppose that (4) holds. We note that (2) is equivalent to

$$\left|\frac{zf'(z)}{f(z)}+p\right|<\left|\frac{zf'(z)}{f(z)}-p+2\alpha\right|,\quad z\in U^*. \tag{6}$$

But f satisfies (6) because

$$|zf'(z)+f(z)|-|zf'(z)-pf(z)+2\alpha f(z)|=$$

$$\left|-p\frac{a}{z^p}+\sum_{n=1}^{\infty}(p+n-1)a_{p+n-1}z^{p+n-1}+p\frac{a}{z^p}+\sum_{n=1}^{\infty}pa_{p+n-1}z^{p+n-1}\right|-$$

$$-\left|-\frac{pa}{z^p}+\sum_{n=1}^{\infty}[(p+n-1)-p+2\alpha]a_{p+n-1}+\frac{a(2\alpha-p)}{z^p}\right|=$$

$$\left|\sum_{n=1}^{\infty}(2p+n-1)a_{p+n-1}z^{p+n-1}\right|-\left|z^{-p}\right|\left|2a\,(\alpha-p)+\sum_{n=1}^{\infty}(n-1+2\alpha)a_{p+n-1}z^{p+n-1}\right|$$

$$\le\sum_{n=1}^{\infty}(2p+n-1)a_{p+n-1}+2a(\alpha-p)+\sum_{n=1}^{\infty}(n-1+2\alpha)a_{p+n-1}=$$

$$=\sum_{n=1}^{\infty}2(p+n-1+\alpha)a_{p+n-1}-2a(p-\alpha)\le 0.$$

The inequality above implies that $f\in\Sigma^*(p,\alpha)$.

Conversely, we suppose that f satisfies (6). If we denote

$$A=\left|\frac{\frac{zf'(z)}{f(z)}+p}{\frac{zf'(z)}{f(z)}-p+2\alpha}\right|,$$

then we have

$$A = \left| \frac{\sum_{n=1}^{\infty}(2p+n-1)a_{p+n-1}z^{2p+n-1}}{-2a(p-\alpha)+\sum_{n=1}^{\infty}(n-1+2\alpha)a_{p+n-1}z^{2p+n-1}} \right| < 1, \quad z \in U^*. \tag{7}$$

For real z, $z \in (0,1)$ the inequality (7) can be rewritten

$$-1 < \frac{\sum_{n=1}^{\infty}(2p+n-1)a_{p+n-1}z^{2p+n-1}}{2a(p-\alpha)-\sum_{n=1}^{\infty}(n-1+2\alpha)a_{p+n-1}z^{2p+n-1}} < 1. \tag{8}$$

We note that

$$E(z) = 2a(p-\alpha) - \sum_{n=1}^{\infty}(n-1+2\alpha)a_{p+n-1}z^{2p+n-1} > 0$$

when $z \in [0,1)$, because $E(z)$ is a continuous function of z, $E(z) \neq 0$ and $E(0) > 0$. Upon clearing the denominator in (8) and by letting $z \to 1$ through positive values, we obtain (4).

Corollary 1. *Let the function f defined by (1) be in the class $\Sigma^*(p,\alpha)$. Then*

$$a_{p+n-1} \leq \frac{a(p-\alpha)}{p+n-1+\alpha}, \quad n \in \mathbb{N}. \tag{9}$$

The equality in (9) is attained for the function $f = f_{p+n-1}$ given by (5).

3. Distortion Theorem

Theorem 2. *Let the function f defined by (1) with the coefficient a fixed be in the class $\Sigma^*(p,\alpha)$. Then, for fixed a,*

$$\frac{a}{r^p} - \frac{a(p-\alpha)}{p+\alpha} r^p \leq |f(z)| \leq \frac{a}{r^p} + \frac{a(p-\alpha)}{p+\alpha} r^p,$$

for $0 < |z| = r < 1$. The result is sharp for the function $f = f_p$ given by (see(5))

$$f_p(z) = \frac{a}{z^p} + \frac{a(p-\alpha)}{p+\alpha} z^p.$$

Proof. It follows from Theorem 1 that

$$(p+\alpha)\sum_{n=1}^{\infty} a_{p+n-1} \leq \sum_{n=1}^{\infty}(p+n-1+\alpha)a_{p+n-1} \leq a(p-\alpha)$$

and then

$$\sum_{n=1}^{\infty} a_{p+n-1} \leq \frac{a(p-\alpha)}{p+\alpha}.$$

Hence we have

$$|f(z)| \leq \frac{a}{r^p} + \sum_{n=1}^{\infty} a_{p+n-1} r^{p+n-1} \leq \frac{a}{r^p} + \sum_{n=1}^{\infty} a_{p+n-1} r^p \leq \frac{a}{r^p} + \frac{a(p-\alpha)}{p+\alpha} r^p$$

and

$$|f(z)| \geq \frac{a}{r^p} - \sum_{n=1}^{\infty} a_{p+n-1} \geq \frac{a}{r^p} - \frac{a(p-\alpha)}{p+\alpha} r^p.$$

4. Closure Theorems

Let the functions f_j be defined, for $j \in \{1,2,...,m\}$, by

$$f_j(z) = \frac{a_j}{z^p} + \sum_{n=1}^{\infty} a_{p+n-1,j} z^{p+n-1}, \quad a_j > 0, \quad a_{p+n-1,j} \geq 0 \tag{10}$$

for $z \in U^*$.

We shall prove the following results for the closure of functions in the classes $\Sigma^*(p,\alpha)$.

Theorem 3. *Let the functions f_j, $j \in \{1,2,...,m\}$ defined by (10) be in the class $\Sigma^*(p.\alpha)$. Then the function h defined by*

$$h(z) = \sum_{j=1}^{m} d_j f_j(z), \qquad d_j \geq 0 \tag{11}$$

where

$$\sum_{j=1}^{m} d_j = 1 \tag{12}$$

is also in the same class $\Sigma^*(p,\alpha)$.

Proof. According to the definition of h we can write

$$h(z) = \frac{b}{z^p} + \sum_{n=1}^{\infty} b_{p+n-1}\, z^{p+n-1},$$

where

$$b = \sum_{j=1}^{m} a_j\, d_j \ \text{ and } \ b_{p+n-1} = \sum_{j=1}^{m} d_j a_{p+n-1,j}, \quad j \in \{1,2,...,m\}.$$

Since $f_j(z)$ are in $\Sigma^*(p,\alpha)$, $(j \in \{1,2,...,m\})$, by means of Theorem 1, we have

$$\sum_{n=1}^{\infty} \left[\frac{p+n-1+\alpha}{p-\alpha}\right] a_{p+n-1,j} \leq a_j$$

for every $j \in \{1,2,...,m\}$. Therefore we have

$$\sum_{n=1}^{\infty} \left[\frac{p+n-1+\alpha}{p-\alpha}\right] \left(\sum_{j=1}^{m} d_j\, a_{p+n-1,j}\right) =$$

$$= \sum_{j=1}^{m} d_j \left(\sum_{n=1}^{\infty} \left[\frac{p+n-1+\alpha}{p-\alpha}\right] a_{p+n-1,j}\right) \leq \sum_{j=1}^{m} d_j\, a_j = b$$

which shows that $h \in \Sigma^*(p,\alpha)$. This completes the proof.

Corollary 3.1. *The class $\Sigma^*(p,\alpha)$ is closed under convex linear combinations.*

Theorem 4. *Let define $f_{p-1}(z) = a\, z^{-p}$ and let f_{p+n-1}, $n \in \mathbb{N}$ be defined by (5). Then f is in the class $\Sigma^*(p,\alpha)$ if and only if it can be expressed in the form*

$$f(z) = \sum_{n=0}^{\infty} \lambda_{p+n-1}\, f_{p+n-1}(z), \tag{13}$$

where $\lambda_{p+n-1} \geq 0 \ \ (n \in \mathbb{N} \cup \{0\})$ *and*

$$\sum_{n=0}^{\infty} \lambda_{p+n-1} = 1. \tag{14}$$

Proof. Assume that f defined by (1) satisfies (13); then

$$f(z) = \sum_{n=0}^{\infty} \lambda_{p+n-1}\, f_{p+n-1}(z) = \lambda_{p-1}\, f_{p-1}(z) + \sum_{n=1}^{\infty} \lambda_{p+n-1}\, f_{p+n-1}(z) =$$

$$= \frac{a\,\lambda_{p-1}}{z^p} + \sum_{n=1}^{\infty} \frac{a\,\left[p+n-1+\alpha+(p-\alpha)\,z^{2p+n-1}\right]\lambda_{p+n-1}}{z^p\,(p+n-1+\alpha)} =$$

$$= \frac{a}{z^p}\left[\lambda_{p-1} + \sum_{n=1}^{\infty} \frac{(p+n-1+\alpha)\,\lambda_{p+n-1}}{p+n-1+\alpha}\right] + \sum_{n=1}^{\infty} \frac{a\,(p-\alpha)\,\lambda_{p+n-1}\,z^{p+n-1}}{p+n-1+\alpha}.$$

It follows that

$$\sum_{n=1}^{\infty}(p+n-1+\alpha)a_{p+n-1} = \sum_{n=1}^{\infty}(p+n-1+\alpha)\,\frac{a\,(p-\alpha)\lambda_{p+n-1}}{p+n-1+\alpha} =$$

$$= a\,(p-\alpha)\sum_{n=1}^{\infty} \lambda_{p+n-1} = a\,(p-\alpha)\,(1-\lambda_{p-1}) \leq a\,(p-\alpha)..$$

So by Theorem 1 we have that $f \in \Sigma^*(p,\alpha)$.

Conversely, assume that the function f defined by (1) belongs to the class $\Sigma^*(p,\alpha)$. Then

$$a_{p+n-1} \leq \frac{a\,(p-\alpha)}{p+n-1+\alpha}, \quad n \in \mathbb{N}.$$

Setting

$$\lambda_{p+n-1} = \frac{p+n-1+\alpha}{p-\alpha}\,a_{p+n-1}, \quad n \in \mathbb{N}$$

and

$$\lambda_{p-1} = 1 - \sum_{n=1}^{\infty} \lambda_{p+n-1},$$

we can see that f can be expressed in the form (13). This completes the proof.

5. Radius of Convexity

In this section we determine the radius of convexity of order δ, $0 \leq \delta < p$, for the class $\Sigma^*(p,\alpha)$.

Theorem 5. *Let the function* f *defined by (1) be in the class* $\Sigma^*(p,\alpha)$*; then* f *is convex of order* δ, $0 \leq \delta < p$, *in the punctured disc* $\{z : 0 < |z| < r^*(p,\alpha,\delta)\}$, *where*

$$r^*(p,\alpha,\delta) = \inf_n \left\{ \frac{p(p-\delta)(p+n-1+\alpha)}{(p-\alpha)(p+n-1)(3p+n-1-\delta)} \right\}^{\frac{1}{2p+n-1}}, \quad n \in \mathbb{N}.$$

The result is sharp.

Proof. It is sufficient to demonstrate that (see (3))

$$\left|1+p+\frac{zf''(z)}{f'(z)}\right| \le p-\delta,\ \ 0\le\delta<p,\ \ \text{for}\ \ 0<|z|<r^*(p,\alpha,\delta).$$

We have

$$\left|\frac{(zf'(z))'+p\,f'(z)}{f'(z)}\right| \le \frac{\sum_{n=1}^{\infty}(p+n-1)(2p+n-1)\,a_{p+n-1}|z|^{2p+n-1}}{p\,a-\sum_{n=1}^{\infty}(p+n-1)\,a_{p+n-1}|z|^{2p+n-1}}.$$

Thus

$$\left|\frac{(zf'(z))'+p\,f'(z)}{f'(z)}\right| \le p-\delta$$

if

$$(15)\qquad \sum_{n=1}^{\infty}(p+n-1)(3p+n-1-\delta)\,a_{p+n-1}|z|^{2p+n-1} \le p\,(p-\delta)\,a$$

But Theorem 2 ensures

$$\sum_{n=1}^{\infty}\frac{p+n-1+\alpha}{p-\alpha}\,a_{p+n-1} \le a.$$

We note that (15) holds if

$$\frac{(p+n-1)(3p+n-1-\delta)}{p\,(p-\delta)\,a}\,|z|^{2p+n-1} \le \frac{p+n-1+\alpha}{(p-\alpha)\,a}$$

or if

$$|z| \le \left\{\frac{p(p-\delta)(p+n-1+\alpha)}{(p-\alpha)(p+n-1)(3p+n-1-\delta)}\right\}^{\frac{1}{2p+n-1}},\quad n\in\mathbb{N}.$$

Hence f is convex of order δ, $(0\le\delta<p)$ in $0<|z|<r^*(p,\alpha,\delta)$.

The sharpness for the class $\Sigma^*(p,\alpha)$ follows by taking the appropriate function f given by (5). This completes the proof of the theorem.

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