LIBERTAS MATHEMATICA, VOL XX (2000)

LOEWNER CHAINS AND UNIVALENCE CRITERIA

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In this note we obtain a sufficient condition for univalence of a holomorphic mapping by using the method of subordination chains.

Let $\mathbb{C}^n$ denote the space of n complex variables $z = (z_1, \dots, z_n)$ with the usual inner product, $\langle \cdot, \cdot \rangle$ euclidian norm $\| \cdot \|$ and open unit ball B^n.

We denote by $\mathcal{L}(\mathbb{C}^n)$ the space of continuous linear operators from $\mathbb{C}^n$ into $\mathbb{C}^n$, i.e. the $n \times n$ complex matrices $A = (A_{jk})$ with the standard operator norm:

$$\|A\| = \sup\{\|Az\| : \ \|z\| \le 1\}, \quad a \in \mathcal{L}(\mathbb{C}^n).$$

The class of holomorphic mappings $f(z) = (f_1(z), \dots, f_n(z)), \ z \in B^n$, from B^n into $\mathbb{C}^n$ is denoted by $\mathcal{H}(B^n)$. We say that $f \in \mathcal{H}(B^n)$ is locally biholomorphic (locally univalent) in B^n if f has a local holomorphic inverse at each point in B^n or equivalently if the derivative:

$$Df(z) = \left(\frac{\partial f_k(z)}{\partial z_j} \right), \quad 1 \le j, k \le n,$$

is nonsingular at each point $z \in B^n$.

If $f, g \in \mathcal{H}(B^n)$, we say that f is subordinate to g (in B^n) if there exists a Schwarz function v such that $f(z) = g(v(z)), \ z \in B^n$, and we shall write $f \prec g$ to indicate that f is subordinate to g.

The function $L : B^n \times I \to \mathbb{C}^n, \ I = [0, \infty)$ is a Loewner chain (subordination chain) if for all $t \in I$ the function $L(\cdot, t) \in \mathcal{H}(B^n)$ and for all $0 \le s < t$ we have $L(\cdot, s) \prec L(\cdot, t)$.

We shall use the following theorem to prove our results.

Theorem 1 [1]. *Let $L(z, t) = a_1(t)z + \dots, \ a_1(t) \ne 0$ be a function from $B^n \times [0, \infty)$ into $\mathbb{C}^n$ such that:*

(i) For each $t \ge 0, \ L(\cdot, t) \in \mathcal{H}(B^n)$.

(ii) $L(z, t)$ is absolutely continuous of t, locally uniformly with respect to B^n.

Let $h(z, t)$ be a function from $B^n \times [0, \infty)$ into $\mathbb{C}^n$ such that:

(iii) For each $t \ge 0, \ h(\cdot, t) \in \mathcal{H}(B^n)$.

(iv) For each $t \ge 0, \ h(0, t) = 0$ and Re $\langle h(z, t), z \rangle \ge 0, \ \forall \ z \in B^n$.

(v) For each $T > 0$ and $r \in (0, 1)$ there is a number $K = K(r, T)$ such that $\|h(z, t)\| \le K(r, T)$ where $\|z\| \le r$ and $0 \le t \le T$.

Suppose $h(z,t)$ satisfies:

$$\frac{\partial L(z,t)}{\partial t} = DL(z,t)h(z,t) \quad a.e. \quad t \geq 0, \ \forall\ z \in B^n. \tag{1}$$

Further, suppose:

(a) $|a_1(t)| \to \infty$ where $t \to \infty$, $a_1(\cdot) \in C^1([0,\infty))$.

(b) There is a sequence $\{t_m\}$, $t_m > 0$, $t_m \to \infty$ such that

$$\lim_{n\to\infty} \frac{L(z,t)}{a_1(t_m)} = F(z) \tag{2}$$

locally uniformly in B^n, where $F \in \mathcal{H}(B^n)$.

Then for each $s \geq 0$, $L(\cdot, s)$ is univalent on B^n.

1. Main results

Theorem 2. *Let $f \in \mathcal{H}(B^n)$, $f(0) = 0$, $Df(0) = I_n$ be locally univalent in B^n, let $c \in \mathbb{C} \setminus \{-1\}$ with $|c| \leq 1$ and let α be a real number, $\alpha \geq 2$.*

If

$$\left\| c\|z\|^\alpha I_n + (1 - \|z\|^\alpha)[Df(z)]^{-1} D^2 f(z)(z,\cdot) - \left(\frac{\alpha}{2} - 1\right) I_n \right\| < \frac{\alpha}{2} \quad \text{for all } z \in B^n, \tag{3}$$

then the function f is univalent on B^n.

Proof. We shall show that (2) enables us to imbed f as the initial element $f(z) = L(z,0)$ in a suitable subordination chain.

We define

$$L(z,t) = f(e^{-t}z) + \frac{1}{1+c}(e^{\alpha t} - 1)e^{-t} Df(ze^{-t})(z), \quad t \in [0,\infty), \ z \in B^n. \tag{4}$$

Since $a_1(t) = \dfrac{e^{(\alpha-1)t}(1 + ce^{-\alpha t})}{1+c}$ we deduce that $a_1(t) \neq 0$, $|a_1(t)| \to \infty$ when $t \to \infty$ and $a_1(t) \in C^1([0,\infty))$.

It is easy to check that:

$$L(z,t) = a_1(t)z + (\text{holomorphic term}),$$

thus

$$\lim_{t\to\infty} \frac{L(z,t)}{a_1(t)} = z$$

locally uniform with respect to B^n and thus (2) holds with $F(z) = z$.

Obviously $L(z,t)$ satisfies the absolute continuity requirements of Theorem 1.

From (4) we obtain:

$$\begin{aligned} DL(z,t) \ &= \ \frac{1}{1+c} e^{(\alpha-1)t} \frac{\alpha}{2} Df(ze^{-t}) \left[I_n + \frac{2}{\alpha} ce^{-\alpha t} I_n + \right. \\ &+ \ \left. \frac{2}{\alpha}(1 - e^{-\alpha t})[Df(ze^{-t})]^{-1} D^2 f(ze^{-t})(ze^{-t},\cdot) - I_n \left(1 - \frac{2}{\alpha}\right) \right]. \end{aligned} \tag{5}$$

If we let, for each fixed $(z,t) \in B^n \times [0,\infty)$, $E(z,t)$ the linear operator

$$E(z,t) = -\frac{2}{\alpha} ce^{-\alpha t} I_n - \frac{2}{\alpha}(1 - e^{-\alpha t})[Df(ze^{-t})]^{-1} D^2 f(ze^{-t})(ze^{-t},\cdot) + I_n \left(1 - \frac{2}{\alpha}\right) \tag{6}$$

then (5) becomes:

$$DL(z,t) = \frac{1}{1+c}e^{(\alpha-1)t}\frac{\alpha}{2}Df(ze^{-t})[I_n - E(z,t)]. \tag{7}$$

Next, we shall show that for each $z \in B^n$ and $t \in [0,\infty)$, $I_n - E(z,t)$ is an invertible operator.

For $t = 0$, $E(z,0) = \left(1 - \frac{2}{\alpha} - \frac{2}{\alpha}c\right) I_n$, we have $I_n - E(z,t) = \frac{2}{\alpha}(1+c)I_n$ and since $1 + c \neq 0$ it follows that $I_n - E(z,t)$ is an invertible operator.

For $t > 0$, since $E(\cdot,t) : \overline{B^n} \to \mathcal{L}(\mathbb{C}^n,\mathbb{C}^n)$ is holomorphic, by using the weak maximum modulus theorem [2], we obtain that $\|E(z,t\|$ can have no maximum in B^n unless $\|E(z,t)\|$ is of constant value throughout $\overline{B^n}$.

If $z = 0$ and $t > 0$, since $\alpha \geq 2$, we have

$$\|E(0,t)\| = \left\|\left(1 - \frac{2}{\alpha} - \frac{2}{\alpha}ce^{-\alpha t}\right) I_n\right\| = \left|1 - \frac{2}{\alpha} - \frac{2}{\alpha}ce^{-\alpha t}\right| < 1. \tag{8}$$

Also, we have

$$\|E(z,t)\| \leq \max_{\|w\|=1} \|E(w,t)\|. \tag{9}$$

Let now $u = e^{-t}w$, where $\|w\| = 1$, then $\|u\| = e^{-t}$ and so:

$$E(w,t) = -\frac{2}{\alpha}c\|u\|^{\alpha}I_n - \frac{2}{\alpha}(1 - \|u\|^{\alpha})[Df(u)]^{-1}D^2f(u)(u,\cdot) + I_n\left(1 - \frac{2}{\alpha}\right). \tag{10}$$

Using (3), (8), (9) and (10) it follows:

$$\|E(z,t)\| < 1, \quad t > 0. \tag{11}$$

Hence for $t > 0$, $I_n - E(z,t)$ is an invertible operator, too.

Further calculation shows that

$$\frac{\partial L(z,t)}{\partial t} = \frac{1}{1+c}e^{(\alpha-1)t}\frac{\alpha}{2}Df(ze^{-t})\left[I_n - \frac{2}{\alpha}ce^{-\alpha t}I_n - \right.$$
$$\left. -\frac{2}{\alpha}(1 - e^{-\alpha t})[Df(ze^{-t})]^{-1}D^2f(ze^{-t})(ze^{-t},\cdot) + I_n\left(1 - \frac{2}{\alpha}\right)\right](z)$$

and

$$\frac{\partial L(z,t)}{\partial t} = \frac{1}{1+c}e^{(\alpha-1)t}\frac{\alpha}{2}Df(ze^{-t})(I_n + E(z,t)](z). \tag{12}$$

Using (7) and (12) we obtain:

$$\frac{\partial L(z,t)}{\partial t} = DL(z,t)[I_n - E(z,t)]^{-1}[I_n + E(z,t)](z).$$

Hence $L(z,t)$ satisfies the differential equation (1) for all $z \in B^n$ and $t \geq 0$ where

$$h(z,t) = [I_n - E(z,t)]^{-1}[I_n + E(z,t)](z). \tag{13}$$

It remains to show that the function defined by (13) satisfies the conditions of Theorem 1. Clearly $h(z,t)$ satisfies the holomorphy and measurability requirements and $h(0,t) = 0$.

Furthermore, the inequality

$$\|h(z,t)\| = \|E(z,t)[h(z,t) + z]\| \leq \|E(z,t)\|\,\|h(z,t) + z\| < \|h(z,t) + z\|$$

implies that Re $\langle h(z,t), z\rangle \geq 0$, $\forall\ z \in B^n$, $t \geq 0$.

By using the inequality:

$$\|[I_n - E(z,t)]^{-1}\| \leq [1 - \|E(z,t)\|]^{-1}$$

we obtain:

$$\|h(z,t)\| \leq \frac{1+\|E(z,t)\|}{1-\|E(z,t)\|} \cdot \|z\|.$$

Hence $h(z,t)$ satisfies the condition (v) of Theorem 1 and it follows that the functions $L(z,t)$ $(t \geq 0)$ are univalent in B^n.

In particular $f(z) = L(z,0)$ is univalent in B^n.

Remark. For $\alpha = 2$, Theorem 2 becomes the n-dimensional version of Ahlfors and Becker's univalence criterion [2]. For $\alpha = 2$ and $c = 0$, Theorem 2 becomes the n-dimensional version of Becker's univalence criterion [3].

References

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