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ON CERTAIN LIMITS RELATED TO THE NUMBER e

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1. In the very interesting web pages by Steven Finch ([3]) there appears also the following limit, submitted by Felix A. Keller:

$$\lim_{n\to\infty}\left[\frac{n^n}{(n-1)^{n-1}}-\frac{(n-1)^{n-1}}{(n-2)^{n-2}}\right]=e. \tag{1}$$

In what follows we will show how certain expressions which are similar to the bracket in (1), are intimately related to certain special means of two arguments, namely the logarithmic and identric means.

Let $a, b > 0$ be two positive real numbers. The logarithmic, resp. identric means of a and b are defined by

$$L = L(a,b) = \frac{b-a}{\ln b - \ln a} \qquad (a \neq b), \qquad L(a,a) = a$$

and

$$I = I(a,b) = \frac{1}{e}(b^b/a^a)^{1/(b-a)} \qquad (a \neq b), \qquad I(a,a) = a.$$

For many properties, generalizations, etc. of these means we quote e.g. [1], [2], [5], [8], [10].

The following inequalities, which will be used here, are due to K.B. Stolarsky [10]:

$$\sqrt{ab} < L(a,b) < I(a,b) < \frac{a+b}{2} \qquad (a \neq b). \tag{2}$$

These relations can be much improved, see e.g. [8].

2. In what follows, we will use the following notations:

$$f(x) = \left(1+\frac{1}{x}\right)^x, \qquad F(x) = (x+1)\left(1+\frac{1}{x}\right)^x,$$

$$g(x) = \ln\left(1+\frac{1}{x}\right) - \frac{1}{x+1} \qquad (x > 0)$$

Since $\ln I(x, x+1) = (x+1)\ln(x+1) - x\ln x - 1$ and

$$\frac{1}{L(x,x+1)} = \ln(x+1) - \ln x, \tag{3}$$

we easily can see that $\ln(F(x)) = \ln(x+1) + \ln f(x) = \ln eI(x, x+1)$, so

$$F(x) = eI(x,x+1) = (x+1)f(x) \tag{4}$$

and

$$F'(x) = \frac{F(x)}{L(x, x+1)} \tag{5}$$

where F' denotes the derivative of F.

3. We note that by (3) we have $f(x) = \frac{e}{x} I(x, x+1)$, so by inequalities (2) applied to $a := x,\ b := x+1$, we get the following double-inequality

$$e\sqrt{\frac{x}{x+1}} < f(x) < e\frac{2x+1}{2x+2}. \tag{6}$$

As a consequence, we can write

$$\frac{x}{\sqrt{x}+\sqrt{x+1}}e < x[e - f(x)] < \frac{x}{2x+2}e, \tag{7}$$

yielding the following limit:

$$\lim_{x\to\infty} x[e - f(x)] = \frac{e}{2}. \tag{8}$$

Of course, this limit may be obtained by l'Hospital's rule, too. However, the inequalities provide a much better information. For another proof of (7), based on Hadamard's inequality, see [7].

Remark. By $nf(n) - (n-1)f(n-1) = n[f(n) - e] + (n-1)[e - f(n-1)] + e$, (8) implies

$$\lim_{n\to\infty} [nf(n) - (n-1)f(n-1)] = e \tag{8'}$$

proved by other arguments in [6]. The limit (8') is due to M. Ghermănescu [4].

4. We now prove first the following result:

Theorem 1. *For $x > 0$, F is a strictly increasing, strictly concave function.*

Proof. Clearly, $F'(x) = \dfrac{F(x)}{L(x, x+1)} > 0$, and, on the other hand since

$$F''(x) = F(x)\left[(\ln(x+1) - \ln x)^2 - \frac{1}{x(x+1)}\right] = F(x)\left[\frac{1}{L^2(x, x+1)} - \frac{1}{x(x+1)}\right],$$

by the left side of (2) we clearly get $F''(x) < 0$. Thus F is strictly concave.

Corollary 1. *For all positive integers $n > 1$ one has*

$$f(n)\frac{2n}{2n+1} < F(n+1) - F(n) < f(n-1)\sqrt{\frac{n}{n+1}}. \tag{9}$$

Proof. By Lagrange's mean value theorem one has

$$F(n+1) - F(n) = F'(\xi) \qquad (\xi \in (n, n+1)),$$

and by Theorem 1, F' is strictly decreasing, so we have

$$\frac{F(n+1)}{L(n+1, n+2)} < F(n+1) - F(n) < \frac{F(n)}{L(n, n+1)}. \tag{10}$$

Now, (9) follows by (10) and (2), by remarking that $F(n) = (n+1)f(n)$ and

$$\frac{2n}{2n+1} < \frac{n}{L(n, n+1)} < \sqrt{\frac{n}{n+1}}.$$

Corollary 2. $\lim_{n\to\infty} [F(n+1) - F(n)] = e.$ (11)

This follows from (10), or (9), if we remark that $f(n) \to e$ $(n \to \infty)$. Thus the limit (1) follows.

Remark. Clearly, (9)-(10) are valid for all positive real numbers $n > 1$.

5. The bracket in limit (1) can be written also as

$$nf(n-1) - (n-1)f(n-2) = n[f(n-1) - f(n-2)] + f(n-2).$$

Since $f(n-2) \to e$ $(n \to \infty)$, we clearly get (from (1) or (11)) that

$$\lim_{n\to\infty} n[f(n-1) - f(n-2)] = 0. \tag{12}$$

This could be proved also by l'Hospital's rule, but we are interested for more precise relations of type (6) or (9), (10). These in turn allow us to study interesting problems.

Theorem 2. *The function g is strictly positive, strictly decreasing for $x > 0$, and for all $x > 0$, the following inequality is valid:*

$$(g(x))^2 + g'(x) < 0. \tag{13}$$

Proof. Since $g(x) = \dfrac{1}{L(x, x+1)} - \dfrac{1}{x+1}$, the inequality $g(x) > 0$ is a simple consequence of the fact that L is a mean (or the right-side of (2)). A simple computation shows that

$$g'(x) = \frac{-1}{x(x+1)^2} < 0.$$

In order to prove (13), calculate

$$(g(x))^2 + g'(x) = \left[\frac{1}{L(x, x+1)} - \frac{1}{x+1}\right] - \frac{1}{x(x+1)^2} < 0 \Leftrightarrow$$

$$\frac{1}{L(x, x+1)} - \frac{1}{x+1} < \frac{1}{(x+1)\sqrt{x}} \Leftrightarrow$$

$$L(x, x+1) > \frac{(x+1)\sqrt{x}}{\sqrt{x}+1}.$$

This is true, since by $L(x, x+1) > \sqrt{x(x+1)}$ (see (2)) it is sufficient to prove that $\sqrt{x(x+1)} > \dfrac{(x+1)\sqrt{x}}{\sqrt{x}+1}$, or equivalently $\sqrt{x} + 1 > \sqrt{x+1}$.

Corollary 3. *The function f is strictly increasing and strictly concave for all $x > 0$.*

Proof. Since $f'(x) = f(x)g(x)$, $f''(x) = f(x)\{[g(x)]^2 + g'(x)\}$, this is a simple consequence of Theorem 2 (i.e. (13)).

Corollary 4. *Let (a_n) be a strictly positive sequence such that*

$$\lim_{n\to\infty} a_n g(n) = l. \tag{14}$$

Then

$$\lim_{n\to\infty} a_n [f(n) - f(n-1)] = le. \tag{15}$$

Proof. By the Lagrange mean-value theorem one has

$$f(n) - f(n-1) = f'(\xi) = f(\xi)g(\xi),$$

and since f is strictly concave, fg is strictly decreasing, yielding the double inequality

$$f(n)g(n) < f(n) - f(n-1) < f(n-1)g(n-1) \qquad (n > 1). \tag{16}$$

Now (15) follows from (16), (14), and the remark that $\lim_{n\to\infty} \frac{g(n-1)}{g(n)} = 1$ (this is implied by (2), but also by simple application of l'Hospital's rule).

Examples. 1) Let $a_n = n$. Then $a_n g(n) \to 0$ (more generally, $xg(x) \to 0$ as $x \to \infty$), so from (15) we reobtain (12).

2) Let $a_n = n^2$. Then, since $x^2 g(x) \to \frac{1}{2}$ $(x \to \infty)$, we obtain

$$\lim_{n\to\infty} n^2[f(n) - f(n-1)] = \frac{e}{2}. \tag{17}$$

Remark. This limit is due to Gh. Stoica [9].

6. Since f is strictly concave, we have that the sequence $(f(n+1) - f(n))$ is strictly decreasing. Similarly, the sequence $(F(n+1) - F(n))$ is strictly descreasing, too. We consider the sequence given by the limit (8).

Theorem 3. *Let $h(x) = f(x)[1 + xg(x)]$ $(x > 0)$. Then the function h is strictly increasing.*

Proof. Since $f'(x) = f(x)g(x)$, after a simple calculus we get

$$h'(x) = f(x)[2g(x) + x(g^2(x) + g'(x))]. \tag{18}$$

Since $g(x) = \frac{1}{L} - \frac{1}{x+1}$ (where $L = L(x, x+1)$ for simplified notation), the square bracket in (18) is

$$P(y) = xy^2 + 2y - \frac{1}{(x+1)^2},$$

where

$$y = \frac{1}{L} - \frac{1}{x+1}.$$

Since the roots of the polynomial $P(y)$ are

$$y_{1,2} = \frac{-(x+1) \pm \sqrt{x + (x+1)^2}}{x(x+1)},$$

it is sufficient to prove that

$$\frac{1}{L} - \frac{1}{x+1} > \frac{\sqrt{x+(x+1)^2} - (x+1)}{x(x+1)},$$

or

$$L < \frac{x(x+1)}{\sqrt{x+(x+1)^2} - 1}. \tag{19}$$

We shall use (see (2)) the inequality $L < \frac{2x+1}{2}$ and prove that

$$\frac{2x+1}{2} < \frac{x(x+1)}{\sqrt{x+(x+1)^2} - 1}. \tag{20}$$

After certain easy (but tedious) computations, this becomes equivalent to $0 < 3x^2 + x$, and this finishes the proof of the theorem.

Corollary 5. *Let $u(x) = x[e - f(x)]$. Then u is a strictly increasing concave function.*

Proof. $u'(x) = e - f(x)(1 + xg(x)) < 0$, since $f(x)(1+xg(x)) = h(x) < e$ by Theorem 3 and the fact that $\lim_{x\to\infty} h(x) = e$. By $u''(x) = -h'(x) < 0$, we get that u is concave.

Corollary 6. *For all $n > 1$ we have*

a) $2nf(n) < (n-1)f(n-1) + (n+1)f(n+1)$

b) $2(n+1)f(n) > nf(n-1) + (n+2)f(n+1)$
c) $2f(n) > f(n-1) + f(n+1)$.

Proof. Since u is strictly concave, we have

$$u(n+1) - u(n) < u(n) - u(n-1),$$

which transforms into a).

On the other hand, by Theorem 1, $F(x) = (x+1)f(x)$ is strictly concave, too, so $F(n+1) - F(n) < F(n) - F(n-1)$, giving b). Finally, relation c) is a consequence of Corollary 2 (i.e., the function f is strictly concave).

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Note added in proof. After completing this paper we have learned that Professors H.J. Brothers and J.A. Knox have proved relation (1) by using certain series expansions for e. The author is grateful to the authors for calling his attention to the following papers: 1) H.J. Brothers and J.A. Knox, New closed form approximations to the logarithmic constant e, Mathematical Intelligencer **20**(1998), no.4, 25-29; 2) H.J. Brothers and J.A. Knox, Novel series-based approximations to e, College Math. Journal, vol.30, 1999 (to appear).