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ON SOME GRONWALL-BIHARI-TYPE INEQUALITIES

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Abstract

Based on operatorial inequalities, Riccati, homogeneous and Bernoulli-type inequalities are established.

1. INTRODUCTION

In the paper [5] (Stetsenko and Shaaban, 1986) it has been proved an operatorial inequality analogous to those of Gronwall and Bihari for monotonical and continuous operators. If E is a Banach space and K a semiordered cone, then $x \geq y$ means $x - y \in K$.

Theorem 1 from [5] states: if u verifies the inequality

$$u \leq Au + f, \tag{1}$$

where f is a fixed element and $A : E \to E$ is a monotonically increasing operator, and if the equation $y = Ay + f$ has a unique solution y^*, the limit of the sequence (y_n) defined by $y_{n+1} = Ay_n + f$, then $u \leq y^*$.

Gronwall's and Bihari's inequalities follow immediately from (1). There has been proved also a Bernoulli-type inequality analogous to the corresponding equation of the same name, in a particular case. In the present paper we use this method to derive a Riccati-type inequality analogous to the equation of the same type, a homogeneous inequality analogous to the homogeneous equation of Euler-type, and generalized Bernoulli-type inequality.

2. MAIN RESULTS

1. Riccati-type inequality

THEOREM 1. *Suppose that the following conditions are fulfilled:*
(i) $u \in \mathbb{C}[a, b]$, $a(t) \in \mathbb{C}[a, b]$.
(ii) $a(t) \geq 0$, $t \in [a, b]$.
(iii) $p, q, r \in \mathbb{R}_+$ *and* $q^2 \leq 4pr$

(iv) $u(t) \geq \dfrac{-q}{2p}$,

$$u(t) \leq C + \int_a^t [pa(s)u^2(s) + qa(s)u(s) + ra(s)]ds, \quad c > 0.$$

Then the function $u(t)$ *satisfies the inequality:*

$$u(t) \leq \exp \int_a^t (2pa(s)y_1(s) + qa(s))ds \left[C - \int_a^t 2pa(s) \exp \left(\int_a^s (2pa(\tau)y_1(\tau) + \right.\right.$$

$$\left.\left. + qa(\tau))d\tau \right) ds \right]^{-1}, \tag{1}$$

on the interval where

$$C - \int_a^t 2pa(s) \exp \left(\int_a^s (2pa(\tau)y_1(\tau) + qa(\tau))d\tau \right) ds \neq 0,$$

and $y_1(t)$ *is a particular solution of the equation of Riccati-type*

$$y'(t) = pa(t)y^2(t) + qa(t)y(t) + ra(t).$$

Proof. Let the operator A be defined by

$$Au = \int_a^t V(u(s))ds, \quad t \in [a, b],$$

where $V(u) = pau^2 + qau + ra$ is positive and increasing. If $y^*(t)$ is the solution of the equation

$$y(t) = C + \int_a^t [pa(s)y^2(s) + qa(s)y(s) + ra(s)]ds, \quad C > 0 \tag{2}$$

then

$$y^*(t) = \exp \int_a^t (2pa(s)y_1(s) + qa(s))ds \left[C - \int_a^t 2pa(s) \exp \left(\int_a^s (2pa(\tau)y_1(\tau) + \right.\right.$$

$$\left.\left. + qa(\tau))d\tau \right) ds \right]^{-1}, \tag{3}$$

from where (2) is comming out directly.

2. Homogeneous inequality

THEOREM 2. *Let* $y \in \mathbb{C}[a, b]$. *Suupose further that, for some* $C > 0$, $\alpha > 0$, *we have*

$$y(t) \leq C + \int_{t_0}^t \alpha\varphi\left(\frac{y(s)}{s}\right) ds, \quad t_0, t \in [a, b], \quad t_0 \leq t. \tag{4}$$

If $\varphi(u) > 0$ *is continuous, monotonically increasing, locally Lipschitzian and* $\alpha\varphi(u) \neq u$, *then* $y(t)$ *verifies the inequality* $y(t) \leq y^*(t)$, *where* $y^*(t) = tF^{-1}(\ln C|t|)$, F *is the primitive of the function* $\dfrac{1}{\alpha\varphi(u) - u}$ *and* F^{-1} *is its inverse.*

Proof. Define the operator A via

$$Ay = \int_{t_0}^{t} \alpha\varphi(y(s)/s)ds, \quad t_0, t \in [a, b], \quad t_0 \leq t.$$

The operator A is increasing and verifies the conditions of Theorem 1 from [5]. Note that $y^*(t)$ is the solution of equation

$$y(t) = C + \int_{t_0}^{t} \alpha\varphi(y(s)/s)ds. \tag{5}$$

Using the fact that $y^*(t)$ is the solution of differential equation $y'(t) = \alpha\varphi(y/t)$, then $y^*(t) = tF^{-1}(\ln C|t|)$, and by Theorem 1 from [5] it results $y(t) \leq y^*(t)$.

3. Bernoulli-type inequality

THEOREM 3. *Suppose that the following conditions hold:*
(i) $u \in \mathbb{C}[a, b]$, $\alpha(t), \beta(t) \in \mathbb{C}[a, b]$,
(ii) $\alpha(t), \beta(t) > 0$, $t \in [a, b]$,
(iii) $\alpha(t) + \beta(t)u^{n-1} > 0$, $t \in [a, b]$, $n > 0$, $n \neq 1$,
(iv) $u(t) \leq C + \int_a^t [\alpha(s)u(s) + \beta(s)u^n(s)]ds$, $C > 0$.
Then the function $u(t)$ *satisfies the inequality:*

$$u(t) \leq \left[(1-n)\int_a^b \beta(s)e^{(n-1)\int_a^s \alpha(\tau)d\tau}ds + C\right]^{\frac{1}{n-1}} \cdot e^{\int_a^t \alpha(s)ds}, \tag{6}$$

on the open interval where $(1-n)\int_a^b \beta(s)e^{(n-1)\int_a^s \alpha(\tau)d\tau}ds + C > 0$.

Proof. Let the operator A be defined by

$$Au(t) = \int_a^t [\alpha(s)u(s) + \beta(s)u^n(s)]ds, \quad t \in [a, b], \quad n > 0, \quad n \neq 1.$$

The operator A is increasing and verifies the conditions of Theorem 1 from [5]. Note that $y^*(t)$ is the solution of equation

$$y'(t) = \alpha(t)y(t) + \beta(t)y^n(t),$$

then

$$y^*(t) = \left[(1-n)\int_a^t \beta(s)e^{(n-1)\int_a^s \alpha(\tau)d\tau}ds + C\right]^{\frac{1}{n-1}} \cdot e^{\int_a^t \alpha(s)ds}, \tag{7}$$

and by Theorem 1 from [5] it results

$$u(t) \leq y^*(t).$$

References

[1] Corduneanu, C., *Differential and Integral Equations*, Al.I. Cuza University, Iaşi, 1977 (Rum.).

[2] Krasnoselski, M.A., *Positive Solutions of Operatorial Equations*, Fizmatgiz, Moscow, 1962 (Russ.).

[3] Laksmikantham, V., Leela, S., Martynyuk, A.A., *Stability Analysis of Nonlinear Systems*, Marcel Dekker, Inc., New York, 1989.

[4] Lungu, N., *Gronwall-Wendorff-Type Inequalities*, International Meeting on Ordinary Differentiaal Equations and their Applications, Firenze (Italy) September 20-24, 1993, 86-88.

[5] Stetsenko, V.Ya, Shaaban, M., *On Operatorial Inequalities Analogous to Gronwall-Bihari Ones*, D.A.N. Tadj., XXIX, No.7, 1986, 393-398 (Russ.).