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SINGULAR TWO-POINT BOUNDARY VALUE PROBLEMS

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Abstract. This paper uses the Sinc-Galerkin method to find the approximate solution of a certain class of singular two-point boundary value problems. Sinc basis functions are used and error bounds are given which show the exponential convergence rate of the method. The matrices necessary for the formulation of the discrete system are easily assembled. Numerical approximation is obtained with an exponential accuracy and compares its performance with the finite difference method.

Keywords: Sinc Functions, Boundary Value Problems, Collocation.

1 Introduction

In this paper we consider the boundary value problem

$$\frac{1}{p(x)}\left(p(x)y'\right)' - q(x)y = f(x), \quad x \in (0,1) \tag{1.1}$$

$$\lim_{x\to 0^+} p(x)y'(x) = 0, \quad y(1) = 0. \tag{1.2}$$

where $p(x) \geq 0, p(0) = 0$ and $p(x)$ is increasing in a neighborhood $[0,\delta]$ of 0, $p^{-1}(x) \in L^1_{loc}(0,1], q(x), f(x) \in C[0,1]$ and $q(x) \geq 0$. Problems of this form are encountered in the study of radially or axially symmetric problems in which case the problem can be reduced to a one-dimensional problem with $p(x) = x^\alpha$. More general functions $p(x)$ are encountered in connection with string equations in dynamics and statistics [11]. Such problems are also encountered in stochastic control problems when studying the steady state properties of systems driven by noise which is proportional to the state, or which are nonlinear functions of the state [10].

Abu-Zaid and M. El-Gebeily [1] provide a finite difference approximation to the solution of the above problem. In this paper the method of solving (1.1), (1.2) is based on using a Sinc-Galerkin method, by errors of the form $\mathcal{O}(\exp(-c/h))$ where c is a constant and $h > 0$ is a step size. Our method is an improvement over errors of the $\mathcal{O}(h^2)$ as in [1]. The existence of a unique solution of (1.1), (1.2) has been discussed by [1, 6].

Stenger [8] originally proposed the numerical solution of ordinary differential equations with the Sinc-Galerkin method. The Sinc function (known in engineering as the band-limited function) is given by

$$Sinc(x) = \frac{\sin(\pi x)}{\pi x}, \quad x \in \mathbb{R}$$

Beginning with $Sinc(x)$ define an orthogonal basis $\{S(k,h)\}_{k=-\infty}^{\infty}, h>0$, where

$$S(k,h)(x) = Sinc\left(\frac{x-kh}{h}\right), \quad x \in \mathbb{R}$$

A basis element may be transformed to any connected subset of the real line via a composition with a suitable conformal map. In conjunction with the Galerkin method for differential equations, perhaps the most distinctive feature of the basis is its resulting exponential convergence rate of the error $\mathcal{O}(\exp(-c\sqrt{N}))$, where $c > 0$ and $2N+1$ basis functions are used to build the approximation. Moreover, the convergence rate maintains when the solution of the differential equation has boundary singularities. Of equal practical significance is that the technique's implementation requires no modification in the presence of singularities. Specifically, the statement of the quadrature, the mesh definition and the resulting matrix structure depend only on the parameters of the differential equation whether it is singular or nonsingular.

In this paper, we express the exact solution of the differential equation via the use of Green's functions as an integral type, the method of this paper enables us to replace the resulting integral equation by a system of algebraic equations, whose solution yields an accurate approximate solution to the differential equation. The approximation of Volterra-type integrals or of integral equations has been attempted by many other people. Excellent exposition of such methods may be found in Stenger [9] and in Linz [2], and Lund [4]. Also the work of Young [3] who studied the approximation of integrals of the form

$$p(x) = \int_0^x \frac{k(x,t)}{(x-t)^\alpha} g(t)dt, \quad x \in (0,b)$$

where g is continuous on $(0,b)$, and with k continuous on $(0,b)\times(0,b)$. And to the use of Sinc quadrature and Sinc collocation by Riley [5] and by Stromberg [7], who were able to achieve arbitrary accuracy in their approximation of the convolution integral

$$p(x) = \int_a^x f(x-t)g(t)dt, \quad x \in (a,b)$$

even though f or g (or both) could have singularities at endpoints of their respective intervals of definition.

This paper is organized as follows. The Sinc solution together with the Galerkin method and the convergence of the scheme is treated in section 2. Section 3 provides numerical examples that demonstrate the exponential convergence of the scheme and compare its performance with the finite difference methods.

2 Approximation Procedure

In this section we develop the Sinc-Galerkin scheme that we will employ to approximate the solution to the problem (1.1), (1.2). Approximations on $(0,1)$ are obtained from corresponding approximations on $\mathbb{R}$ via a conformal map. For a function $f \in C^\infty\mathbb{R}$ to be approximable on $\mathbb{R}$, f must obey certain analyticity and boundedness conditions in a strip in the complex plane $\mathbb{C}$ which contains $\mathbb{R}$. Though the conformal map, we obtain a corresponding "eye-shaped" region (see Figure 1.7, p. 68, of [8]) containing the interval $(0,1)$, in which our integrand must obey certain analyticity and boundedness conditions. With this in mind, we state the following.

Definition 2.1. *Let $d > 0$ and $\mathcal{D}_d$ denote the open strip*

$$\mathcal{D}_d = \{z \in \mathbb{C} : |\Im(z)| < d\} \tag{2.1}$$

Let $B(\mathcal{D}_d)$ denote the family of all functions f that are analytic in $\mathcal{D}_d$ and which satisfy

$$\int_{-d}^{d} |f(x+iy)|dy \to 0, \text{ as } x \to \mp\infty$$

and such that

$$\mathcal{N}(f,\mathcal{D}) = \lim_{x \to d^-} \left\{ \int_{\mathbb{R}} |f(x+iy)|dx + \int_{\mathbb{R}} |f(x-iy)|dx \right\} < \infty.$$

If $f \in B(\mathcal{D}_d)$, we approximate f by a truncated "Whittaker cardinal series"

$$f(x) = C_N(f,h)(x) = \sum_{k=-N}^{N} f(kh)S(k,h)(x).$$

To obtain approximations over $(0,1)$, we make use of the following conformal map:

$$\phi(z) = \log\left(\frac{z}{1-z}\right),\ \phi'(z) = \frac{1}{z(1-z)} \tag{2.2}$$

For any d such that $0 < d \leq \pi/2$, ϕ maps the region

$$\bar{\mathcal{D}}_d = \{z : |\arg[z/(1-z)]| < d\} \tag{2.3}$$

onto $\mathcal{D}_d$ (Note that $\partial\bar{\mathcal{D}}_d$ consists of two circular arcs intersecting symmetrically with an angle of $2d$ at 0 and 1). Define

$$z = \psi(w) = \phi^{-1}(w) = \frac{e^w}{1+e^w},\ \psi'(w) = \psi(w)[1-\psi(w)] \tag{2.4}$$

Let h denote a fixed positive number, and let the Sinc points be defined on $(0,1)$ by $z_k = \psi(kh)$, $k \in \mathbb{Z}$. Let e_k defined by $e_k = 1/2 + \sigma_k, k \in \mathbb{Z}$ where

$$\sigma_k = \int_0^k Sinc(x)dx.$$

Note that $\sigma_{-k} = -\sigma_k$, and we therefore only need to compute σ_k for positive integers k. For a positive integer N set $m = 2N+1$, and for a given function y defined in $(0,1)$, define a diagonal matrix $\mathcal{D}(y) = diag[y(z_{-N}), \ldots, y(z_N)]^T$, the superscript "$T$" denotes the transpose. Let $I^{(-1)}$ be a square matrix of order m having e_{i-j} as its $(i,j)th$ element, $i,j = -N, \ldots, N$. We state the following theorem without proof.

Theorem 2.1. *(see, [8, p. 219]) Let* $f/\phi' \in B(\mathcal{D}_d)$*, with* $d > 0, \alpha > 0$*, and let* $h = [\pi d/(\alpha N)]^{1/2}$*. Then there exists a constant,* C_1*, which is independent of* N*, such that*

$$\Big| \int_0^{z_k} f(t)dt - h \sum_{\ell=-N}^{N} e_{k-\ell} \frac{f(z_k)}{\phi'(z_\ell)} \Big| \leq C_1 e^{-\sqrt{2\pi d\alpha N}} \tag{2.5}$$

The boundary value problem (1.1), (1.2), can be readily transformed into an integral equations problem. As in [1], rewriting (1.1) in the form

$$(p(x)y')' = p(x)(q(x)y(x) + f(x)) \tag{2.6}$$

then integrating and applying the boundary conditions (1.2) we get

$$y(x) = -\int_x^1 \frac{1}{p(\tau)} \int_0^\tau p(t)\left[q(t)y(t) + f(t)\right] dt d\tau \tag{2.7}$$

Interchanging the order of integration in (2.7) we get

$$\begin{aligned} y(x) = &-\int_0^x \left(\int_x^1 \frac{1}{p(\tau)} d\tau\right) p(t)[q(t)y(t) + f(t)]dt \\ &-\int_x^1 \left(\int_t^1 \frac{1}{p(\tau)} d\tau\right) p(t)[q(t)y(t) + f(t)]dt \end{aligned}$$

or,

$$y(x) = -\int_0^1 k(x,t)p(t)[q(t)y(t) + f(t)]dt \tag{2.8}$$

where $k(x,t)$ is the kernel and is given by

$$k(x,t) = \begin{cases} \int_x^1 \frac{1}{p(\tau)} d\tau, \; t \leq x \\ \int_t^1 \frac{1}{p(\tau)} d\tau, \; t > x \end{cases} \tag{2.9}$$

Equation (2.8) can also be written as

$$y(x) = \int_0^1 [v(x,t)y(t) + w(x,t)]dt \tag{2.10}$$

where $v(x,t) = -k(x,t)p(t)q(t)$, and $w(x,t) = -k(x,t)p(t)f(t)$.

Let us now construct an approximate solution via Sinc method. To this end, we assume that $v(x,t)/\phi' \in B(\mathcal{D}_d)$. The approach is a direct discretization of (2.10) via the use of the indefinite integration formula (see, [8, p. 219]).

Take $h = \sqrt{\pi d/(\alpha N)}$, define $\mathbf{V} = (v_{-N}(t), ..., v_N(t))^T$, where $v_k(t) = v(x_k, t)$, $k = -N, ..., N$, and $\mathcal{D}(1/\phi') = diag[1/\phi'(z_{-N}), ..., 1/\phi'(z_N)]^T$. Also we define $\mathbf{W} = (w(z_{-N}), ..., w(z_N))^T$, then using equation (2.5) we form approximations y_i to the solution $y(x)$ evaluated at the Sinc points z_k as solutions of the system of $2N + 1$ non-linear equations.

$$\mathbf{Y} = hI^{(-1)}\mathcal{D}(v/\phi')\mathbf{Y} + hI^{(-1)}\mathcal{D}(1/\phi')\mathbf{W} \tag{2.11}$$

where $\mathbf{Y} = (y(z_{-N}), \ldots, y(z_N))^T$. We may attempt to solve the system of equations (2.11) by successive approximations, that is, by means of iterative scheme

$$\mathbf{Y}^{(r+1)} = hI^{(-1)}\mathcal{D}(v/\phi')\mathbf{Y}^{(r)} + hI^{(-1)}\mathcal{D}(1/\phi')\mathbf{W} \tag{2.12}$$

$\mathbf{Y}^{(0)}$ being the null vector. In practice it is nearly always more efficient to use the Seidel iteration, instead of the Neumann iteration described in (2.12). In using the Seidel version, we let A denote the matrix multiplying the vector $\mathbf{Y}^{(r)}$ in (2.12), and we set $A = L + U$, where the matrix L has only zero entries above the diagonal, and where U has only entries on and below the diagonal. Then, letting $\mathbf{b}$ be the vector $hI^{(-1)}\mathcal{D}(1/\phi')\mathbf{W}$, the Seidel iteration scheme then takes the form

$$(I - L)\mathbf{Y}^{(r+1)} = U\mathbf{Y}^{(r)} + \mathbf{b} \tag{2.13}$$

It is easy to show that the convergence of scheme (2.13) depends on the ℓ^∞ norm of $\mathbf{V}$. Recalling that e_k defined above satisfies the inequality $|e_k| \leq 1.1$ (see, [8, p. 172]), we have

$$\begin{aligned}
\| hI^{(-1)}\mathcal{D}(v/\phi') \| &= \max_i \sum_{j=-N}^{N} |e_{i-j} v(z_j)/\phi'(z_j)| \\
&\leq 1.1h \sum_{j=-N}^{N} |v(z_j)/\phi'(z_j)| \\
&\approx 1.1 \int_0^1 |v| dt \\
&\leq \sup_{t\in(0,1)} |v(x,t)| \\
&= 1.1 \| v \|_\infty
\end{aligned}$$

It follows, therefore, that we can always achieve convergence of the scheme provided that $\| v \|_\infty$ is small.

It remains to show that the approximate solution $\mathbf{Y}^*$ of node values of equation (2.11) converges to the node values of the exact solution $\mathbf{Y}$. To this end, choose R so that $\mathbf{Y}, \mathbf{Y}^*$ belongs to the ball $\mathcal{B} = \{X : \| X \|_\infty < 1.1 \| v \|_\infty < R/2\}$. To prove the convergent of the approximate solution $\mathbf{Y}^*$ it suffices to show that $|\mathbf{Y} - \mathbf{Y}^*|$ is small. Suppose that $\mathbf{Y}^*$ is the approximate solution which therefore satisfies:

$$\mathbf{Y}^* = A\mathbf{Y}^* + \mathbf{b}$$

where $A = hI^{(-1)}\mathcal{D}(v/\phi')$, $\mathbf{b} = hI^{(-1)}\mathcal{D}(1/\phi')$. Subtract this from (2.11) to get

$$(\mathbf{Y} - \mathbf{Y}^*) - (\mathbf{Y} - \mathbf{Y}^*)A = E.$$

or

$$\| \mathbf{Y} - \mathbf{Y}^* \| \leq \| (\mathbf{Y} - \mathbf{Y}^*)A \| + \| E \| . \tag{2.14}$$

But from the definition of the matrix A, we can find a small s $(0 < s < 1)$ such that $\| d(A) \| < s < 1$. Here dA denotes the Jacobian of the matrix A, and so, by the Mean Value

N	ERR(N)	
	FDM	Sinc-Method
16	7.8E-04	2.4E-04
32	2.0E-04	5.4E-06
64	5.0E-05	1.5E-06

Table 1. Results for Example 3.1.

Theorem, we see that $\| (\mathbf{Y} - \mathbf{Y}^*)A \| \leq s \| \mathbf{Y} - \mathbf{Y}^* \|$. Substitute this back into equation (2.14) we get

$$\| \mathbf{Y} - \mathbf{Y}^* \| \leq \frac{1}{1-s} \| E \| .$$

Which shows that the approximate solution is sufficiently close to the exact solution.

3 Numerical Results

In this section we provide numerical examples which verify the exponential convergence of the Sinc-Galerkin scheme employed in section 2. The computer application program "Maple" was used to execute the algorithms that were used with the numerical examples. Choosing examples with known solutions allows for a more complete error analysis. We consider the same examples were used in [1] and compare their solutions using finite difference method (FDM) with our solution. The Tables 1, 2, 3 show results for the three examples. In all examples, sequences of runs with $N = 16, 32, 64$ are reported and displayed in the first column of each Table. The supremum norm error between the numerical approximation y_i and the true solutions $y(x_i)$ at the nodes is determined and reported as $ERR(N) = \sup_i |y_i - y(x_i)|$. The error of our method is displayed in the third column of each Table. For comparison the error of the finite difference method taken from [1] are recorded in the second column, this comparison indicated that Sinc-Galerkin method is much better than the finite difference method. In all cases d was taken to be $\pi/2$, and $\alpha = 1$. To get values at $x = 1, \bar{x} = 1 - 2^{-24}$ was used.

Example 3.1.

$$\left((\sin\frac{\pi}{2}x)y'\right)' - \frac{\pi^2}{2}\left(\sin\frac{\pi}{2}x\right)y = \frac{-\pi^2}{2}\sin\pi x,\ 0 \leq x \leq 1 \tag{3.1}$$

$$\lim_{x\to 0^+}(\sin\frac{\pi}{2}x)y' = 0,\ y(1) = 0. \tag{3.2}$$

It is easily checked that the exact solution for the problem in (3.1), (3.2) is given by $y(x) = \cos\frac{\pi}{2}x$. Table 1 below shows $\mathcal{O}(h^2)$ converges of the FDM [1], and an exponential convergence of Sinc method.

Example 3.2.

$$\left((2x - x^2)^{3/2}y'\right)' - (2x - x^2)^{3/2}y = f(x),\ 0 \leq x \leq 1$$

N	ERR(N)	
	FDM	Sinc-Method
16	2.2E-03	6.0E-04
32	6.1E-04	2.6E-05
64	1.6E-04	1.7E-06

Table 2. Results for Example 3.2.

N	ERR(N)	
	FDM	Sinc-Method
16	2.1E-03	1.4E-03
32	5.8E-04	2.4E-04
64	1.5E-04	6.0E-06

Table 3. Results for Example 3.3.

$$\lim_{x \to 0^+} (2x - x^2)^{3/2} y' = 0, \; y(1) = 0.$$

where we can calculate the appropriate $f(x)$ by applying the differential operator to the exact solution $y(x) = x^2(1-x)$. Numerical results are given on Table 2.

Example 3.3.

$$(p(x)y')' - q(x)p(x)y = p(x)f(x), \; 0 \leq x \leq 1.$$

$$\lim_{x \to 0^+} p(x)y' = 0, \; y(1) = 0.$$

where $p(x) = \sqrt{x}(\sqrt{x}+1)\ln(\sqrt{x}+1)$, $q(x) = 1$, and $f(x)$ can be computed exactly if we know that the exact solution is given by $y(x) = x^2(1-x)$.

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