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THE "D.V. IONESCU METHOD OF FUNCTION φ" AND SPLINE FUNCTIONS

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Abstract. Dumitru Victor Ionescu (1901–1985) known shortly as "D.V. Ionescu" was one of the greatest Romanian mathematicians of the 20$^{\text{th}}$ century. The aim of this paper is to underline his comprehensive contributions to the modern applied mathematics by the ocasion on his 100 years anniversary. His known mathematical method as "*method of function* φ" is shown to be a versatil method to construct the classical **spline functions**.

1 Introduction

D.V. Ionescu was one of the most outstanding representatives of Romanian mathematics of the 20$^{\text{th}}$ century. In more than 65 years of intensive mathematical creation he has investigated various fields of mathematics as: functional equations, ordinary integral and partial differential equations, algebra, geometry, rational mechanics and numerical mathematics. His favorite research fields have been Numerical mathematics and Theory of approximation, where he can regarded as one of its *masters*.

In more than 40 years of intensive research in approximation formulas of analysis and numerical solution of differential equations, D.V. Ionescu created a new general method of constructing of such formulas called by its author **the method of function** φ. This method is known nowadays in the mathematical literature as "*D.V. Ionescu - method*". Our aim here is to underline that the D.V. Ionescu - method is also a general method to generate the various kind of **spline functions** in one or several variables.

2 The D.V. Ionescu's method as a constructing method of spline functions

The starting point of D.V. Ionescu method in numerical analysis is the classical Green's formula. Similar approaches were attempted by J. Radon [43] and A. Ghizzetti [9], but the method of D.V. Ionescu distinguishes itself through its general character being applicable to all linear approximating formulas of Analysis in one or more variables, such as: Quadrature formulas, Interpolation formulas, Numerical differentiation formulas, Divided differences, Numerical procedures of ordinary and partial differerential equations, and also the efficient Construction of Spline functions.

D.V. Ionescu's method consists mainly in associating to any approximating formula of a boundary-value problem on an ordinary or partial differential equation, the boundary conditions being suitable chosen according to the formula to be established.

The solution of this boundary value problem, usually denoted by φ, generates the coefficients, the knots of the formula as well as its remainder R expressed in the form:

$$R[f] = \int_a^b \varphi(x) f^{(n)}(x) dx, f \in C^n[a,b] \tag{2.1}$$

The kernel function φ is always a **spline function**. The representation (2.1) was also found, but only for particular cases of formulas by G. Peano [41], L. Tchakaloff [45], G. Kowalevski [32] and R. von Misess [37]. However, the D.V. Ionescu's method differs from these by its generality and constructiveness. The components of the approximating formulas are here obtained by means of the boundary value problem which is ingeniously solved.

In detail the D.V. Ionescu's method is presented in the monographs [14], [15] which have been long time unique in the mathematical literature.

Shortly we shall present here the D.V. Ionescu's method to construct the spline functions in the representation of the divided difference of a given function as a finite intervall.

Let $f \in C^n[x_0, x_n]$ be a given function and

$$x_0 < x_1 < \cdots < x_n \tag{2.2}$$

a system of simple knots of the intervall $[x_0, x_n]$.

One denots usually by

$$[x_0, x_1, \ldots, x_n; f]$$

the *divided difference of order n* of the function f on the knots (2.2).

To the intervalls $[x_0, x_1], [x_1, x_2], \ldots, [x_{n-1}, x_n]$ one associate the polynomials $\varphi_1, \varphi_2, \ldots, \varphi_n$, respectively, of degree $\leq n-1$, i.e. the solutions of the following differential equations:

$$\varphi_1^{(n)} = 0, \varphi_2^{(n)} = 0, \cdots, \varphi_n^{(n)} = 0 \tag{2.3}$$

By the formula of integrating by parts one gets:

$$\begin{aligned} \int_{x_0}^{x_1} \varphi_1 f^{(n)} ds &= [\varphi_1 f^{(n-1)} - \varphi_1' f^{(n-2)} + \cdots + (-1)^n \varphi_1^{(n-1)} f]_{x_0}^{x_1} \\ \cdots \quad \cdots & \quad \cdots \quad \cdots \\ \int_{x_{n-1}}^{x_n} \varphi_n f^{(n)} ds &= [\varphi_n f^{(n-1)} - \varphi_n' f^{(n-2)} + \cdots + (-1)^n \varphi_n^{(n-1)} f]_{x_{n-1}}^{x_n} \end{aligned} \tag{2.4}$$

Adding equalities (2.4) and denoting the left - hand side by $\mathcal{P}[f]$ we observe that this functional depends on the values of $f, f', \cdots, f^{(n-1)}$ on the knots $x_0, x_1, \cdots, x_n$.

To the differential equations (2.3) we attach the following boundary conditions:

$$\begin{array}{llll}
\varphi_1(x_0) = 0, & \varphi_1'(x_0) = 0, & \cdots, \varphi_1^{(n-2)}(x_0) = 0 \\
\varphi_2(x_1) = \varphi_1(x_1), & \varphi_2'(x_1) = \varphi_1'(x_1), & \cdots, \varphi_2^{(n-2)}(x_1) = \varphi_1^{(n-2)}(x_1) \\
\cdots & \cdots & \cdots \quad \cdots \\
\varphi_n(x_{n-1}) = \varphi_{n-1}(x_{n-1}), & \varphi_n'(x_{n-1}) = \varphi_{n-1}'(x_{n-1}), & \cdots, \varphi_n^{(n-2)}(x_{n-1}) = \varphi_{n-1}^{(n-2)}(x_{n-1}) \\
\varphi_n(x_n) = 0, & \varphi_n'(x_n) = 0, & \cdots, \varphi_n^{(n-2)}(x_n) = 0 \\
\mathcal{P}[x^n] = 1 & &
\end{array} \tag{2.5}$$

By a through computation one deduces that the boundary value problem (2.3)–(2.5) has a unique solution φ such that $\varphi|_{[x_{i-1},x_i]} = \varphi_i$, $i = 1, 2, \cdots, n$.

Theorem 2.1. *If $f \in C^n[x_0, x_n]$, then the divided difference of order n of the function f on the knots x_0, x_1, $\cdots$, x_n can be expressed by the formula:*

$$[x_0, x_1, \cdots, x_n; f] = \int_{x_0}^{x_n} \varphi(s) f^{(n)}(s) ds \tag{2.6}$$

where the function φ coincides on each subinterval $[x_{i-1}, x_i]$ with the functions φ_i, ($i = 1, 2, \cdots, n$) which are the solution of the boundary value problem (2.2)–(2.5).

The proof of this theorem is in the paper [24] of D.V. Ionescu.

The function φ from (2.6) can be effectively written as:

$$\varphi(x) = c_0 \frac{(x - x_0)_+^{n-1}}{(n-1)!} + c_1 \frac{(x - x_1)_+^{n-1}}{(n-1)!} + \cdots + c_n \frac{(x - x_n)_+^{n-1}}{(n-1)!} \tag{2.7}$$

where $u_+ := \max\{u, 0\}$ and $c_i := \dfrac{V(x_0, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)}{V(x_0, \ldots, x_n)}$, $i = 0, 1, \ldots, n$ and V is the usual Vandermonde determinant.

It is clear from (2.7) that the function φ is a *polynomial spline function* of degree $n - 1$ and of class C^{n-2} with the knots x_0, x_1, $\ldots$, x_n.

Theorem 2.2. *The spline function φ from (2.7) is positive on the interval $[x_0, x_n]$ and*

$$\int_{x_0}^{x_n} \varphi(s) ds = \frac{1}{n!}$$

Corollary 2.1. *The divided difference from (2.6) can be written by a mean theorem as:*

$$[x_0, x_1, \ldots, x_n; f] = \frac{f^{(n)}(c)}{n!}, \; c \in]x_0, x_n[.$$

Remark. The D.V. Ionescu's method can be used in a similar way to express the divided difference of the function f on the multiple knots $x_0, x_1, \ldots, x_k$ with the orders of multiplicity respectively $n_0, n_1, \ldots, n_k$.

One obtaines

$$[\underbrace{x_0,\ldots,x_0}_{n_0},\underbrace{x_1,x_1,\ldots,x_1}_{n_1},\ \ldots,\underbrace{x_k,x_k,\ldots,x_k}_{n_k};f] = \int_{x_0}^{x_k}\varphi(s)f^{(n)}(s)ds \qquad (2.8)$$

where $n_0+n_1+\cdots+n_k=n$ and φ is here a *polynomial spline function of degree* n *with the multiple knots* of multiplicities $n_0,n_1,\ldots,n_k$, respectively.

The greatest efficiency of the D.V. Ionescu's method in Numerical Analysis is its posibilities to be naturally extended for the several variable problems. Shortly, we insert here some of the most important results concerning to divided differences of the functions in two, respectively in p variables.

Let $f\in C^{2n}(D)$ be a given function defined on a rectangle $D:=\{(x,y)\in\mathbb{R}^2|x_0\le x\le x_n;y_0\le y\le y_n\}$, $n\in\mathbb{N}$ fixed and the knots $M_{ik}(x_i,y_k)$ where

$$x_0<x_1<\cdots<x_{n-1}<x_n;\ y_0<y_1<\cdots<y_{n-1}<y_n$$

Definition 2.1. *The divided difference of order* n *of the function* f *on the knots* M_{ik} $(i,k=0,1,\ldots,n)$ *is defined by*

$$\begin{bmatrix} x_0,x_1,\ldots,x_n \\ y_0,y_1,\ldots,y_n \end{bmatrix};f\Big] := \sum_{i=0}^{n}\sum_{k=0}^{n} C_{ik}f(x_i,y_k) \qquad (2.9)$$

where the coefficients C_{ik} *are defined by:*

$$C_{ik}:=(-1)^{i+k}\frac{V(x_0,x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_n)}{V(x_0,x_1,\ldots,x_n)}\cdot\frac{V(y_0,y_1,\ldots,y_{k-1},y_{k+1},\ldots,y_n)}{V(y_0,y_1,\ldots,y_n)}$$

Theorem 2.3. *The divided difference of order* n *of the function* $f\in C^{2n}(D)$ *on the knots* $M_{ik}(i,k=0,1,\ldots,n)$ *is represented by the formula:*

$$\left[\begin{matrix} x_0,x_1,\ldots,x_n \\ y_0,y_1,\ldots,y_n \end{matrix};f\right] = \int\int_D \varphi(x,y)\frac{\partial^{2n}f(x,y)}{\partial x^n\partial y^n}dxdy$$

where the kernel function $\varphi:D\to\mathbb{R}$ *is the spline function of two variables which coincides on each rectangle* $D_{ik}:=\{(x,y)\in D|x_i\le x\le x_{i+1};\ y_k\le y\le y_{k+1}\}$ *with the function* φ_{ik} *which is the unique solution of the following boundary value problem*

$$\frac{\partial^{2n}\varphi_{i,k}}{\partial x^n\partial y^n}=0,\quad i,k=0,1,\ldots,n$$

$$\frac{\partial^{2(n-r)-1}(\varphi_{i,k}-\varphi_{i,k-1})}{\partial x^{n-r}\partial y^{n-r-1}}(x,y_k)=0;\quad \frac{\partial^{2(n-r)-1}(\varphi_{i,k}-\varphi_{i-1,k})}{\partial x^{n-r-1}\partial y^{n-r}}(x_i,y)=0$$

$$\frac{\partial^{2(n-r-1)}(\varphi_{i,k}+\varphi_{i-1,k}-\varphi_{i,k-1}-\varphi_{i-1,k-1})}{\partial x^{n-r-1}\partial y^{n-r-1}}(x_i,y_k)=\begin{cases} C_{i,k}, & r=0 \\ 0, & r\neq 0\end{cases}$$

$$r=0,1,\ldots,n-1;\quad i,k=0,1,\ldots,n$$

The proof of this Theorem is in [15] where the *spline function* φ is effectively written as a unique solution of the above boundary value problem and it is shown that this function is positive on the rectangle D and vanishes ont the boundary of D.

The divided difference of order n *of a given function* f *of* p *variables* $x_1, x_2, \ldots, x_p$ *on the prescribed knots* $M_{i_1,i_2,\ldots,i_p}$ *was for the first time defined by D.V. Ionescu* [15]. He proved, under convenient assumptions on the function f that the following integral representation of the divided difference holds:

$$\begin{bmatrix} x_1^{(0)} & x_1^{(1)}, & \ldots, & x_1^{(n)} \\ x_2^{(0)} & x_2^{(1)}, & \ldots, & x_2^{(n)} \\ \ldots & \ldots & \ldots & \ldots \\ x_p^{(0)} & x_p^{(1)}, & \ldots, & x_p^{(n)} \end{bmatrix}; f \Bigg] = \iint_{\Omega} \cdots \int \varphi(x_1, \ldots, x_p) \frac{\partial^{pn} f(x_1, \ldots, x_p)}{\partial x_1^n \ldots \partial x_p^n} dx_1 dx_2 \ldots dx_p$$

where the domain $\Omega \subset \mathbb{R}^p$ is defined by

$$\Omega = \{(x_1, \ldots, x_p) | x_1^0 \leq x_1 \leq x_1^n, \ldots, x_p^0 \leq x_p \leq x_p^{(n)}\}$$

One proves that the function φ is also the unique solution of a boundary value problem and it is the *spline function of* p *variables*, positive on Ω and it vanishes on the boundary of Ω.

The divided difference theory for the function of several variables is investigated in detail in [15].

D.V. Ionescu has introduced his method of function φ in the years 1950, especially investigate and to construct new *quadrature formulas* and new *numerical differentiation formulas*. His important original results in both these fields have been published in the monograph [14], probably the first book with this subject in the mathematical literature. Applying this method to construct any quadrature formula of the form:

$$\int_a^b f(x)dx = C_1 f(x_1) + C_2 f(x_2) + \cdots + C_n f(x_n) + R[f]$$

D.V. Ionescu has determined effectively the coefficients C_1, C_2, $\cdots$, C_n, the knots x_1, x_2, $\cdots$, x_n and represented the remainder $R[f]$ under the form:

$$R[f] = \int_a^b \varphi(s) f^{(k)}(s) ds$$

The kernel function φ is a *monospline function*. D.V. Ionescu ([22], [23], [24]) extended his method for the several variable constructing many kind of practical cubature formulas of the form:

$$\int\int_D f(x,y)dxdy = \sum_{i=0}^{n} \sum_{j=0}^{m} A_{ij} f(x_i, y_j) + R[f; x, y]$$

where the remainder is represented by a double definite integral on the domain $D \subset \mathbb{R}^2$. The kernel function in this case is a *monospline function* of two variables.

V.A. Fadeeva [8] established the following numerical differentiation formula:

$$\Delta^2 f = \frac{h}{2}[f'(x_3) - f'(x_1)] + R[f]$$

where the knots x_1, x_2, x_3 are equidistant with the step size h.

D.V. Ionescu [14], [24] generalized this formula constructing the following numerical differentiation formula:

$$\Delta^{n-1} f = A_1 f'(x_1) + A_2 f'(x_2) + \cdots + A_n f'(x_n) + R[f]$$

where $\Delta^{n-1} f$ denotes the finite difference of order $n-1$ of a given function $f \in C^{n+1}[x_1, x_n]$ in the point x_1 on the equidistant knots $x_1, x_2, \ldots, x_n$.

It can be shown that the remainder

$$R[f] = \int_{x_1}^{x_n} \varphi(x) f^{(n+1)}(x) dx$$

where φ is a *spline function* which is negative on $[x_1, x_n]$ and also that:

$$\int_{x_1}^{x_n} \varphi(s) ds = -\frac{h^{n+1}}{12} \quad \text{and} \quad R[f] = -\frac{h^{n+1}}{12} f^{(n+1)}(c), c \in]x_1, x_n[$$

3 The D.V. Ionescu's generalization of Runge-Kutta methods

To assess the contribution of D.V. Ionescu in the numerical solution of differential equation we need a short hystory of the Runge-Kutta methods in the 20^{th} century (see the paper of J.C. Butcher and G. Wanner [3] and of J.C. Butcher [2]).

Notable are in 1883 paper of Bashford and Adams [1], and the 1895 paper of Runge. The paper of Runge is recognized as the starting point of modern one-step method. The 20^{th} century began with the paper of Heun [12] whose contribution was to raise the order of the method from two to three, as in Runge's paper to four. Of the various four stages, fourth-order methods derived by Kutta [33] was the most widely, before D.V. Ionescu's method.

The set of conditions for fifth-order methods is more complicated than Kutta realised, because there are 17 conditions. The reason for the discrepance is that for orders five or greather, the conditions became different. The additional conditions to make the method applicable to high - dimensional problems only happens to be satisfied.

Nyström [38] in 1925 was able to correct some of the fifth-order methods of Kutta and he also showed how to apply the Runge-Kutta method for second order differential equation systems.

The essential contribution of the D.V. Ionescu is the constructing of Runge-Kutta methods of an arbitrary order. This is especially significant contribution because, for the first time numerical methods for differential equations went beyond the use of what are essentially quadrature formulas.

The basic approach to the analysis of Runge-Kutta methods is to obtain the Taylor expansions for the exact and numerically computed solutions at the end of a single step and to compare these series term by term.

Let consider the following initial scalar differential equation problem:

$$\begin{aligned} y'(x) &= f(x, y(x)), \\ y(x_0) &= y_0 \end{aligned} \tag{3.1}$$

where $f : D \to \mathbb{R}$, $D := \{(x, y) \in \mathbb{R}^2,\ |x - x_0| \le r,\ |y - y_0| \le R\}$ and f is an analytic function. To simplify the notation we will denote x, y, f and the various partial derivatives as being evaluated at the initial point (x_0, y_0) in a step and we will then find Taylor expansions in turn for $k_1,\ k_2, \ldots, k_s$, and finally for $y_1 \approx y(x_0 + h)$. The definitions of $k_1,\ k_2, \ldots, k_s$ and y_1 are the following:

$$\begin{aligned}
k_1 &= f(x_0, y_0)\\
k_2 &= f(x_0 + c_2 h, y_0 + a_{21} h k_1)\\
k_3 &= f(x_0 + c_3 h, y_0 + a_{31} h k_1 + a_{32} h k_2)\\
k_4 &= f(x_0 + c_4 h, y_0 + a_{41} h k_1 + a_{42} h k_2 + a_{43} h k_3)\\
&\vdots\\
k_p &= f(x_0 + c_p h, y_0 + a_{p1} h k_1 + \ldots + a_{p,p-1} h k_{p-1})
\end{aligned} \tag{3.2}$$

$$y_1 = y_0 + h \sum_{i=1}^{s} b_i k_i \tag{3.3}$$

The various terms in the Taylor series expansion of both the exact solution and the approximation computed y_1 by an above Runge-Kutta method leads to an algebraic system for the free coefficients c_i and a_{ij}, generally at the form:

$$\sum_{j=1}^{s} a_{ij} c_j^{q-1} = \frac{c_i^q}{q}; \quad i = 1, 2, \ldots, s; \quad q = 1, 2, \ldots, p \tag{3.4}$$

If $s = p$, which turns out to be possible for order up to 4 the obtained algebraic system is always solvable. Even for higher order, where the system is more complicated, it is assuming that

$$\sum_{j=1}^{s} a_{ij} = c_i, \quad i = 1, 2, \ldots, s \tag{3.5}$$

where we adopt the convention that $a_{ij} = 0$ for $j \ge i$ in explicit methods, and for more general implicit methods the (3.5) is also assumed.

J.C. Butcher [4] deeply and exhaustive investigated the theory of Runge-Kutta methods introducing the graphs of the numbers c_i and a_{ij} as *"rooted trees"* or *"arborescenses"*.

For explicit Runge-Kutta methods with s stages, there are $s(s+1)/2$ free parameters to choose. It is easy to show that an order p is possible only if $s \ge p$. Up to order 4, $s = p$ is possible, for $p > 4$ the relationship between the minimum s to obtain order p is very complicated and is partly given in the following table:

Minimum s to obtain order p

p	1	2	3	4	5	6	7	8
s	1	2	3	4	6	7	9	11

For implicit Runge-Kutta methods order p can be obtained with s stages if and only if $p \le 2s$.

Rigorous error estimations and convergence proofs where given by Runge himself but the complete and explicite estimation of various order has been accomplished only in 1951 by Bieberbach for first order equation, and in 1956 by D.V. Ionescu for the systems of differential equations and for higher order differential equations.

The following Table, quated from [3] and expanded shows the chronology of attempts methods of increasingly high order with close to the minimal numbers of stages.

Succesive derivation of high order Runge-Kutta methods.

p	s	Author	Year	Reference
2	2	Coriolis	1837 (Trapezoidal rule method)	[4]
2	2	Runge	1895 (Midpoint rule method)	[43]
3	4	Runge	1895	[43]
3	3	Heun	1900	[12]
4	8	Heun	1900	[12]
4	4	Kutta	1901	[33]
5	6	Kutta	1901	[33]
5	6	Nyström	1925 (Correction to a method of Kutta)	[38]
6	8	Huta	1956	[13]
p	s	Ionescu	1954 (Scalar differential equation)	[28]
p	s	Ionescu	1956 (Systems of differential equations)	[28]
6	7	Butcher	1964	[3]
7	9	Butcher	1968	[4]
8	11	Curtis	1970	[6]
8	11	Cooper & Verner	1972	[5]
10	18	Curtis	1975	[7]
10	17	Hairer	1978	[10]

Coming now to the D.V. Ionescu contributions and generalization of the Runge-Kutta methods, we underline here that his general method is valid for any p and s, arbitrary, under the condition that the system (3.4) to be solved. We shall very shortly present here the D.V. Ionescu method [28]:

If the function f in (3.1) is smooth enough we have the Taylor expansion:

$$y(x+h) - y(x) = \frac{h}{1!}y'(x) + \frac{h^2}{2!}y''(x) + \frac{h^3}{3!}y'''(x) + \cdots \tag{3.6}$$

where this serie is convergent for any $h \leq r_1 < r$.

The equation (3.1) with the initial condition $y(x) = y$ allow to compute the coefficients $y'(x), y''(x), \ldots$ in (3.6).

Let consider the knots

$$x + \lambda_i h, \quad i = 1, 2, \ldots, s \tag{3.7}$$

and the quadrature formula:

$$\int_x^{x+h} f(s)ds = \sum_{i=1}^{s} A_i f(x + \lambda_i h) \tag{3.8}$$

where the numbers A_i and λ_i are choosen so that the quadrature formula (3.8) has the degree of exactness $p \geq s - 1$. Writting that (3.8) is exact for any polynomial function

$f(t) = (t-x)^j$, $j = 0, 1, \ldots, p$, we have:

$$\sum_{i=1}^{s} A_i \lambda_i^j = \frac{h^{j+1}}{j+1}, \quad j = 0, 1, \ldots, p \tag{3.9}$$

In (3.8) replacing the function f by y' and in the second side the $y'(x+\lambda_i h)$ by

$$y'(x+\lambda_i h) = \sum_{j=0}^{p} \frac{\lambda_i^j h^j}{j!} y^{(j+1)}(x) + R_i$$

we get:

$$y(x+h) - y(x) = \sum_{j=0}^{p} \frac{h^j}{j!} y^{(j+1)}(x) \sum_{i=1}^{s} A_i \lambda_i^j + \sum_{i=1}^{s} A_i R_i$$

Taking in considerations the formulas (3.9) we have the expansions:

$$y(x+h) - y(x) = \frac{h}{1!} y'(x) + \frac{h^2}{2!} y''(x) + \cdots + \frac{h^{p+1}}{(p+1)!} y^{(p+1)} + \cdots \tag{3.10}$$

Now replacing in (3.8) $f(s)$ by $y'(s) = f(s, y(s))$ and choosing the knots $x + \lambda_i h$ we obtain:

$$y(x+h) - y(x) = \sum_{i=1}^{s} A_i f(x+\lambda_i h, y_i) \tag{3.11}$$

where $y_i := y(x+\lambda_i h)$.

If we denote

$$k_i := y(x+\lambda_i h) - y(x)$$

it follows:

$$k_i = \frac{\lambda_i h}{1!} y' + \frac{(\lambda_i h)^2}{2!} y'' + \cdots + \frac{(\lambda_i h)^p}{p!} y^{(p)} + \cdots$$

and denoting

$$K := \sum_{i=1}^{s} A_i f(x+\lambda_i h, y+k_i)$$

we obtain the expansion

$$K = \frac{h}{1!} y' + \frac{h^2}{2!} y'' + \cdots + \frac{h^{p+1}}{(p+1)!} y^{(p+1)} + \cdots$$

i.e. the functions of h, $y(x+h) - y(x)$ and K possess the first $p+1$ terms in the Taylor expansions identically.

It follows now, for $p = 0$, $p = 1$, $p = 2$, $p = 3$, ..., the calculation of K and the corresponding Runge-Kutta methods for each one.

For instance, for $p = 3$, the Sympson's quadrature rule

$$\int_x^{x+h} f(s)ds = \frac{h}{6}\left[f(x) + 4f\left(x + \frac{h}{2}\right) + f(x+h)\right],$$

we have

$$\lambda_1 = 0, \quad \lambda_2 = \frac{1}{2}, \quad \lambda_3 = 1$$
$$A_1 = \frac{h}{6}, A_2 = \frac{4h}{6}, A_3 = \frac{h}{6}$$

and

$$K = \frac{h}{6}\left[f(x,y) + 4f\left(x + \frac{h}{2}, y + k_2\right) + f(x + h, y + k_3)\right]$$

with the calculated k_2 and k_3.

In this case we have the classical Runge's formula with $p = 3$ and $s = 3$ with the error $O(h^4)$.

For the case $p = 4$, using the following quadrature formula of Mikeladze ($s = 4$):

$$\int_x^{x+h} f(s)ds = \frac{h}{24}[gf(x) + 19f(x+h) - 5f(x+2h) + f(x+3h)]$$

with:

$$\lambda_1 = 0, \quad \lambda_2 = 1, \quad \lambda_3 = 2, \quad \lambda_4 = 4$$
$$A_1 = \tfrac{9h}{24}, A_2 = \tfrac{19h}{21}, A_3 = -\tfrac{5h}{24}, A_4 = \tfrac{h}{24}$$

the value of K is

$$K = \frac{1}{24}(9l_1 + 19l_2 - 5l_3 + l_4)$$

and $l_1 = hf(x,y)$; $l_2 = hf(x+h, y+k_2)$, $l_3 = hf(x+2h, y+k_3)$, $l_4 = hf(x+3h, y+k_4)$, with the corresponding calculated values of k_2, k_3 and k_4.

In this case for $p = 4$, $s = 4$ it is obtained the classical Kutta's formula with the estimated error $O(h^5)$.

For the cases $p = 4, 5, \ldots$, the D.V. Ionescu method shows that the most important things is the choosing of a suitable quadrature formula. Depending of the quadrature formula we can get an any order with a minimal stages Runge-Kutta method and with a desired order of the error.

In 1956 D.V. Ionescu [28] extended his method for the system of differential equations and also for the differential equations of higher order. Unfortunatelly, the both papers [28 I], [28 II] of D.V. Ionescu have been published in romanian language in the review Buletin Ştiinţific, Secţia de Ştiinţe Matematice şi Fizice a Filialei Cluj a Acad. R.P.R., an enough isolated review in that time, these papers were remained unknow for very long time.

Nowadays, probably after the knowledges of the D.V. Ionescu method, the Runge-Kutta methods have been adapted to the solution of more general problem classes, each of wich has been the subject of specialized research in recent years. The relevant areas are, for differential–algebraic equations, for delay differential equation, for other functional differential equations, for Volterra integral equations and for stochastic differential equations.

In addition to these generalizations, the Runge-Kutta methods have been applied high efficiency to partial differential equations through so-called *method of lines*.

Concluding, the D.V. Ionescu method is useful to get also other general procedures for the numerical solution of differential equation problems, as the Adams-Bashforth algorithms, or succesive approximating methods, in which the definite integrals are approximated by suitable chosen quadrature formulas.

References

1. F. Bashford; J.C. Adams, *An attempt to test the theories of capilarry action by comparing the theoretical and measured forms*, Cambridge Univ. Press, Cambridge 1883.
2. J.C. Butcher, *Numerical methods for ordinary differential equations in the 20th century* J. Comput. Appl. Math., 125 (2000), pp. 1–29.
3. J.C. Butcher, G. Wanner, *Runge-Kutta methods; Some historical notes*, Appl. Numer. Math., 22 (1996), pp. 113–151.
4. J.C. Butcher, *The Numerical Analysis of Ordinary Differential Equations; Runge-Kutta and General Linear Methods*, Wiley, Chichester, 1987.
5. G.J. Cooper; J.H. Werner, *Some explicit Runge-Kutta methods of high order*, SIAM J. Numer. Anal. 9 (1972), pp. 389–405.
6. A.R. Curtis, *An eighth order Runge-Kutta process with eleven function evaluations per step.*, Numer. Math. 16 (1970), pp. 268-277.
7. A.R. Curtis, *High order explicit Runge-Kutta formulae their uses and limitations*, J. Inst. Math. Appl. 16 (1975), pp. 35–55.
8. V.N. Fadeeva, *La méthode des droites appliquée à quelques problèmes aux limites*, Trudi Mat-Inst., (1949), pp. 73–103.
9. A. Ghizetti, *Sulla formule di quadratura*, Rend. del Sem. Mat. e Fisica di Milano, (1954–1955), pp. 237–268.
10. A. Hairer, *Runge-Kutta method of order 10*, J. Inst. Math. Appl. 21 (1978), pp. 47–59.
11. E. Hairer, S.P. Norsett, G. Wanner, *Solving Ordinary Differential Equations, I, II*, Springer, Berlin 1993.
12. K. Heun, *Neue Methode zur approximativen Integration der Differentialgleichungen einer unabhängigen Veränderlichen*, Z. Math. Phys. 45 (1900), pp. 23–38.
13. A. Huta, *Un amelioration de la methode de Runge-Kutta-Nystrom pour la resolution numerique des equations differentielles*, Acta Math. Univ. Comenian. 1 (1956), pp. 201–224.
14. D.V. Ionescu, *Cuadraturi numerice*, Ed. tehnică, Bucureşti, 1957.
15. D.V. Ionescu, *Diferenţe divizate*, Ed. Academiei, Bucureşti, 1978.
16. D.V. Ionescu, *Sur le formule de cuadrature de Gauss*, Mathematica, (1965), pp. 43–46.
17. D.V. Ionescu, *Restes des formules de quadrature de Gauss et de Turan.*, Acta. Math. Sci. Hungaricae, (1967), pp. 283–295.
18. D.V. Ionescu, *Câteva formule de cuadratură mecanică*, St. Cerc. Şt., Acad. R. P. R., Filiala Cluj, 1951, pp. 16–37.
19. D.V. Ionescu, *La représentation de la différence divisée d'une fonction de deux variables par une integrale double*, Mathematica, (1961), pp. 57–79, 231–271.
20. D.V. Ionescu, *La diffèrence divisée d'une fonction de trois variables et sa représentation par une integrale triple*, Rev. Roum. Math. Pures Appl. (1970), pp. 1425–1436.
21. D.V. Ionescu, *Sur une classe de formules de cubature*, C. R. Acad. Sci. Paris, 266 (1968), pp. 1155–1158.
22. D.V. Ionescu, *L'extension d'une formule de cubature*, Bull. de la Classe des Sciences, Acad. Royale de Belgique, 56 (1970), pp. 661–690.
23. D.V. Ionescu, *Extension de la formule de quadrature de Gauss à une classe de formule de cubature*, C. R. Acad. Sci. Paris, 269 (1969), pp. 655–657.
24. D.V. Ionescu, *La méthode de la fonction φ en analyse numérique*, Litografia Univ. Babeş-Bolyai, Cluj-Informatică, 1977, p. 57.
25. D.V. Ionescu, A. Coţiu, *Une extension de la formule de quadrature de Lacroix*, Mathematica, 9 (1967), pp. 49–52.
26. D.V. Ionescu, A. Coţiu, *Une nouvelle extension de la formule de quadrature de Lacroix*, Anal. Şt. Univ. "Al.I. Cuza" Iaşi, XX (1971), pp .161–165.

27. D.V. Ionescu, *La representation de la difference divisée generalisée par une integrale definie I, II*, Anal. Şt. Univ. "Al.I. Cuza" Iaşi, 11 (165–184); Bull. Math. Soc. Math. R.S. Roumanie, 9 (1965), pp. 31–57.
28. D.V. Ionescu, *O generalizare a unei proprietăţi care intervine în metoda Runge-Kutta pentru integrarea numerică a ecuaţiilor diferenţiale I,II*, Bull. Şt. Acad. R.P.R., 6 (1754), pp. 229–241, 8 (1956), pp. 67–103.
29. D.V. Ionescu, *Restul în formulele de integrare numerică a lui Nyström şi Störmer*, St. Cerc. Matem. Acad. R.P.R., Filiala Cluj, 14 (1963), pp. 43–56.
30. D.V. Ionescu, *L'application de la méthode des approximation succesives à l'intégration numérique des équations différentielles*, Bull. Math. Loc. Sci. Math. et Phys. de R.P.R., t. 9, 4 (1959), pp. 423–431.
31. D.V. Ionfscu, *Aplicarea metodei aproximaţiilor succesive la integrarea numerică a ecuaţiilor diferenţiale de ordinul I*, St. Cerc. Mat. Cluj, XI, supplement (1960), pp. 701–116.
32. G. Kowalewski, *Interpolation und genährte Quadratur*, B.G. Teubner, Leipzig, 1032.
33. W. Kutta, *Beitrag zur näherungsweisen Integration Differentialgleichungen*, Z. Math. Phys. 46 (1901), pp. 435–553.
34. G. Micula, *Funcţii spline şi aplicaţii*, Edit. tehnică, Bucureşti, 1958.
35. G. Micula, M. Micula, *Sur une formule de quadrature de Tricomi*, Bull. Math. Soc. Math. de la R.S. Roumanie, 12(9968), 95-105.
36. G. Micula, *On "D.V. Ionescu's Method" in Numerical Analysis as a constructing method of spline functions*, Rev. Roum. Math. Pures Appl. 25 (1981), pp. 1137–1148.
37. R. von Mises, *Über allgemeine Quadraturformeln*, J. Reine und Angew. Math., 172 (1936), pp. 56–57.
38. E.J. Nyström, *Ueber die numerische Integration von Differentialgleichungen*, Acta Soc. Sci. Fenn. 80, (1928), pp. 1–34.
39. P. Pavel, *Formule de cuadratură de tip Gauss*, Doctoral thesis, Univ. Babeş- Bolyai, Uluj, 1967.
40. P. Pavel, *Despre o formulă de derivare numerică de tip Fadeeva*, Studia Univ. Babbeş-Bolyai, Series Mathematica, 1 (1972).
41. G. Peano, *Resto nelle formule di quadratura expresso con un integrale definito*, Atti Acad. Naz. Lincei, Rend. Classe Sai. Fis-Math-Nat., (2913), pp. 562–569.
42. I. Radon, *Restaussdrücke bei Interpolation und Quadraturformeln durch bestimmte Integralen*, Monatshefte für Math. und Phys. 42 (1935), pp. 289–396.
43. C. Runge, *Ueblr die numerische Aufloesung von Differentialgleichungen*, Math. Ann. 46 (8894), pp. 167–178.
44. I.J. Schoenberg, *Spline functions, convex curves and mechanical quadrature*, Bull. Amer. Math. Soc., 64 (1928), pp. 352–357.
45. L. Tschakaloff, *Über eine Dartstellung des Newton Differenzenquotienten und ihre Anwendungen*, Congès Internat. Math. Oslo, 1936.
46. P. Turan, *On the theory of mechanical quadrature*, Acta Sci. Math. Szeged, 12 (1950), pp. 30–37.