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EXISTENCE AND UNIQUENESS OF WEAK SOLUTION IN THE LINEAR THEORY OF ELASTIC SHELLS WITH VOIDS

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Abstract

This paper is concerned with the linear theory of elastic materials with voids [2]. We consider thin porous shells modelled as Cosserat surfaces. In the context of the linear theory of elastic shells with voids established in [4], we study the existence and uniqueness of weak solution for the mixed initial-boundary value problem.

1. Introduction

The theory of elastic materials with voids has been developed by Nunziato and Cowin [1,2]. They have studied nonlinear and linear porous solids in which the skeletal or matrix material is elastic and the interstices are void of material. In this theory, the bulk theory is written as the product of two fields, the matrix material density field and the volume fraction field. This representation introduces an additional degree of kinematical freedom. The theory is expected to find applications in the treatment of the mechanics of granular materials and manufactured porous bodies. There has been much written in recent years on the subject of the theory of elastic materials with voids.

In [3] Naghdi has presented a detailed analysis of the theory of shells and plates described as Cosserat surfaces. Following the approach of [3], we have derived in [4] a theory of thermoelastic shells with voids. First, a nonlinear theory has been considered and then we have established the linear theory of porous thermoelastic shells. We have considered thin shells modelled as Cosserat surfaces.

In the present paper we restrict our attention to the linear isothermal theory of elastic shells with voids and consider the existence and uniqueness of weak solutions for the mixed initial-boundary value problem.

Existence of weak solutions for linear elastic Cosserat shells has also been studied by other authors, but only for some classes of static boundary value problems. Davini [5] has considered homogeneous and isotropic Cosserat surfaces, while Chiorescu [6] has established an existence theorem under certain restrictive conditions for anisotropic Cosserat surfaces with non-uniform thickness. We also mention that a uniqueness theorem for the classical solution has been obtained in [4], without any definitness assumption on the constitutive coefficients.

After an introductory section in which the basic equations of porous elastic shells are reviewed and the mixed initial–boundary value problem is formulated, we proceed in Section 3 to investigate the existence and uniqueness of weak solution for static problems. Under positive definitness assumption for the strain energy function, we use the methods presented in Nečas [7] and Hlaváček [8,9] to obtain the desired results.

In Section 4 dynamic problems are considered by reducing them to an operational e-quation studied by Duvaut and Lions [10]. The existence and uniqueness of weak solution is obtained under the same definitness assumption as in the static case. Throughout this paper we deal with inhomogeneous and anisotropic porous Cosserat shells with uniform thickness in the reference configuration.

2. The basic equations of porous elastic shells

The linear theory of thermoelastic shells with voids, modelled as Cosserat surfaces, was deduced in [4, Section 5].

In this paper we confine our attention to the isothermal theory.Let $\mathcal{S}$ be the reference configuration of a Cosserat surface and let θ^α $(\alpha = 1,2)$ denote a curvilinear material coordinate system on $\mathcal{S}$.

We suppose that $(\theta^1, \theta^2) \in \Sigma$, where $\Sigma \in \mathbf{R}^2$ is an open bounded set of the Euclidean 2-space which has a Lipschitzian boundary Γ (see [7, page 15] or [9] for the definition of a Lipschitzian boundary). The surface $\mathcal{S}$ is defined by an injective mapping $\mathbf{R} \in C^2(\overline{\Sigma})$ from $\overline{\Sigma}$ into the Euclidean 3-space.

The motion of a Cosserat surface is given by

$$\mathbf{r} = \mathbf{r}(\theta^\alpha, t), \ \mathbf{d} = \mathbf{d}(\theta^\alpha, t), \ (\theta^\alpha) \in \overline{\Sigma}, \quad t \in [0,T],$$

where $\mathbf{r}$ is the position vector of any material point at time t and $\mathbf{d}$ is the deformable director assigned to every point of the surface. By $\mathbf{R}(\theta^\alpha)$ and $\mathbf{D}(\theta^\alpha)$ we denote the position vector and the deformable director in the reference configuration, respectively.

Using the same notations as in [4, Section 5], we have the covariant base vectors and the unit normal to the surface $\mathcal{S}$

$$\mathbf{A}_\alpha = \frac{\partial \mathbf{R}}{\partial \theta^\alpha} \ (\alpha = 1,2), \quad \mathbf{A}_3 = \frac{\mathbf{A}_1 \times \mathbf{A}_2}{|\mathbf{A}_1 \times \mathbf{A}_2|}$$

respectively. The first and second fundamental forms of $\mathcal{S}$ are

$$A_{\alpha\beta} = \mathbf{A}_\alpha \cdot \mathbf{A}_\beta, \quad B_{\alpha\beta} = B_{\beta\alpha} = -\mathbf{A}_\alpha \cdot \mathbf{A}_{3,\beta} = \mathbf{A}_3 \cdot \mathbf{A}_{\alpha,\beta}$$

where a comma denotes partial differentiation with respect to the surface coordinates (θ^α). Throughout this paper we employ the usual indicial notation: Greek indices takes values $\{1,2\}$, while Latin indices the values $\{1,2,3\}$. The summation convention is also used.

We remind that in the theory of materials with voids the mass density ρ has the decomposition (see [1,2])

$$\rho = \nu\gamma \quad (\rho_\circ = \nu_\circ \gamma_\circ \text{ in the reference configuration})$$

where γ is the density field of the matrix material and ν is the volume fraction field $(0 < \nu \leq 1)$.

The infinitesimal displacement $\mathbf{u}$, director displacement $\boldsymbol{\delta}$ and change in the volume fraction field φ are defined by (see [4, Section 5])

$$\mathbf{r} = \mathbf{R} + \mathbf{u},\ \mathbf{d} = \mathbf{D} + \boldsymbol{\delta},\ \nu = \nu_\circ + \varphi.$$

In this paper we shall deal with elastic porous shells with constant thickness in the reference configuration. According to [3], this class of shells is characterized by the fact that the reference director coincides with the unit normal to the reference surface, i.e.$\mathbf{D} = \mathbf{A}_3$.

Next, we recall the basic equations of elastic porous shells and we formulate the initial–boundary value problem. The strain measures $e_{\alpha\beta}, \gamma_i$ and $\kappa_{i\alpha}$ are expressed in terms of u_i and δ_i by the geometrical equations

$$\begin{aligned} e_{\alpha\beta} &= \tfrac{1}{2}(u_{\alpha|\beta} + u_{\beta|\alpha}) - B_{\alpha\beta}u_3 \\ \gamma_\alpha &= \delta_\alpha + u_{3,\alpha} + B^\gamma_\alpha u_\gamma, \quad \gamma_3 = \delta_3 \\ \kappa_{\alpha\beta} &= \delta_{\beta|\alpha} - B_{\alpha\beta}\delta_3 - B^\gamma_\alpha u_{\gamma|\beta} + B^\gamma_\alpha B_{\beta\gamma}u_3, \\ \kappa_{3\alpha} &= \delta_{3,\alpha} - B^\gamma_\alpha \delta_\gamma + B^\gamma_\alpha u_{3,\gamma} + B^\beta_\alpha B^\gamma_\beta u_\gamma, \end{aligned} \tag{1}$$

where $\mathbf{u} = u^i\mathbf{A}_i,\ \boldsymbol{\delta} = \delta^i\mathbf{A}_i$ and a vertical bar stands for covariant differentiation with respect to the metric tensor $A_{\alpha\beta}$.

If we denote the material time derivative by a superposed dot, then the equations of motion are

- equations of linear momentum

$$\begin{aligned} N^{\alpha\beta}{}_{|\alpha} - B^\beta_\alpha N^{\alpha3} + F^\beta &= \rho_\circ \ddot{u}^\beta, \\ N^{\alpha3}{}_{|\alpha} + B_{\alpha\beta}N^{\alpha\beta} + F^3 &= \rho_\circ \ddot{u}^3, \end{aligned} \tag{2}$$

- equations of director momentum

$$\begin{aligned} M^{\alpha\beta}{}_{|\alpha} - B^\beta_\alpha M^{\alpha3} - m^\beta + L^\beta &= \rho_\circ \alpha \ddot{\delta}^\beta, \\ M^{\alpha3}{}_{|\alpha} + B_{\alpha\beta}M^{\alpha\beta} - m^3 + L^3 &= \rho_\circ \alpha \ddot{\delta}^3, \end{aligned} \tag{3}$$

- equations of moment of momentum

$$\begin{aligned} N^{12} + B^2_\gamma M^{\gamma1} &= N^{21} + B^1_\gamma M^{\gamma2} \\ N^{\alpha3} - m^\alpha - B^\alpha_\gamma M^{\gamma3} &= 0 \end{aligned} \tag{4}$$

- equations of equilibrated force

$$h^\alpha{}_{|\alpha} - g + P = \rho_\circ k \ddot{\varphi} \tag{5}$$

In the above relations $N^{\alpha i}, M^{\alpha i}, m^i$ are the components of the contact force, the contact director couple and the intrinsic director couple, respectively, while h^α and g are the equilibrated stress and the intrinsic equilibrated body force, respectively. The field quantities per unit area F^i, L^i and P stand for the assigned force, the assigned director couple and the (external) equilibrated body force, respectively. The scalar fields α and k are inertia coefficients assumed to be prescribed and independent of time. We also suppose that $\rho_\circ, \alpha$ and k are positive measurable bounded functions of (θ^α).

If we denote by

$$N'^{\alpha\beta} = N^{\alpha\beta} + B^\beta_\gamma M^{\gamma\alpha}, \tag{6}$$

then the equation $(4)_1$ is equivalent to the symmetry $N'^{\alpha\beta} = N'^{\beta\alpha}$ while the equation $(4)_2$ can be regarded as a constitutive relation for $N^{\alpha3}$.

The linear constitutive equations are

$$\mathcal{A}=\mathcal{A}(e_{\alpha\beta},\kappa_{i\alpha},\gamma_i,\varphi,\varphi_{,\beta}),\quad N'^{\alpha\beta}=N'^{\beta\alpha}=\frac{\partial\mathcal{A}}{\partial e_{\alpha\beta}}$$

$$M^{\alpha i}=\frac{\partial\mathcal{A}}{\partial\kappa_{i\alpha}},\quad m^i=\frac{\partial\mathcal{A}}{\partial\kappa_{i\alpha}},\quad h^\alpha=\frac{\partial\mathcal{A}}{\partial\varphi_{,\alpha}},\quad g=\frac{\partial\mathcal{A}}{\partial\varphi},\tag{7}$$

where $\mathcal{A}$ is the energy of deformation per unit area of $\mathcal{S}$. In the linear theory, $\mathcal{A}$ is a quadratic form of the apropriate variables

$$\begin{aligned}\mathcal{A}=&\tfrac{1}{2}\,{}_1C^{\alpha\beta\gamma\delta}e_{\alpha\beta}e_{\gamma\delta}+\tfrac{1}{2}\,{}_2C^{\alpha\beta\gamma\delta}\kappa_{\alpha\beta}\kappa_{\gamma\delta}+\,{}_3C^{\alpha\beta\gamma\delta}e_{\alpha\beta}\kappa_{\gamma\delta}+\\&+{}_1C^{\alpha\beta\gamma}\kappa_{3\alpha}\kappa_{\beta\gamma}+\,{}_2C^{\alpha\beta\gamma}e_{\alpha\beta}\gamma_\gamma+\,{}_3C^{\alpha\beta\gamma}e_{\alpha\beta}\kappa_{3\gamma}+\\&+{}_4C^{\alpha\beta\gamma}\gamma_\alpha\kappa_{\beta\gamma}+\,{}_5C^{\alpha\beta\gamma}e_{\alpha\beta}\varphi_{,\gamma}+\,{}_6C^{\alpha\beta\gamma}\kappa_{\alpha\beta}\varphi_{,\gamma}+\\&+\tfrac{1}{2}\,{}_1C^{\alpha\beta}\gamma_\alpha\gamma_\beta+\tfrac{1}{2}\,{}_2C^{\alpha\beta}\kappa_{3\alpha}\kappa_{3\beta}+\,{}_3C^{\alpha\beta}\gamma_\alpha\kappa_{3\beta}+\,{}_4C^{\alpha\beta}e_{\alpha\beta}\gamma_3+\\&+\,{}_5C^{\alpha\beta}\kappa_{\alpha\beta}\gamma_3+\,{}_6C^{\alpha\beta}\gamma_\alpha\varphi_{,\beta}+\,{}_7C^{\alpha\beta}\kappa_{3\alpha}\varphi_{,\beta}+\tfrac{1}{2}\,{}_8C^{\alpha\beta}\varphi_{,\alpha}\varphi_{,\beta}+\\&+\,{}_9C^{\alpha\beta}e_{\alpha\beta}\varphi+\,{}_{10}C^{\alpha\beta}\kappa_{\alpha\beta}\varphi+\,{}_1C^{\alpha}\gamma_\alpha\gamma_3+\,{}_2C^{\alpha}\kappa_{3\alpha}\gamma_3+\\&+\,{}_3C^{\alpha}\varphi_{,\alpha}\gamma_3+\,{}_4C^{\alpha}\varphi_{,\alpha}\varphi+\,{}_5C^{\alpha}\gamma_\alpha\varphi+\,{}_6C^{\alpha}\kappa_{3\alpha}\varphi+\\&+\tfrac{1}{2}\,{}_1C\,(\gamma_3)^2+\tfrac{1}{2}\,{}_2C\varphi^2+\,{}_3C\gamma_3\varphi.\end{aligned}\tag{8}$$

The above constitutive coefficients ${}_nC^{\alpha\cdots}$ are bounded measurable functions of (θ^α) defined on $\overline{\Sigma}$ and depend in addition only on the reference values $\{A_{\alpha\beta},B_{\alpha\beta},\nu_o\}$. Some of the constitutive coefficients satisfy certain symmetry conditions

$$\begin{aligned}&{}_1C^{\alpha\beta\gamma\delta}=\,{}_1C^{\gamma\delta\alpha\beta},\ {}_2C^{\alpha\beta\gamma\delta}=\,{}_2C^{\gamma\delta\alpha\beta},\\&{}_1C^{\alpha\beta}=\,{}_1C^{\beta\alpha},\ {}_2C^{\alpha\beta}=\,{}_2C^{\beta\alpha},\ {}_8C^{\alpha\beta}=\,{}_8C^{\beta\alpha}.\end{aligned}\tag{9}$$

We suppose that the quadratic form $\mathcal{A}$ defined by (8) is positive definite, i.e. there exists a constant $c>0$ such that for each $(\theta^\alpha)\in\overline{\Sigma}$

$$\mathcal{A}(e_{\alpha\beta},\kappa_{i\alpha},\gamma_i,\varphi,\varphi_{,\beta})\geq c[e_{\alpha\beta}e_{\alpha\beta}+\kappa_{i\alpha}\kappa_{i\alpha}+\gamma_i\gamma_i+\varphi\varphi+\varphi_{,\beta}\varphi_{,\beta}]\tag{10}$$

Let $\mathcal{C}$ be the boundary of $\mathcal{S}$ and let $\mathcal{C}_u,\mathcal{C}_N,\mathcal{C}_\delta,\mathcal{C}_M,\mathcal{C}_\varphi,\mathcal{C}_h\subset\mathcal{C}$ be either open in $\mathcal{C}$ or empty sets such that the following mutually disjoint decompositions hold (equalities being valid except for sets of zero measure on $\mathcal{C}$) $\mathcal{C}=\mathcal{C}_u\cup\mathcal{C}_N=\mathcal{C}_\delta\cup\mathcal{C}_M=\mathcal{C}_\varphi\cup\mathcal{C}_h$. We consider the mixed boundary conditions

(11) $u_i=\tilde{u}_i$ on $\mathcal{C}_u\times[0,T]$, $\delta_i=\tilde{\delta}_i$ on $\mathcal{C}_\delta\times[0,T]$, $\varphi=\tilde{\varphi}$ on $\mathcal{C}_\varphi\times[0,T]$, and

(12) $N^{\alpha i}n_\alpha=\tilde{N}^i$ on $\mathcal{C}_N\times[0,T]$, $M^{\alpha i}n_\alpha=\tilde{M}^i$ on $\mathcal{C}_M\times[0,T]$, $h^\alpha n_\alpha=\tilde{h}$ on $\mathcal{C}_h\times[0,T]$, where $\mathbf{n}=n_\alpha\mathbf{A}^\alpha$ is the unit outward normal in the surface to the curve $\mathcal{C}$.

The initial conditions are

$$\begin{aligned}u_i(\theta^\alpha,0)&=u_{oi}(\theta^\alpha),\quad\dot{u}_i(\theta^\alpha,0)=u_{1i}(\theta^\alpha)\\\delta_i(\theta^\alpha,0)&=\delta_{oi}(\theta^\alpha),\quad\dot{\delta}_i(\theta^\alpha,0)=\delta_{1i}(\theta^\alpha),\\\varphi_i(\theta^\alpha,0)&=\varphi_o(\theta^\alpha),\quad\dot{\varphi}(\theta^\alpha,0)=\varphi_1(\theta^\alpha),\qquad(\theta^\alpha)\in\overline{\Sigma}.\end{aligned}\tag{13}$$

All the functions in the right–hand sides of the initial and boundary conditions are prescribed and their regularity will be precisely stated later on.

3. Static problems

We begin this section by defining some function spaces that are used in the weak formulation of boundary value problems. They are usually given on open sets of $\mathbf{R}^n$, but we shall extend them to function spaces on $\mathcal{S}$ without difficulties.

Let $L^2(\mathcal{S})$ be the closure of the set of real functions f of class $C^\infty(\overline{\mathcal{S}})$ with respect to the norm

$$\|f\|_0^2 = \int_{\mathcal{S}} |f|^2 \, d\sigma$$

If we consider $L^2(\Sigma)$ with the usual norm $\|\cdot\|_{L^2}$ and identify a function $f \in L^2(\mathcal{S})$ with the function $f \circ \mathbf{R} \in L^2(\Sigma)$, then the spaces $L^2(\mathcal{S})$ and $L^2(\Sigma)$ coincide and their norms are equivalent.

Consider now the real Sobolev space $H^1(\mathcal{S})$ obtained from $C^\infty(\overline{\mathcal{S}})$ by closure with respect to the norm

$$\|f\|_1^2 = \int_{\mathcal{S}} |f|^2 \, d\sigma + \int_{\mathcal{S}} f_{,\alpha} \; f_{,\alpha} d\sigma$$

where we have denoted $f_{,\alpha} = \partial f / \partial \theta^\alpha$.

With the identification of functions on $\mathcal{S}$ with the corresponding functions on Σ, we find that $H^1(\mathcal{S})$ coincides with the ordinary Sobolev space $H^1(\Sigma)$ and that $\|\cdot\|_1$ is equivalent with the usual norm on $H^1(\Sigma)$, denoted by $\|\cdot\|_{H^1}$.

In the case of the static boundary value problem, the prescribed functions in (11), (12) do not depend on time and we consider

$$\tilde{N}^i \in L^2(\mathcal{C}_N), \; \tilde{M}^i \in L^2(\mathcal{C}_M), \; \tilde{h} \in L^2(\mathcal{C}_h). \tag{14}$$

We also suppose that the prescribed functions $\tilde{u}_i, \tilde{\delta}_i$ and $\tilde{\varphi}$ defined in (11) are the restrictions to $\mathcal{C}_u, \mathcal{C}_\delta$ and $\mathcal{C}_\varphi$ of some functions $\bar{u}_i, \bar{\delta}_i, \bar{\varphi} \in H^1(\mathcal{S})$, respectively.

In order to derive existence and uniqueness results for weak solutions of the static boundary value problem, we shall use the same methods as in [9]. The general theorems that are used (see [9, Section 1]) are proved in [7] and [8] for an arbitrary open set $\Omega \subset \mathbf{R}^n$ (in our case $\Sigma \subset \mathbf{R}^2$). Since the spaces $(L^2(\mathcal{S}), \|\cdot\|_\circ)$ and $(H^1(\mathcal{S}), \|\cdot\|_1)$ are topologically isomorphic with the ordinary spaces $(L^2(\Sigma), \|\cdot\|_{L^2})$ and $(H^1(\Sigma), \|\cdot\|_{H^1})$, respectively, then we can easily deduce that these theorems also hold in our functional framework.

Let $W = (H^1(\mathcal{S}))^7$ and let us denote the elements of W by v or w, where $v = \{u_i, \delta_i, \varphi\} \in W$ and $w = \{s_i, \eta_i, \psi\} \in W$ stand for arbitrary displacement fields. The norm on W is

$$\|v\|_W^2 = \sum_{i=1}^{3} (\|u_i\|_1^2 + \|\delta_i\|_1^2) + \|\varphi\|_1^2 . \tag{15}$$

We also denote by $L = (L^2(\mathcal{S}))^7$ and $f = \{F^i, L^i, P\}$. We have $f \in L$ and

$$\|f\|_L^2 = \sum_{i=1}^{3} (\|F^i\|_0^2 + \|L_i\|_0^2) + \|P\|_0^2 . \tag{16}$$

We define the subspace V of W of all elements $v \in W$ which satisfy the homogeneous boundary conditions (11), i.e.

$$u_i = 0 \text{ on } C_u, \quad \delta_i = 0 \text{ on } C_\delta, \;\; \varphi = 0 \text{ on } C_\varphi, \tag{17}$$

for each $v = \{u_i, \delta_i, \varphi\} \in V$, in the sense of traces.

Let $A(v,w)$ be the bilinear form defined on $W \times W$ by

$$
\begin{aligned}
A(v,w) = \int_S [&{}_1C^{\alpha\beta\gamma\delta} e_{\alpha\beta}(v) e_{\gamma\delta}(w) + {}_2C^{\alpha\beta\gamma\delta} \kappa_{\alpha\beta}(v)\kappa_{\gamma\delta}(w) + \\
&+ {}_3C^{\alpha\beta\gamma\delta}(e_{\alpha\beta}(v)\kappa_{\gamma\delta}(w) + e_{\alpha\beta}(w)\kappa_{\gamma\delta}(v)) + {}_1C^{\alpha\beta\gamma}(\kappa_{3\alpha}(v)\kappa_{\beta\gamma}(w) + \kappa_{3\alpha}(w)\kappa_{\beta\gamma}(v)) + \\
&+ {}_2C^{\alpha\beta\gamma}(e_{\alpha\beta}(v)\gamma_\gamma(w) + e_{\alpha\beta}(w)\gamma_\gamma(v)) + {}_3C^{\alpha\beta\gamma}(e_{\alpha\beta}(v)\kappa_{3\gamma}(w) + e_{\alpha\beta}(w)\kappa_{3\gamma}(v)) + \\
&+ {}_4C^{\alpha\beta\gamma}(\gamma_\alpha(v)\kappa_{\beta\gamma}(w) + \gamma_\alpha(w)\kappa_{\beta\gamma}(v)) + {}_5C^{\alpha\beta\gamma}(e_{\alpha\beta}(v)\psi_{,\gamma} + e_{\alpha\beta}(w)\varphi_{,\gamma}) + \\
&+ {}_6C^{\alpha\beta\gamma}(\kappa_{\alpha\beta}(v)\psi_{,\gamma} + \kappa_{\alpha\beta}(w)\varphi_{,\gamma}) + {}_1C^{\alpha\beta}\gamma_\alpha(v)\gamma_\beta(w) + {}_2C^{\alpha\beta}\kappa_{3\alpha}(v)\kappa_{3\beta}(w) + \\
&+ {}_3C^{\alpha\beta}(\gamma_\alpha(v)\kappa_{3\beta}(w) + \gamma_\alpha(w)\kappa_{3\beta}(v)) + {}_4C^{\alpha\beta}(e_{\alpha\beta}(v)\gamma_3(w) + e_{\alpha\beta}(w)\gamma_3(v)) + \\
&+ {}_5C^{\alpha\beta}(\kappa_{\alpha\beta}(v)\gamma_3(w) + \kappa_{\alpha\beta}(w)\gamma_3(v)) + {}_6C^{\alpha\beta}(\gamma_\alpha(v)\psi_{,\beta} + \gamma_\alpha(w)\varphi_{,\beta}) + \\
&+ {}_7C^{\alpha\beta}(\kappa_{3\alpha}(v)\psi_{,\beta} + \kappa_{3\alpha}(w)\varphi_{,\beta}) + {}_8C^{\alpha\beta}\varphi_{,\alpha}\psi_{,\beta} + {}_9C^{\alpha\beta}(e_{\alpha\beta}(v)\psi + e_{\alpha\beta}(w)\varphi) + \\
&+ {}_{10}C^{\alpha\beta}(\kappa_{\alpha\beta}(v)\psi + \kappa_{\alpha\beta}(w)\varphi) + {}_1C^{\alpha}(\gamma_\alpha(v)\gamma_3(w) + \gamma_\alpha(w)\gamma_3(v)) + \\
&+ {}_2C^{\alpha}(\kappa_{3\alpha}(v)\gamma_3(w) + \kappa_{3\alpha}(w)\gamma_3(v)) + {}_3C^{\alpha}(\varphi_{,\alpha}\gamma_3(w) + \psi_{,\alpha}\gamma_3(v)) + \\
&+ {}_4C^{\alpha}(\varphi_{,\alpha}\psi + \psi_{,\alpha}\varphi) + {}_5C^{\alpha}(\gamma_\alpha(v)\psi + \gamma_\alpha(w)\varphi) + {}_6C^{\alpha}(\kappa_{3\alpha}(v)\psi + \kappa_{3\alpha}(w)\varphi) + \\
&+ {}_1C\gamma_3(v)\gamma_3(w) + {}_2C\varphi\psi + {}_3C(\gamma_3(v)\psi + \gamma_3(w)\varphi)]d\sigma
\end{aligned} \tag{18}
$$

for each $v = \{u_i, \delta_i, \varphi\} \in W$ and $w = \{s_i, \eta_i, \psi\} \in W$, where $e_{\alpha\beta}(v), \kappa_{i\alpha}(v)$ and $\gamma_i(v)$ are expressed in terms of u_i and δ_i by the geometrical relations (1).

We see that the bilinear form $A(v,w)$ is continuous in both arguments and

$$
A(v,w) = A(w,v), \quad A(v,v) = 2\int_S \mathcal{A}(e_{\alpha\beta}(v), \kappa_{i\alpha}(v), \gamma_i(v), \varphi, \varphi_{,\beta})d\sigma, \quad \forall v, w \in W. \tag{19}
$$

Let us define the functionals f and g on W by

$$
(f,v) = \int_S (F^i u_i + L^i \delta_i + P\varphi)d\sigma, \tag{20}
$$

$$
(g,v) = \int_{C_N} \tilde{N}^i u_i ds + \int_{C_M} \tilde{M}^i \delta_i ds + \int_{C_h} \tilde{h}\varphi\, ds. \tag{21}
$$

The theorems of embedding imply that the functions f and g are continuous on W. In the sequel we denote by $\tilde{v} \equiv \{\tilde{u}_i, \tilde{\delta}_i, \tilde{\varphi}\} \in W$.

Definition 3.1. We say that $v \in W$ is a *weak solution* of the boundary value problem if

$$
v - \tilde{v} \in V
$$

and if

$$
A(v,w) = (f,w) + (g,w), \quad \text{for each } w \in V. \tag{22}
$$

We define the operators N_l $(l = 1, 2, ..., 16)$ mapping W into $L^2(\mathcal{S})$

$$
\begin{gathered}
N_1 v = e_{11}(v),\ N_2 v = e_{12}(v),\ N_3 v = e_{21}(v),\ N_4 v = e_{22}(v), \\
N_5 v = \kappa_{11}(v),\ N_6 v = \kappa_{12}(v),\ N_7 v = \kappa_{21}(v),\ N_8 v = \kappa_{22}(v), \\
N_9 v = \kappa_{31}(v),\ N_{10} v = \kappa_{32}(v),\ N_{11} v = \gamma_1(v),\ N_{12} v = \gamma_2(v), \\
N_{13} v = \gamma_3(v), \quad N_{14} v = \varphi, \quad N_{15} v = \varphi_{,1}, \quad N_{16} v = \varphi_{,2},
\end{gathered} \tag{23}
$$

Then, the hypothesis of positive definitness (10) can be put in the form

$$
A(v,v) \geq c \sum_{l=1}^{16} \|N_l v\|_0^2, \tag{24}
$$

for each $v \in W$, where $c > 0$ is a constant.

Using the Theorem 7.7 of [7, page 192] we prove without difficulties that $N_l v$ form a coercive system of operators on W, i.e. (see [9]) there exists a constant $c_1 > 0$ such that for each $v \in W$

$$\left(\sum_{l=1}^{16} \|N_l v\|_0^2 \right) + \|v\|_L^2 \geq c_1 \, |v|_W^2 \,. \tag{25}$$

We define the subspace

$$\mathcal{P} = \{ v \in V \mid \|N_l v\|_\circ = 0, \; l = 1, 2, ..., 16 \} \tag{26}$$

and the factor–space $W/\mathcal{P}$ of the classes $\hat{v} = \{ v + p \mid v \in V, \; p \in \mathcal{P} \}$ endowed with the norm

$$|\hat{v}|_{W/\mathcal{P}} = \inf_{p \in \mathcal{P}} |v + p|_W \,.$$

The elements of $\mathcal{P}$ are characterized by

$$e_{\alpha\beta}(v) = \kappa_{i\alpha}(v) = \gamma_i(v) = 0 \text{ and } \varphi = 0 \tag{27}$$

almost everywhere in $\mathcal{S}$. The relations (18) and (27) show us that we can define a bilinear form on $W/\mathcal{P} \times W/\mathcal{P}$ by

$$[\hat{v}, \hat{w}] \equiv A(v, w). \tag{28}$$

Theorem 1.2 of [9] can be written in our notations as follows

Theorem 1 *Let*

$$A(v, w) \equiv [\hat{v}, \hat{w}]$$

define a bilinear form for each $\hat{v}, \hat{w} \in W/\mathcal{P}$, $v \in \hat{v}$, $w \in \hat{w}$. Let (24), (25) hold. Then the necessary and sufficient condition for the existence of a weak solution to the boundary value problem is

$$p \in \mathcal{P} \Rightarrow (f, p) + (g, p) = 0. \tag{29}$$

The weak solution is determined except for an element $p \in \mathcal{P}$. Further, for each $\hat{w} \in W/\mathcal{P}$ there is

$$A(\hat{w}, \hat{w}) \geq c_3 |w|_{W/\mathcal{P}}^2, \qquad c_3 > 0 \tag{30}$$

which we call the inequality of Korn's type.

Since in our case relations (24), (25) and (28) hold, we find that all the suppositions of Theorem 1 are satisfied and we obtain the existence and uniqueness of weak solution for the static equilibrium of elastic Cosserat shells with voids.

Let us confine our attention to two important cases of boundary value problems. The first case is $\mathcal{P} = \{0\}$. Then the condition (29) holds and we obtain

Theorem 2 *Let $\mathcal{P} = \{0\}$. Then there exists one and only one weak solution $v \in W$. For each $w \in W$ we have*

$$A(w, w) \geq c_3 |w|_W^2,$$

where $c_3 > 0$ is a constant, which is an inequality of Korn's type.

Necessary and sufficient conditions for $\mathcal{P} = \{0\}$ are given by

Theorem 3 *Let $\mathcal{S}$ be a simple connected surface. $\mathcal{P} = \{0\}$ if and only if the following two conditions are satisfied:*

(i) C_u is non–empty;

(ii) $\mathcal{C}_u$ *and* $\mathcal{C}_\delta$ *are such that they do* **not** *meet the following situation:*

"$\mathcal{C}_u$ *is included in a straight line* Δ *and* $\mathcal{C}_\delta$ *is included in a plane orthogonal to* Δ*."*

The proof of this theorem can be deduced using simple arguments of vector analysis and the expressions for the displacement of a Cosserat surface due to infinitesimal rigid body motions (see [3, page 463]).

Another important case is when the contact forces and couples are given all over the boundary $\mathcal{C}$. Denoting by $\mathbf{F} = F^i\mathbf{A}_i$, $\mathbf{L} = L^i\mathbf{A}_i$, $\widetilde{\mathbf{N}} = \tilde{N}^i\mathbf{A}_i$, $\widetilde{\mathbf{M}} = \tilde{M}^i\mathbf{A}_i$, then from Theorem 1, we get

Theorem 4 *Let* $\mathcal{C}_u = \mathcal{C}_\delta = \emptyset$ *and* $\mathcal{C}_N = \mathcal{C}_M = \mathcal{C}$. *Then the necessary and sufficient conditions for the existence of a weak solution are*

$$\int_{\mathcal{S}} \mathbf{F} d\sigma + \int_{\mathcal{C}} \widetilde{\mathbf{N}} ds = \mathbf{0}, \quad \text{and} \tag{31}$$

$$\int_{\mathcal{S}} (\mathbf{R}\times\mathbf{F} + \mathbf{D}\times\mathbf{L}) d\sigma + \int_{\mathcal{C}} (\mathbf{R}\times\widetilde{\mathbf{N}} + \mathbf{D}\times\widetilde{\mathbf{M}}) ds = \mathbf{0}.$$

The weak solution is uniquely determined except for an element $p \in \mathcal{P}$, *where*

$$\mathcal{P} = \{v \in W \mid \mathbf{u} = \mathbf{c} + \boldsymbol{\omega}\times\mathbf{R},\ \boldsymbol{\delta} = \boldsymbol{\omega}\times\mathbf{D},\ \varphi = 0\}, \tag{32}$$

$\mathbf{c}$ *and* $\boldsymbol{\omega}$ *being arbitrary constant vectors.*

Remark 1 *The conditions (31) express the total equilibrium for external forces and couples. The elements of* $\mathcal{P}$ *in (32) represent rigid body displacements of the porous Cosserat surface* $\mathcal{S}$. *Further simplifications could be made in (31) and (32) if we remind that* $\mathbf{D} = \mathbf{A}_3$ *in our theory.*

4. Dynamic problems

In this section we obtain existence and uniqueness results in the dynamic case using a theorem presented in [10].

We begin by giving the weak formulation of the mixed initial–boundary value problem formulated in Section 2.

Taking the scalar product of the equations of motion (2), (3) and (5) with the function s_i, η_i and ψ, respectively, where $w = \{s_i, \eta_i, \psi\}$ is a test function assumed to be regular (as always in the first stage of weak formulation), we obtain after making use of the divergence theorem and of the geometrical equations (1) that

$$\begin{aligned} &\int_{\mathcal{S}} \rho_o(\ddot{u}^i s_i + \alpha\ddot{\delta}^i\eta_i + k\ddot{\varphi}\psi) d\sigma + \int_{\mathcal{S}} [N'^{\alpha\beta}(v) e_{\alpha\beta}(w) + \\ &+ M^{\alpha i}(v)\kappa_{i\alpha}(w) + m^i(v)\gamma_i(w) + h^\alpha(v)\psi_{,\alpha} + g(v)\psi] d\sigma = \\ &= \int_{\mathcal{S}} (F^i s_i + L^i\eta_i + P\psi) d\sigma + \int_{\mathcal{C}_N} \tilde{N}^i s_i ds + \int_{\mathcal{C}_M} \tilde{M}^i\eta_i ds + \\ &+ \int_{\mathcal{C}_h} \tilde{h}\psi ds + \int_{\mathcal{C}_u} N^i s_i ds + \int_{\mathcal{C}_\delta} M^i\eta_i ds + \int_{\mathcal{C}_\varphi} h\psi ds \end{aligned} \tag{33}$$

For each $t \in [0,T]$, we shall denote by $v = \{u^i, \delta^i, \varphi\}$ the function $(\theta^\alpha) \to v(\theta^\alpha, t)$ and by $v'(t) = \partial v(t)/\partial t$. We shall use the notation $v_\rho(t)$ for the function with components $v_\rho(t) \equiv \{\rho_o u^i(t), \rho_o \alpha \delta^i(t), \rho_o k\varphi(t)\} \in W$. We presume that the coefficients ρ_o, α and k are positive function of class $C^1(\overline{\Sigma})$, independent of time. If we remind the definition of functionals $f(t)$ and $g(t)$ in (20), (21), then the above relation (33) can be written as

$$(v_\rho''(t), w) + A(v(t), w) = (f(t), w) + (g(t), w) +$$

$$+ \int_{C_u} N^i s_i ds + \int_{C_\delta} M^i \eta_i ds + \int_{C_\varphi} h\psi ds \tag{34}$$

Let $\widetilde{v} = \{\widetilde{u}_i, \widetilde{\delta}_i, \widetilde{\varphi}\} \in W$ and let $v_o = \{u_o, \delta_{oi}, \varphi_o\}$, $v_1 = \{u_{1i}, \delta_{1i}, \varphi_1\}$ be the initial data defined in (13). The mode of dependence of $\widetilde{v}$ on t, as well as the regularity of v_o and v_1 will be precisely stated later on.

Relation (34) justifies the following

Definition 4.1. *A* ***weak solution*** *of the initial-boundary value problem is a function $t \to v(t)$ of $[0,T] \to W$ such that*

$$v(t) - \widetilde{v}(t) \in V$$

and

$$(v_\rho''(t), w) + A(v(t), w) = (f(t), w) + (g(t), w), \quad \forall w \in V \tag{35}$$

with the initial conditions

$$v(0) = v_{oi}, \ v'(0) = v_1.$$

Here $W = (H^1(S))^7$, $L = (L^2(S))^7$ and $V \subset W$ are the spaces defined in the previous section. Let (,) be the scalar product in L.

We note that $V \subset L$ and V is dense in L. By identifying L with its dual, we can write

$$V \subset L \subset V', \tag{36}$$

where V' is the dual of V. For the exact meaning of the inclusions (36) see [11, page 82]. We denote also by (,) the scalar product of V' and V, which is compatible with the scalar product on L.

If we replace $v(t)$ by $v(t) - \widetilde{v}(t)$ in Definition 1 and keep the notation $v(t)$ (and the same for $v_o - \widetilde{v}(0)$ and $v_1 - \widetilde{v}'(0)$), then our problem is equivalent to the following:

we search a function $t \to v(t)$ of $[0,T] \to V$ such that

$$(v_\rho''(t), w) + A(v(t), w) = (\Psi(t), w), \quad \forall w \in V \tag{37}$$

and

$$v(0) = v_o, \ v'(0) = v_1, \tag{38}$$

where we have defined the functional $\Psi(t)$ on V by

$$(\Psi(t), w) = (f(t), w) + (g(t), w) - (\widetilde{v}_\rho''(t), w) - A(\widetilde{v}(t), w). \tag{39}$$

The existence and uniqueness of the weak solutions for the mixed initial–boundary value problem is stated in the following

Theorem 5 *We assume that*

$$\Psi, \Psi' \in L^2(0,T;V') \tag{40}$$

and

$$v_o \in V, \ v_1 \in L. \tag{41}$$

There exists one and only one function v such that

$$v \in L^\infty(0,T;V)$$

$$\begin{aligned} v' &\in L^{\infty}(0,T;L) \\ v'' &\in L^{\infty}(0,T;V') \end{aligned} \tag{42}$$

and which satisfies (37) and (38).

Proof. Let us remind some properties of the bilinear form A established in Section 3. From $(19)_1$ we see that

$$A(v,w) = A(w,v), \quad \forall v,w \in V. \tag{43}$$

Next, we want to prove that

$$\left| \begin{array}{l} \forall \lambda > 0, \text{ there exists a constant } a > 0 \text{ such that} \\ \qquad A(v,v) + \lambda \|v\|_L^2 \geq a\, |v|_V^2\,, \quad \forall v \in V \end{array} \right. \tag{44}$$

Indeed, let $\lambda > 0$ be an arbitrary constant. By (24), we have for each $v \in V$

$$A(v,v) + \lambda \|v\|_L^2 \geq c \left(\sum_{l=1}^{16} \|N_l v\|_0^2 \right) + \lambda \|v\|_L^2 \geq$$

$$\geq \min(c,\lambda) \left[\left(\sum_{l=1}^{16} \|N_l v\|_0^2 \right) + \|v\|_L^2 \right]$$

and using (25), we get

$$A(v,v) + \lambda \|v\|_L^2 \geq c_1 \min(c,\lambda)\, |v|_V^2\,,$$

which shows that (44) is true for $a = c_1 \min(c,\lambda)$.

We shall now use Theorem 4.1 from [10, page 124] which is proved in an abstract setting where only the spaces V, L and V' are involved and the bilinear form $A(v,w)$ which satisfies (43) and (44).

The only difference between our problem and the problem in [10] is the apparition of the coefficients ρ_o, $\rho_o \alpha$ and $\rho_o k$ (from $v''_\rho(t) = \{\rho_o u^{i\prime\prime}(t),\, \rho_o \alpha \delta^{i\prime\prime}(t),\, \rho_o k \varphi''(t)\}$), in equation (37), but this is not essential and the proof of Theorem 4.1 in [10, page 124] can be adapted without difficulties. ■

We close this paper by indicating sufficient conditions for which assumptions (40) and (41) hold, in terms of prescribed functions on the boundary and of initial data. The following remark can be proved in the same manner as in [10].

Remark 2 *Let us assume that*

$$F^i, L^i, P, F^{i\prime}, L^{i\prime}, P' \in L^2(\mathcal{S} \times [0,T]),$$

$$\tilde{N}^i, \tilde{N}^{i\prime} \in L^2(\mathcal{C}_N \times [0,T]), \quad \tilde{M}^i, \tilde{M}^{i\prime} \in L^2(\mathcal{C}_M \times [0,T]), \quad \tilde{h}, \tilde{h}' \in L^2(\mathcal{C}_h \times [0,T]),$$

and

$$\left| \begin{array}{c} \{\tilde{u}_i, \tilde{\delta}_i, \tilde{\varphi}\} \text{ defined in (11) is the restriction to} \\ (\mathcal{C}_u \times [0,T]) \times (\mathcal{C}_\delta \times [0,T]) \times (\mathcal{C}_\varphi \times [0,T]) \text{ of a function } \tilde{v} \text{ such that} \\ \tilde{v}, \tilde{v}', \tilde{v}'', \tilde{v}''' \in L^2(0,T;W). \end{array} \right.$$

Then the function Ψ defined in (39) satisfies the conditions (40).

In this case, the suppositions (41) are satisfied if the initial data $v_o = \{u_{oi}, \delta_{oi}, \varphi_o\}$ and $v_1 = \{u_{1i}, \delta_{1i}, \varphi_1\}$ defined in (13) meet the conditions

$$u_{oi}, \delta_{oi}, \varphi_o \in H^1(\mathcal{S}) \quad \text{and} \quad u_{1i}, \delta_{1i}, \varphi_1 \in L^2(\mathcal{S}).$$

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