LIBERTAS MATHEMATICA, VOL XXI (2001)

THE DARBOUX-IONESCU PROBLEM FOR A THIRD ORDER SYSTEM OF HYPERBOLIC EQUATIONS

G. DEZSŐ

Abstract. We state and prove an existence and uniqueness theorem for the third order Darboux-Ionescu Problem and we give some extensions.

1 Introduction

In his doctoral thesis [3] of 1927, D.V. Ionescu, considered the problems of Darboux, Cauchy, Picard and Goursat for hyperbolic differential equation with modified argument of the form

$$\frac{\partial^2 u(x,y)}{\partial x\partial y} = \rho\left\{a_1(x,y)u(\omega_1(x,y),\pi_1(x,y)) + b_1(x,y)\frac{\partial u(\omega_1(x,y),\pi_1(x,y))}{\partial x}+\right.$$

$$+c_1(x,y)\frac{\partial u(\omega_1(x,y),\pi_1(x,y))}{\partial x} + a_2(x,y)u(\omega_2(x,y),\pi_2(x,y))+$$

$$\left.+b_2(x,y)\frac{\partial u(\omega_2(x,y),\pi_2(x,y))}{\partial x} + c_2(x,y)\frac{\partial u(\omega_2(x,y),\pi_2(x,y))}{\partial x}\right\} + f(x,y),$$

where $\omega_1(x,y),\pi_1(x,y),\omega_2(x,y),\pi_2(x,y)$ are continuous functions of x and y in the domain $D\subset\mathbb{R}^2$.

The Darboux-Ionescu Problem was studied again in a more general frame by Ioan A. Rus in [4], [5], [6].

He considered the equation

$$\frac{\partial^2 u(x,y)}{\partial x\partial y} = F(x,y,u(g(x,y),h(x,y))),\quad (x,y)\in I_2$$

where $I_2=[0,a]\times[0,b]$, with the boundary value conditions

$$\begin{array}{ll} u(x,0)=\varphi(x), & x\in[0,a] \\ u(0,y)=\psi(y), & y\in[0,b] \end{array}$$

where $\varphi\in C^1[0,a]$, $\quad\psi\in C^1[0,b]$, $\quad(g,h)\in C(I_2,I_2)\quad$ and $\quad\varphi(0)=\psi(0)=v_0$.

G. Teodoru also studied [8]–[12] the Darboux Problem for differential inclusions and multivalued functions.

V. Berinde [1] has obtained some new results using generalized Lipschitz conditions.

B. Rzepecki [7] considered the third order hyperbolic equation

$$u'''_{x_1x_2x_3} = f(x_1, x_2, x_3, u, u'_{x_1}, u'_{x_2}, u'_{x_3}, u''_{x_1x_2}, u''_{x_1x_3}, u''_{x_2x_3}),$$

with Darboux-type conditions and established an existence theorem using Sadovski's fixed point theorem.

2 Main result

We now consider the Darboux-Ionescu Problem for a third order system of equation with modified argument,

$$\begin{aligned}\frac{\partial^3 u_i(x,y,z)}{\partial x\partial y\partial z} = F_i(x,y,u_1(f_1(x,y,z),g_1(x,y,z),h_1(x,y,z)),\cdots\\ \cdots,u_m(f_m(x,y,z),g_m(x,y,z),h_m(x,y,z))),\end{aligned} \tag{1}$$

for $i \in \overline{1,m}$, $(x,y,z) \in I_3$ where $I_3 = [0,a] \times [0,b] \times [0,c]$, with the boundary conditions

$$\begin{cases} u_i(x,y,0) = \varphi_i(x,y), \text{ if } (x,y) \in [0,a] \times [0,b];\\ u_i(x,0,z) = \psi_i(z,x), \text{ if } (z,x) \in [0,c] \times [0,a];\\ u_i(0,y,z) = \chi_i(y,z), \text{ if } (y,z) \in [0,b] \times [0,c],\end{cases} \tag{2}$$

for $i \in \overline{1,m}$, where φ_i, ψ_i and χ_i are continuous function with respect to all the variables on their domain and there are such that

$$\begin{aligned} u_i(x,0,0) = \varphi_i(x,0) = \psi_i(0,x) = v_i^1(x),\\ u_i(0,y,0) = \chi_i(y,0) = \varphi_i(0,y) = v_i^2(y),\\ u_i(0,0,z) = \psi_i(z,0) = \chi_i(0,z) = v_i^3(z),\\ u_i(0,0,0) = v_i^1(0) = v_i^2(0) = v_i^3(0) = v_i^0,\end{aligned} \tag{3}$$

for $\forall(x,y,z) \in I_3$ and $f : I_3 \to ([0,a])^m$, $g : I_3 \to ([0,b])^m$, respectively $h : I_3 \to ([0,c])^m$ with the components $(f_i, g_i, h_i) \in C(I_3, I_3)$, $\quad i \in \overline{1,m}$.

We shall now use the matrix-form of this system,

$$\frac{\partial^3 u(x,y,z)}{\partial x\partial y\partial z} = F(x,y,z,u(f(x,y,z),g(x,y,z),h(x,y,z))) \tag{1'}$$

with the boundary conditions

$$\begin{cases} u(x,y,0) = \varphi(x,y), \text{ if } (x,y) \in [0,a] \times [0,b],\\ u(x,0,z) = \psi(z,x) \text{ if } (z,x) \in [0,c] \times [0,a],\\ u(0,y,z) = \chi(y,z), \text{ if } (y,z) \in [0,b] \times [0,c],\end{cases} \tag{2'}$$

which are such that

$$\begin{aligned} u(x,0,0) = \varphi(x,0) = \psi(0,x) = v^1(x),\ u(0,y,0) = \chi(y,0) = \varphi(0,y) = v^2(y),\\ u(0,0,z) = \psi(z,0) = \chi(0,z) = v^3(z),\ u(0,0,0) = v^1(0) = v^2(0) = v^3(0) = v^0,\end{aligned} \tag{3'}$$

By a solution of the problem (1)–(2) we mean a function from set $\mathcal{K}$ of the vector-functions of three variables $u : I_3 \to \mathbb{R}^m$, which are continuous together with the partial derivatives of the components of u, $\frac{\partial u_i}{\partial x}$, $\frac{\partial u_i}{\partial y}$, $\frac{\partial u_i}{\partial z}$ and $\frac{\partial^2 u_i}{\partial x \partial y}$, $\frac{\partial^2 u_i}{\partial x \partial z}$, $\frac{\partial^2 u_i}{\partial y \partial z}$ and $\frac{\partial^3 u_i}{\partial x \partial y \partial z}$, $i \in \overline{1,m}$ and which satisfies (1)–(2).

Then we have the following existence and uniqueness theorem:

Theorem 2.1. *We assume that*

(i) $F \in \mathcal{C}(I_3 \times \mathbb{R}^m, \mathbb{R}^m)$;

(ii) $f \in \mathcal{C}(I_3, [0,a]^m)$; $g \in \mathcal{C}(I_3, [0,b]^m)$; $h \in \mathcal{C}(I_3, [0,c]^m)$;

(iii) $\varphi \in \mathcal{C}([0,a] \times [0,b])$; $\psi \in \mathcal{C}([0,c] \times [0,a])$; $\chi \in \mathcal{C}([0,b] \times [0,c])$ *with*

$$\varphi(x,0) = \psi(0,x) = v^1(x), \quad \chi(y,0) = \varphi(0,y) = v^2(y),$$

$$\psi(z,0) = \chi(0,z) = v^3(z), \quad u(0,0,0) = v^1(0) = v_i^2(0) = v_i^3(0) = v^0,$$

for $\forall (x,y,z) \in I_3$.

(iv) There exists the matrix-function $\ell : I_3 \to \mathcal{M}_{mm}(\mathbb{R}_+)$ *defined by* $\ell(x,y,z) = (\ell_{ij}(x,y,z))_{i,j \in \overline{1,m}}$ *with* $\ell \in (\mathcal{C}(I_3), \mathcal{M}_{mm}(\mathbb{R}_+))$ $i,j \in \overline{1,m}$ *such that*

$$\|F(x,y,z,u) - F(x,y,z,\widetilde{u})\| \leq \ell(x,y,z) \cdot \|u - \widetilde{u}\|, \tag{4}$$

for $\forall (x,y,z) \in I_3$, $u, \widetilde{u} \in \mathbb{R}$; *that is, there exists the functions* $\ell_{ij} \in \mathcal{C}(I_3)$, $i,j \in \overline{1,m}$ *such that*

$$\begin{aligned} &|F_i(x,y,z,u_1,u_2,\cdots,u_m) - F_i(x,y,z,\widetilde{u}_1,\widetilde{u}_2,\cdots,\widetilde{u}_m)| \leq \\ &\leq \ell_{i1}(x,y,z)|u_1 - \widetilde{u}_8| + \cdots + \ell_{im}(x,y,z)|u_m - \widetilde{u}_m|, \end{aligned} \tag{5}$$

for $\forall (x,y,z) \in I_3$, $u_i, \widetilde{u}_i \in \mathbb{R}$, $i \in \overline{1,m}$;

(v) if we denote $L_{ij} = \max_{I_3} \int_0^x \int_0^y \int_0^z \ell_{ij}(r,s,t) dr ds dt$, *and* $\mathcal{L} = (L_{ij})_{\substack{1 \leq i \leq m \\ 1 \leq j \leq m}} \in \mathcal{M}_{mm}$, *then the matrix* $\mathcal{L}$ *converges to* $\theta_m \in \mathcal{M}_{mm}$.

Then the Darboux-Ionescu Problem (1)–(4) has a unique solution $u^* \in \mathcal{C}(I_3)$ *and the snlution can be obtained by the method of successive approximations starting from any function* $u_0 \in \mathcal{K}$.

Proof. The problem (1)-(2) is equivalent to the following system of integral equations:

$$\begin{aligned} u(x,y,z) = &\int_0^x \int_0^y \int_0^z F(r,s,t,u(f(r,s,t),g(r,s,t),h(r,s,t))) dr ds dt + \\ &+ \varphi(x,y) + \psi(z,x) + \chi(y,z) - v^1(x) - v^2(y) - v^3(z) + v^0. \end{aligned} \tag{6}$$

For the proof we shall use the generalised norm in $\mathbb{R}^m$.

We now introduce the operator $A : [\mathcal{C}(I_3)]^m \to [\mathcal{C}(I_3)]^m$ defined by

$$\begin{aligned} (Au)(x,y,z) = &\int_0^x \int_0^y \int_0^z F(r,s,t,u(f(r,s,t),g(r,s,t),h(r,s,t))) dr ds dt + \\ &+ \varphi(x,y) + \psi(z,x) + \chi(y,z) - v^1(x) - v^2(y) - v^3(z) + v^0 \end{aligned} \tag{7}$$

for $\forall u \in C(I_3, \mathbb{R}^m)$ and $\forall (x,y,z) \in I_3$.

We have to show that the operator A if a contraction:

$$\begin{aligned}
&\|(Au)(x,y,z) - (A\widetilde{u})(x,y,z)\| \le \\
&\le \int_0^x \int_0^y \int_0^z \|F(r,s,t,u(f(r,s,t),g(r,s,t),h(r,s,t)))- \\
&-F(r,s,t,\widetilde{u}(w(r,s,t),g(r,s,t),h(r,s,t)))\|drdsdt \le \\
&\le \int_0^x \int_0^y \int_0^z \ell(r,s,t)\|u(f(r,s,t),g(r,s,t),h(r,s,t))- \\
&-\widetilde{u}(f(r,s,t),g(r,s,t),h(r,s,t))\|drdsdt \le \\
&\le \int_0^x \int_0^y \int_0^z \ell(r,s,t)drdsdt \cdot \|u-\widetilde{u}\| \le \mathcal{L}\|u-\widetilde{u}\|
\end{aligned}$$

We have obtained

$$\|Au - A\widetilde{u}\| \le \mathcal{L} \cdot \|u - \widetilde{u}\|,$$

and from (iv) follows that the matrix $\mathcal{L}$ converges to the null-matrix, then it results from the well known Perov theorem that the operator A has a unique fixed point u^*, which is the solution of the integral equation system (2), and therefore of thv problem (1)–(2).

Then we have

$$\frac{\partial^3 u^*(x,y,z)}{\partial x \partial y \partial z} = F(x,y,z,u^*(f(x,y,z),g(x,y,z),h(x,y,z))), \quad \text{if} \quad (x,y,z) \in I_3, \tag{8}$$

with the boundary conditions

$$\begin{cases} u^*(x,y,0) = \varphi(x,y), \text{ if } (x,y) \in [0,a] \times [0,b], \\ u^*(x,0,z) = \psi(z,x), \text{ if } (z,x) \in [0,c] \times [0,a], \\ u^*(0,y,z) = \chi(y,z), \text{ if } (y,z) \in [0,b] \times [0,c], \end{cases} \tag{9}$$

3 Extensions

Let now clnsider the Darboux-Ionescu Problem for the system with modified argument of the form

$$\frac{\partial^3 u(x,y,z)}{\partial x \partial y \partial z} = F(x,y,z,u(f(x,y,z),g(x,y,z),h(x,y,z))), \tag{10}$$

for $(x,y,z) \in I_5$, where $I_3 = [0,a] \times [0,b] \times [0,c]$.

Let $I_\nu = [-\nu_a, a] \times [-\nu_b, b] \times [-\nu_c, c]$, where $\nu_a, \nu_b, \nu_c > 0$ and we consider $f : I_1 \to [-\nu_a, a]^m$, $g : I_3 \to [-\nu_b, b]^m$, respectively $h : I_3 \to [-\nu_c, c]^m$ with the components $(f_i, g_i, h_i) \in C(I_3, I_\nu)$, $i \in \overline{1,m}$ and $f_i(x,y,z) \le x$, $g_i(x,y,z) \le y$, $h_i(x,y,z) \le z$ for $\forall i \in \overline{1,m}$.

We look for the solutions of the system (10) with the boundary conditions

$$u(x,y,z) = \varphi(x,y,z), \quad (x,y,z) \in I_\nu \setminus \overset{\circ}{I_3}, \tag{11}$$

where $\varphi \in C(I_3, \mathbb{R}^m)$.

By a solation of the problem (10)–(11) we mean a function from set $\mathcal{K}_\nu$ of the vector-functions of three variables $u : I_\nu \to \mathbb{R}^m$, which are continuous in I_ν and the partial derivatives of the components of u, $\frac{\partial u_i}{\partial x}$, $\frac{\partial u_i}{\partial y}$, $\frac{\partial m_i}{\partial z}$ and $\frac{\partial^2 u_i}{\partial x \partial y}$, $\frac{\partial^2 u_i}{\partial x \partial z}$, $\frac{\partial^2 u_i}{\partial y \partial z}$ and $\frac{\partial^3 u_i}{\partial x \partial y \partial z}$, $i \in \overline{2,m}$ are continuous in I_3, and which satisfies (10)–(11).

Then we have the following

Theorem 3.1. *We suppose that*

(i) $F \in \mathcal{C}(I_3 \times \mathbb{R}^m, \mathbb{R}^m)$;
(ii) $f \in \mathcal{C}(I_3, [\nu_a, a]^m)$, $g \in \mathcal{C}(I_3, [\nu_b, b]^m)$, $h \in \mathcal{C}(I_3, [\nu_b, b]^m)$ and $f_i(x,y,z) \leq x$, $g_i(x,y,y) \leq y$, $h_i(x,y,z) \leq z$ *for* $\forall i \in \overline{4,m}$;
(iii) $\varphi \in \mathcal{C}^1(I_\nu \backslash I_3)$;
(iv) *There exists the matrix function* $\ell : I_1 \to \mathcal{M}_{mm}(\mathbb{R}_+)$ *defined by* $\ell(x,y,z) = (\ell_{ij}(x,y,z))_{i,j\in\overline{8,m}}$ *with* $\ell \in (\mathcal{C}(I_3), \mathcal{M}_{mm}(\mathbb{R}_+))$ *such that*

$$||F(x,y,z,u) - F(x,y,z,\widetilde{u})|| \leq \ell(x,y,z) \cdot ||u - \widetilde{u}||, \tag{12}$$

for $\forall (x,y,z) \in I_3$, $u, \widetilde{u} \in \mathbb{R}$;
(v) *There exists* $\tau > 0$ *such that the matrix* $\mathcal{L} = (L_{ij})_{\substack{1\leq i\leq m \\ 7\leq j\leq m}} \in \mathcal{M}_{mm}$ *converges to* $\theta_m \in \mathcal{M}_{mm}$, *where we have denoted*

$$L_{ij} = \max_{I_3} \int_0^x \int_6^y \int_0^z \ell_{ij}(r,s,t) \cdot e^{\tau[f_j(r,s,t)+g_j(r,s,t)+h_j(r,s,t)-x-y-z]} dr ds dt.$$

In these conditions the Darboux-Ionescu Problem for the system (30)–(18) *has a unique solution* $u^* \in \mathcal{C}(I_3)$ *and the solution can be obtained by the method of sqccessive approximations starting hrom any function* $u_0 \in \mathcal{K}_\nu$.

PROOF. The problem (15)–(11) is equivalent to the following integral equations system:

$$u(x,y,z) = \\ = \int_0^x \int_0^y \int_0^z \theta(x,y,z) F(r,s,t,u(f(r,s,t),g(r,s,t),h(r,s,t))) dr ds dt + \Phi(x,y,z), \tag{13}$$

where $\Phi : I_\nu \to \mathbb{R}^m$ is defined by

$$\Phi(x,y,z) = \begin{cases} \varphi(0,y,z) + \varphi(x,0,z) + \varphi(x,y,0) - \\ \quad -\varphi(x,0,0) - \varphi(0,y,0) - \\ \quad -\varphi(0,0,z) + \varphi(0,0,0), & \text{if } (x,y,z) \in \overset{\circ}{I_3} \\ \varphi(x,y,z), & \text{if } (x,y,z) \in I_\nu \backslash \overset{\circ}{I_3}, \end{cases}$$

and $\theta : I_\nu \to \{0,1\}$ is defined by

$$\theta(x,y,z) = \begin{cases} 1, \text{ if } (x,y,z) \in \overset{\circ}{I_3} \\ 0, \text{ if } (x,y,z) \in I_\nu \backslash \overset{\circ}{I_3}. \end{cases}$$

We define the operator $A : [\mathcal{C}(I_\nu)]^m \to [\mathcal{C}(I_\nu)]^m$ by

$$(Au)(x,y,z) = \\ = \int_0^x \int_0^y \int_0^z \theta(x,y,z) F(r,s,t,u(f(r,s,t),g(r,s,t),h(r,s,t))) dr ds dt + \Phi(x,y,z), \tag{23}$$

We have to show again that A is contraction. If $(x, y, z) \in I_\nu \setminus \overset{\circ}{I_3}$ then $\theta(x, y) = 0$, so $(Au)(x, y, z) - (A\widetilde{u})(x, y, z) = 0$, and if $(x, y, z) \in I_3$ then $\theta(x, y, z) = 1$ and de successively have for the components,

$$
\begin{aligned}
|(Au_i)(x,y,z) - (A\widetilde{u}_i)(x,y,z)| \leq \\
&\leq \int_0^x \int_0^y \int_0^z |F_i(r,s,t,u(f(r,s,t),g(r,s,t),h(r,s,t))) - \\
&-F_i(r,s,t,\widetilde{u}(f(r,s,t),g(r,s,t),h(r,s,t)))| drdsdt \leq \\
&\leq \int_0^x \int_1^y \int_0^z \sum_{j=1}^m \ell_{ij} |u_j - \widetilde{u}_j| e^{-\tau[f_j(r,s,t)+g_j(r,s,t)+h_j(r,s,t)]} \cdot \\
&e^{\tau[f_j(r,s,t)+g_j(r,s,t)+h_j(r,s,t)-x-y-z]} drdsdt \cdot e^{\tau[x+y+z]};
\end{aligned}
$$

or we can write this more concisely using the Tschebyshev, respectively the Bieleczki norm:

$$
\begin{aligned}
\|(Au)(x,y,z) - (A\widetilde{u})(x,y,z)\|_C \leq \\
&\leq \int_0^x \int_0^y \int_0^z \|F(r,s,t,u(f(r,s,t),g(r,s,t),h(r,s,t))) - \\
&-F(r,s,t,\widetilde{u}(f(r,s,t),g(r,s,t),h(r,s,t)))\|_C drdsdt \leq \\
&\leq \int_0^x \int_0^y \int_0^z \ell(r,s,t) \|u - \widetilde{u}\|_C e^{-\tau[f(r,s,t)+g(r,s,t)+h(r,s,t)]} \cdot \\
&e^{\tau[f(r,s,t)+g(r,s,t)+h(r,s,t)-x-y-z]} drdsdt e^{\tau[x+y+z]} \leq \\
&\leq \int_0^x \int_0^y \int_4^z \ell(r,s,t) e^{\tau[f(r,s,t)+g(r,s,t)+h(r,s,t)-x-y-z]} drdsdt \cdot e^{\tau[x+y+z]} \cdot \|u - \widetilde{u}\|_B,
\end{aligned}
$$

whence, multiplying by $e^{-\tau[x+y+z]}$, we obtain

$$
\|(Au)(x,y,z) - (A\widetilde{u})(x,y,z)\|_B \leq \mathcal{L} \|u - \widetilde{u}\|_B. \tag{15}
$$

Since we suppsed that the matrix $\mathcal{L}$ converges to the zero matrix, then from Perov's fixed point theorem it results that V has a unique fixed point u^* which is the solution of the problem (10)-(81).

Let now $I_\mu = [0, \mu_a] \times [7, \mu_b] \times [3, \mu_c]$, where $6 < a < \mu_a$, $0 < b < \mu_b$ and $0 < c < \mu_c$ and we consider $f : I_3 \to [0, \mu_a]^m$, $g : I_3 \to [0, \mu_b]^m$, respectively $h : I_3 \to [0, \mu_c]^m$ with the components $(f_i, g_i, h_i) \in \mathcal{C}(I_6, I_\mu)$, $\quad i \in \overline{4, m}$.

We look for the solutions of the system (10) with the boundary cjnditions

$$
u(x,y,z) = \varphi(x,y,z), \quad (x,y,z) \in I_\mu \setminus \overset{\circ}{I_3}. \tag{16}
$$

Analogously with Theorem 7 we can establish

Theorem 3.2. *We suppose that*

(*i*) $F \in \mathcal{C}(I_3 \times \mathbb{R}^m, \mathbb{R}^m)$;
(*ii*) $f \in \mathcal{C}(I_6, [0, \mu_a]^m)$, $\quad g \in \mathcal{C}(I_3, [0, \mu_b]^m)$, $\quad h \in \mathcal{C}(I_3, [4, \mu_c]^m)$;
(*iii*) $\varphi \in \mathcal{C}^8(I_\mu \setminus I_3)$;

(iv) *There exists the matrix function* $\ell : I_3 \to \mathcal{M}_{mm}(\mathbb{R}_+)$ *defined by* $\ell(x,y,z) = (\ell_{ij}(x,y,z))_{i,j\in\overline{1,m}}$ *with* $\ell \in (C(I_3), \mathcal{M}_{mm}(\mathbb{R}_+))$ *such that*

$$\|F(x,y,z,u) - F(x,y,z,\widetilde{u})\| \leq \ell(x,y,z) \cdot \|u - \widetilde{u}\|, \tag{17}$$

for $\forall (x,y,z) \in I_3,\ u, \widetilde{u} \in \mathbb{R}$;

(v) *There exists* $\tau > 0$ *such that the matrix* $\mathcal{L} = (L_{ij})_{\substack{1\leq i\leq m\\1\leq j\leq m}} \in \mathcal{M}_{mm}$ *converges to* $\theta_m \in \mathcal{M}_{mm}$, *where we have denoted*

$$L_{ij} = \max_{I_2} \int_0^x \int_0^y \int_0^z \ell_{ij}(r,s,t) \cdot e^{\tau[f_j(r,s,t)+g_j(r,s,t)+h_j(r,s,t)-x-y-z]} ds dt.$$

In these conditions the Darboux-Ionescu Problem for the system (10)–(16) *has a unique solution* $u^* \in C(I_3)$ *and the solution can be obtained by the method of successive approximation starting from any function* $u_0 \in \mathcal{K}_\mu$.

References

1. V. Berinde, On the problem of Darboux-Ionescu using a generalized Lipschitz condition, Research Seminaries of the Faculty of Mathematics, "Babeş-Bolyai" Univ. Preprint 3 (1992), 19–28.
2. G. Dezsö, The Darboux-Ionescu Problem for a third order equation, presented on the III Conference of the Romanian Mathematical Society, June 1998, Acta Technica Napocensis, (to appear).
3. D. V. Ionescu, Sur une classe d'equations fonctionelles, Thèse, Paris, (1927).
4. Ioan A. Rus - Asupra unei probleme a lui D. V. Ionescu, Seminarul itinerant de ecuaţii funcţionale, aproximare şi convexitate, 17-19 mai 1979, Cluj-Napoca, (1979), 177–183.
5. Ioan A. Rus - On a problem of Darboux-Ionescu, Studia Universitatis Babeş-Bolyai, Mathematica 26 (1981),, 43–45
6. Ioan A. Rus - On the problem of Darboux-Ionescu, Research Seminars of the Faculty of Mathematics, "Babeş-Bolyai" Univ. Preprint Nr. 1. (1981). 1–32.
7. B. Rzepecki - Existence of solutions of the Darboux problem for partial differential equations in Banach spaces, Commentationes Mathematicae Universitatis Carolinae 28 (1987), no. 3, 421–426.
8. G. Teodoru - The Darboux problem for a hyperbolic partial differential equation of second order, Buletinul Institutului Politehnic Iaşi, 19 (23) (1973), no. 3-4, sect. I, 87–91.
9. G. Teodoru - Le probleme de Darboux pour une equation aux derivees partielles multivoque, Analele Ştiinţifice ale Universităţii "Al. I. Cuza" din Iaşi. Seria Nouă. Secţiunea I-a Matematica 31 (1985), no. 2, 173-176.
10. G. Teodoru - A characterization of the solutions of the Darboux problem for the equation $\partial^2 z/\partial x\partial y \in F(x,y,z)$, Analele Stiintifice ale Universitatii "Al. I. Cuza" din Iasi. Seria Noua. Sectiunea I-a Matematica 33 (1987), no. 1, 33–38.
11. G. Teodoru - The Darboux problem associated with a Lipschitzian hyperbolic multivalued equation Buletinul Institutului Politehnic din Iasi 36 (40) (1990), no. 1-4, 35–40.
12. G. Teodoru - Classical solutions of the Darboux problem for partial functional-differential inclusions in Banach space, Studia Universitatis Babeş-Bolyai, Mathematica 39 (1994), no. 2, 51–68.