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AN EXISTENCE RESULT ON NONCOMPACT INTERVALS TO FIRST ORDER INTEGRODIFFERENTIAL INCLUSIONS IN BANACH SPACES

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Abstract

In this paper an extension of Schaefer's theorem to multivalued maps on locally convex topological spaces due to Ma is used to investigate the existence of solutions on the semi infinite interval of first order initial value problems for an integrodifferential inclusion in Banach spaces.

Keywords: Initial value problem, Integrodifferential inclusion, Convex multivalued map, Existence, Fixed point.

Classification: 34 A60, 34 G20, 45 J05

1. Introduction

In the few past years several papers have been devoted to study the existence on compact intervals of mild, strong and classical solutions for differential and integrodifferential equations in abstract spaces. We refer to the books of Goldstein [5], Corduneanu [3], Heikkila and Lakshmikantham [6], Ladas and Lakshmikantham [9] and to the paper of Heikkila and Lakshmikantham [7] . However very few results are available for evolution inclusions on compact intervals see, for instance, the papers of Benchohra [2], Avgerinos and Papageorgiou [1] and Papageorgiou [12], [13].

The fundamental tools used in the proof of the above mentioned works are essentially fixed point arguments or the semigroups method.

In this paper we shall prove a theorem which assures the existence of mild solutions defined on an unbounded real interval J for the initial value problem (IVP for short) of the first order integrodifferential inclusion

$$y' - Ay \in F\left(t, \int_0^t K(t,s,y)ds\right) \quad \text{for a.e. } t \in J := [0, \infty), \tag{1.1}$$

$$y(0) = y_0 \tag{1.2}$$

where $F : J \times E \longrightarrow 2^E$ is a bounded, closed, convex multivalued map, $K : D \times E \longrightarrow E$, $D = \{(t,s) \in J \times J : t \geq s\}$, $y_0 \in E$, A is the infinitesimal generator of a strongly continuous semigroup $T(t), t \geq 0$ and E a real Banach space with the norm $\|\cdot\|$.

The method we are going to use is to reduce the problem (1.1)-(1.2) to the search for fixed points of a suitable multivalued map on the Fréchet space $C(J,E)$ and we make use of an extension due to Ma [11] of Schaefer's theorem [14] to multivalued maps on locally convex topological spaces.

2. Preliminaries

In this section, we introduce notations, definitions, and preliminary facts from multivalued analysis which are used throughout the paper.

J_m is the compact real interval $[0,m]$ $(m \in \mathbf{N})$.

$C(J,E)$ is the linear metric Fréchet space of continuous functions from J into E with the metric (see Corduneanu [3])

$$d(y,z) = \sum_{m=0}^{\infty} \frac{2^{-m}\|y-z\|_m}{1+\|y-z\|_m} \quad \text{for each} \quad y,z \in C(J,E),$$

where

$$\|y\|_m := \sup\{\|y(t)\| : t \in J_m\}.$$

$B(E)$ denotes the Banach space of bounded linear operators from E into E.

A measurable function $y : J \longrightarrow E$ is Bochner integrable if and only if $\|y\|$ is Lebesgue integrable. (For properties of the Bochner integral see Yosida [15]).

$L^1(J,E)$ denotes the Banach space of continuous functions $y : J \longrightarrow E$ which are Bochner integrable normed by

$$\|y\|_{L^1} = \int_0^{\infty} \|y(t)\|dt \quad \text{for all} \quad y \in L^1(J,E).$$

U_p denotes the neighbourhood of 0 in $C(J,E)$ defined by

$$U_p := \{y \in C(J,E) : \|y\|_m \leq p \quad \text{for each} \quad m \in \mathbf{N}\}.$$

The convergence in $C(J,E)$ is the uniform convergence on compact intervals, i.e. $y_j \longrightarrow y$ in $C(J,E)$ if and only if for each $m \in \mathbf{N}$, $\|y_j - y\|_m \longrightarrow 0$ in $C(J_m,E)$ as $j \longrightarrow \infty$.

$M \subseteq C(J,E)$ is a bounded set if and only if there exists a positive function $\phi \in C(J,\mathbf{R}_+)$ such that

$$\|y(t)\| \leq \phi(t) \quad \text{for all} \quad t \in J \quad \text{and all} \quad y \in M.$$

The Ascoli-Arzela theorem says that a set $M \subseteq C(J,E)$ is compact if and only if for each $m \in \mathbf{N}, M$ is a compact set in the Banach space $(C(J_m,E), \|\cdot\|_m)$.

Let $(E, \|\cdot\|)$ be a Banach space. A multivalued map $G : E \longrightarrow 2^E$ has convex (closed) values if $G(x)$ is convex (closed) for all $x \in E$.

G is bounded on bounded sets if $G(B) = \cup_{x \in B} G(x)$ is bounded in E for all bounded set B of E (i.e. $\sup_{x \in B}\{\sup\{\|y\| : y \in G(x)\}\} < \infty$).

G is called upper semicontinuous (u.s.c.) on E if for each $x_0 \in E$ the set $G(x_0)$ is a nonempty, closed subset of E, and if for each open set N of E containing $G(x_0)$, there exists an open neighbourhood M of x_0 such that $G(M) \subseteq N$.

G is said to be completely continuous if $G(B)$ is relatively compact for every bounded subset $B \subseteq E$.

If the multivalued G is completely continuous with nonempty compact values, then G is u.s.c. if and only if G has a closed graph (i.e. $x_n \longrightarrow x_0$, $y_n \longrightarrow y_*$, $y_n \in Gx_n$ imply $y_* \in Gx_0$).

G has a fixed point if there is $x \in E$ such that $x \in Gx$.

In the following $BCC(X)$ denotes the set of all nonempty bounded, closed and convex subsets of X.

A multivalued map $G : J \longrightarrow BCC(E)$ is said to be measurable if for each $x \in E$ the function $Y : J \longrightarrow \mathbb{R}$ defined by

$$Y(t) = d(x, G(t)) = \inf\{|x - z| : z \in G(t)\}$$

belongs to $L^1(J, \mathbb{R})$. For more details on multivalued maps see the books of Deimling [4] and Hu and Papageorgiou [8] .

Let us introduce the following hypotheses:

(H1) A is the infinitesimal generator of a linear semigroup $T(t), t \geq 0$.

(H2) $F : J \times E \longrightarrow BCC(E); (t, y) \longmapsto F(t, y)$ is measurable with respect to t for each $y \in E$, u.s.c. with respect to y for each $t \in J$ and for each fixed $y \in C(J, E)$ the set

$$S_{F,y} := \left\{ g \in L^1(J, \mathbb{R}) : g(t) \in F\Big(t, \int_0^t K(t, s, y(s))ds\Big) \text{ for a.e. } t \in J \right\}$$

is nonempty.

(H3) There exists a function $\alpha \in C(J, \mathbb{R}_+)$, such that

$$\Big|\int_0^t K(t, s, y)ds\Big| \leq \alpha(t)\|y\| \text{ for a.e. } t, s \in J \text{ and } y \in E.$$

(H4) There exists $\beta \in L^1(J, \mathbb{R}_+)$ such that

$$\|F(t, y)\| = \sup\{\|v\| : v \in F(t, y)\} \leq \beta(t)\psi(\|y\|) \text{ for a.e. } t \in J \text{ and } y \in E,$$

where $\psi : \mathbb{R}_+ \longrightarrow (0, \infty)$ is continuous, increasing satisfying

$$\psi(\alpha(t)u) \leq \alpha(t)\psi(u) \text{ for each } t \in J \text{ and } u \in [0, \infty)$$

and

$$M\int_0^m \alpha(s)\beta(s)ds < \int_c^\infty \frac{du}{\psi(u)} \text{ for each } m \in \mathbb{N},$$

with $c = M\|y_0\|$ and $M = \sup\{\|T(t)\|; t \in J\}$.

(H5) $K(t, s, y) \in L^1_{loc}(J, E)$ for each $(t, y) \in J \times E$.

(H6) For each neighbourhood U_p of 0, $y \in U_p$ and $t \in J$ the set

$$\left\{ T(t)y_0 + \int_0^t T(t - s)g(u)duds : g \in S_{F,y} \right\}$$

is relatively compact.

Definition 2.1 *A continuous solution $y(t)$ of the integral inclusion*

$$y(t) \in T(t)y_0 + \int_0^t T(t-s)F\Big(s, \int_0^s K(s,u,y(u))du\Big)ds$$

is called a mild solution of (1.1)-(1.2) *on* J.

Remark 1 *If $dimE < \infty$ and J is a compact real interval then for any $y \in C(J,E)$ the set $S_{F,y}$ is nonempty (see Lasota and Opial* [10] *).*

The following lemmas are crucial in the proof of our main result:

Lemma 1 [10] . *Let I be a compact real interval and E be a Banach space. Let F be a multivalued map satisfying (H2) and let Γ be a linear continuous mapping from $L^1(I,E)$ to $C(I,E)$, then the operator*

$$\Gamma \circ S_F : C(I,E) \longrightarrow BCC(C(I,E)),\ y \longmapsto (\Gamma \circ S_F)(y) := \Gamma(S_{F,y})$$

is a closed graph operator in $C(I,E) \times C(I,E)$.

Lemma 2 [11] . *Let X be a locally convex space. Let $N : X \longrightarrow X$ be a compact convex valued, u.s.c. multivalued map such that there exists a closed neighbourhood U_p of 0 for which $N(U_p)$ is a relatively compact set for each $p \in \mathbb{N}$. If the set*

$$\Omega := \{y \in X : \lambda y \in N(y) \quad for\ some \quad \lambda > 1\}$$

is bounded, then N has a fixed point.

3. Main Result

Now, we are in a position to state and prove our main result.

Theorem 1 *Suppose that the hypotheses (H1)-(H5) are satisfied. Then the initial value problem (1.1)-(1.2) has at least one solution on J.*

Proof. We transform the problem into a fixed point problem. Asolution to (1.1)-(1.2) is a fixed point for the multivalued map$N : C(J,E) \longrightarrow 2^{C(J,E)}$ defined by

$$N(y) := \Big\{h \in C(J,E) : h(t) = T(t)y_0 + \int_0^t T(t-s)g(s)ds \ : g \in S_{F,y}\Big\}$$

where

$$S_{F,y} = \Big\{g \in L^1(J,E) : g(t) \in F\Big(t, \int_0^t K(t,s,y(s))ds\Big) \text{ for a.e. } t \in J\Big\}.$$

We will show that N has a fixed point. The proof will be given in several steps. We first shall show that $N(U_p)$ is relatively compact for each $p \in \mathbf{N}$,and N is u.s.c. with convex closed values.

Step 1: *$N(y)$ is convex for each $y \in C(J,E)$.*

Indeed, if $h_1,\ h_2$ belong to $N(y)$, then there exist $g_1, g_2 \in S_{F,y}$ such that

$$h_i(t) = T(t)y_0 + \int_0^t T(t-s)g_i(s)ds, \quad i = 1, 2, \quad t \in J.$$

Let $0 \le k \le 1$, then for $t \in J$ we have

$$[kh_1 + (1-k)h_2](t) = T(t)y_0 + \int_0^t T(t-s)[kg_1(s) + (1-k)g_2(s)]ds.$$

Since $S_{F,y}$ is convex (because F has convex values) then

$$kh_1 + (1-k)h_2 \in N(y).$$

Step 2: *$N(U_p)$ is bounded in $C(J,E)$ for each $p \in \mathbf{N}$.*

Let $U_p := \{y \in C(J,E) : \|y\|_m \le p\}$ be a neighbourhood of0 in $C(J,E)$ and $y \in U_p$, then for each $h \in N(y)$ there existsg $\in S_{F,y}$ such that for $t \in J$ we have

$$h(t) = T(t)y_0 + \int_0^t T(t-s)g(s)ds.$$

From $(H3)-(H4)$ we have for $t \in J_m$ that

$$\begin{aligned}
\|h(t)\| &\le \|T(t)\|\|y_0\| + \int_0^t \|T(t-s)\|\|g(s)\|ds \\
&\le M\|y_0\| + M\int_0^t \beta(s)\psi\Big(\Big|\int_0^s K(s,u,y(u))du\Big|\Big)ds \\
&\le M\|y_0\| + M\int_0^t \beta(s)\psi(\alpha(s)\|y(s)\|)ds \\
&\le M\|y_0\| + M\int_0^t \beta(s)\alpha(s)\psi(\|y(s)\|)ds \\
&\le M\|y_0\| + M\|\beta\|_{L^1(J_m)}\|\alpha\|_\infty \sup_{y\in U_p} \psi(\|y\|).
\end{aligned}$$

Step 3: *For each p$\in$ $\mathbf{N}$, $N(U_p)$ is equicontinuous set in $C(J,E)$.*

Let $t_1, t_2 \in J_m,\ t_1 < t_2$, then for all $h \in N(y),\ y \in U_p$ we have

$$\begin{aligned}
\|h(t_2) - h(t_1)\| \le\ & \|(T(t_2) - T(t_1))y_0\| \\
&+ \Big\|\int_0^{t_2} \Big[T(t_2 - s) - T(t_1 - s)\Big]g(u)duds\Big\| \\
&+ \Big\|\int_{t_1}^{t_2} T(t_1 - s)g(u)duds\Big\|
\end{aligned}$$

$$\leq \|(T(t_2)-T(t_1))y_0\| + \Big\|\int_0^{t_2}\Big[T(t_2-s)-T(t_1-s)\Big]g(s)ds\Big\| + M(t_2-t_1)\int_0^{m}\|g(s)\|ds.$$

As a consequence of Step 2 and Step 3 and (H5) together with the Ascoli-Arzela theorem we can conclude that $N : C(J,E) \longrightarrow 2^{C(J,E)}$ isa completely continuous multivalued map.

Step 4: *N has a closed graph.*

Let $y_n \longrightarrow y_*$, $h_n \in N(y_n)$ and $h_n \longrightarrow h_0$. We shall prove that $h_0 \in N(y_*)$. $h_n \in N(y_n)$ means that there exists $g_n \in S_{F,y_n}$ such that

$$h_n(t) = T(t)y_0 + \int_0^t T(t-s)g_n(s)ds, \quad t \in J.$$

We must prove that there exists $g_0 \in S_{F,y_*}$ such that

$$h_0(t) = T(t)y_0 + \int_0^t T(t-s)g_0(s)ds, \quad t \in J. \tag{3.1}$$

The idea is then to use the fact that
(i) $h_n \longrightarrow h_0$;
(ii) $h_n - T(t)y_0 \in \Gamma(S_{F,y_n})$ where

$$(\Gamma g)(t) := \int_0^t T(t-s)g(s)ds, \quad t \in J.$$

If $\Gamma \circ S_F$ is a closed graph operator, we would be done. But we don't know whether $\Gamma \circ S_F$ is a closed graph operator. So, we cutthe functions $y_n, h_n - T(t)y_0, g_n$ and we consider them defined on the interval $[k, k+1]$ for any $k \in \mathbf{N} \cup \{0\}$. Then, using Lemma 1, in this case we are able to affirm that (3.1) is true on the compact interval $[k, k+1]$, i.e.

$$h_0(t)\Big|_{[k,k+1]} = T(t)y_0 + \int_0^t T(t-s)g_0^k(s)ds$$

for a suitable L^1-selection g_0^k of $F(t, \int_0^t K(t,s,y_*(s))ds)$ on the interval$[k, k+1]$.
At this point we can paste the functions g_0^k obtaining the selectiong_0 defined by

$$g_0(t) = g_0^k(t) \quad \text{for} \quad t \in [k, k+1).$$

We obtain then that g_0 is an L^1-selection and (3.1) will satisfied.
We give now the details.
Clearly we have that

$$\|(h_n - T(t)y_0) - (h_0 - T(t)y_0)\|_\infty \longrightarrow 0 \quad \text{as} \quad n \longrightarrow \infty.$$

Now, we consider for all $k \in \mathbf{N} \cup \{0\}$, the mapping

$$S_F^k : C([k, k+1], E) \longrightarrow L^1([k, k+1], E)$$

$$u \longmapsto S_{F,u}^k := \{f \in L^1([k, k+1], E) : f(t) \in F\left(t, \int_0^t K(t, s, u(s))ds\right)$$

for a.e. $t \in [k, k+1]\}$.

Also, we consider the linear continuous operators

$$\Gamma_k : L^1([k, k+1], E) \longrightarrow C([k, k+1], E)$$

$$g \longmapsto \Gamma_k(g)(t) = \int_0^t T(t-s)g(s)ds.$$

From Lemma 1, it follows that $\Gamma_k \circ S_F^k$ is a closed graph operator for all $k \in \mathbf{N} \cup \{0\}$. Moreover, we have that

$$\left(h_n(t) - T(t)y_0\right)\Big|_{[k,k+1]} \in \Gamma_k(S_{F,y_n}^k).$$

This, besides to $y_n \longrightarrow y_*$ and Lemma 1, furnishes that

$$\left(h_0(t) - T(t)y_0\right)\Big|_{[k,k+1]} \in \Gamma_k(S_{F,y_*}^k),$$

i.e.

$$\left(h_0(t) - T(t)y_0\right)\Big|_{[k,k+1]} = \int_0^t T(t-s)g_0^k(s)ds$$

for some $g_0^k \in S_{F,y_*}^k$. So the function g_0 defined on$J$ by

$$g_0(t) = g_0^k(t) \quad \text{for} \quad t \in [k, k+1)$$

is in S_{F,y_*} since$g_0(t) \in F\left(t, \int_0^t K(t, s, y_*(s))ds\right)$ for a.e. $t \in J$.

Therefore $N(U_p)$ is relatively compact for each $p \in \mathbf{N}$,and N is u.s.c. with convex closed values.

Step 5: *The set*

$$\Omega := \{y \in C(J, E) : \lambda y \in Ny \quad \textit{for some} \quad \lambda > 1\}$$

is bounded.

Let $\lambda y \in Ny$ for some $\lambda > 1$. Then there exists$g \in S_{F,y}$ such that

$$y(t) = \lambda^{-1}T(t)y_0 + \lambda^{-1}\int_0^t T(t-s)g(s)ds, \quad t \in J.$$

In view of (H3) and (H4) we obtain that for $t \in J_m$

$$\|y(t)\| \le M\|y_0\| + M\int_0^t \alpha(s)\beta(s)\psi(\|y(s)\|)ds.$$

Let us take the right-hand side of the above inequality as $v(t)$, then we have

$$v(0) = M\|y_0\| \quad \text{and} \quad \|y(t)\| \le v(t), \ t \in J_m.$$

Using the increasing character of ψ we get

$$v'(t) \le M\alpha(t)\beta(t)\psi(v(t)), \ t \in J_m.$$

This implies for each $t \in J_m$ that

$$\int_{v(0)}^{v(t)} \frac{du}{\psi(u)} \le M \int_0^m \alpha(s)\beta(s)ds < \int_{v(0)}^{\infty} \frac{du}{\psi(u)}.$$

This inequality implies that there exists a constant b such that $v(t) \le b, \ t \in J_m$,and hence $\|y\|_\infty \le b$ where b depends on m and on the functions $\alpha, \ \beta$ and ψ. This shows that Ω is bounded.

Set $X := C(J, E)$. As a consequence of Lemma 2 we deduce that N has a fixed point which is a mild solution of (1.1)-(1.2). ■

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