LIBERTAS MATHEMATICA, VOL XXI (2001)

EXISTENCE OF SOLUTIONS OF NONLINEAR FUZZY INTEGRAL EQUATIONS IN BANACH SPACES

K. BALACHANDRAN and P. PRAKASH

Abstract. In this paper we prove the existence of solutions of nonlinear fuzzy integral equations. The results are obtained by using regular fuzzy sets and the Darbo fixed point theorem.

Keyword: Regular fuzzy set, Fuzzy integral equation, Darbo's theorem.

1991 AMS Subject Classification: 45G10.

1 Introduction

When a physical problem is transformed into the deterministic initial value problem

$$\frac{d\phi(t)}{dt} = f(t, \phi(t)), \quad \phi(0) = w_0 \tag{$*$}$$

one can not be sure that this modeling is perfect. The initial value may not be known exactly and the function f may contain unknown parameters. Especially, if they are known through some measurements, they are necessarily subject to errors. The analysis of the effect of these errors leads to the study of the qualitative behaviour of the solution of $(*)$ like continuous dependance and several kind of stability problems.

If the nature of the error is random, then instead of the deterministic equation $(*)$ we get a random differential equation with a random initial value and random coefficients. If the underlying structure is not probabilistic, it may be appropriate to use fuzzy numbers instead of real random variables. This leads to a fuzzy initial value problem. Similarly one can describe fuzzy boundary value problems and fuzzy integral equations.

The problem of existence of solutions of fuzzy differential equations and fuzzy integral equations has been studied by many authors [3,4,6,8,13,14,16-21]. Kaleva [5] first investigated the fuzzy differential equations. Ouyang [10] equipped the space of regular fuzzy sets on a Banach space with a uniform operator topology and then embedded it into a locally convex topological vector space. Further he defined a calculus for fuzzy mappings. Subsequently, Oyuang and Wu [11] studied the problem of existence of solutions of fuzzy differential equation $(*)$. Park et al [13] proved the existence of solutions for the fuzzy integral equation

$$\phi(u) = w_0 + \int_{u_0}^{u} F(u, s, \phi(s))ds, \phi(u_0) = w_0$$

and Balachandran and Dauer [1] established the local existence theorem and approximate solutions of the perturbed fuzzy integral equation

$$\phi(u) = w_0 + \int_{u_0}^{u} F(u, s, \phi(s))ds + \int_{u_0}^{u} G(u, s, \phi(s))ds.$$

The purpose of this paper is to prove the local existence theorem and approximate solutions of the nonlinear fuzzy integral equation of the form

$$\phi(u) = F(u, \phi(u)) + \int_{u_0}^{u} G(u, s, \phi(s))ds \qquad (\mathcal{FIE})$$

where $F : J \times T(X) \to T(X)$, $G : J \times J \times T(X) \to T(X)$, are continuous, $J = [u_0, u_0 + d]$, $d > 0$ and $T(X)$ is a regular fuzzy set. The results generalize the results of Park et al [13]. Further a possible generalization for the class of a nonlinear integral equation is indicated.

2 Preliminaries

Let X be a reflexive Banach space. The collection of all convex compact subsets of X will be denoted by $CC(X)$. This set is equipped with Hausdroff metric and it is a complete metric space. Further $CC(X)$ can be embedded into a normed linear space $RCC(X)$[15].

Definition 2.1. *By a regular fuzzy set in X we mean a mapping $w : (0,1] \to CC(X)$ such that:*

(i) $w(t_1) \supset w(t_2)$ *whenever* $0 < t_1 \leq t_2 \leq 1$,
(ii) $w(t) \to w(t_0)$ *if* $t \to t_0$ and
(iii) $\sup p(w) = cl(\bigcup_{0<t\leq 1} w(t)) \in CC(X)$,

where $cl(B)$ denotes the closure of B.

The set of all regular fuzzy sets in X is denoted by $T(X)$ and it becomes a cone for addition and scalar multiplication. Further, it is a subspace of the product topological space $\Pi_{0<t\leq 1} Y_t$, where $Y_t = CC(X)$ for every t. Denote $\Pi_{(0,1]} RCC(X)$ by $\mathcal{R}(\mathcal{X})$. Then $\mathcal{R}(\mathcal{X})$ is a locally convex topological vector space and $T(X) \hookrightarrow \mathcal{R}(\mathcal{X})$. A mapping $F : [a,b] \to T(X)$ is called integrable(differentiable) if it is integrable(differentiable) in the sense of the integration(differentiation) for vector valued function when F is considered as a mapping from $[a,b]$ to $\mathcal{R}(\mathcal{X})$.

For a mapping $F : [a,b] \to T(X)$, define $F_t : [a,b] \to CC(X)$ by

$$F_t(s) = F(s)(t) \qquad \text{if} \qquad 0 < t \leq 1$$

and

$$F_0(s) = \text{supp } F(s).$$

Let A be a bounded set in X, then the Kuratowskii measure for A is given by

$$\alpha(A) = \inf\{\epsilon > 0 : A \text{ can be covered by a finite number of sets each with diameter} \leq \epsilon\}$$

For this measure we have the following properties [7]. Let $\text{clco}(A)$ denote the closed convex hull of A.

Lemma 2.1. *Let A and B be bounded subsets of X. Then:*

(i) $\alpha(A) = \alpha(cl(A)) = \alpha(clco(A))$
(ii) $A \subseteq B$ *implies* $\alpha(A) \leq \alpha(B)$
(iii) a) *If* $A \subseteq C(J, E)$ *is bounded, then*

$$\sup_{u\in J} \alpha(A(u)) \leq \alpha(A(J)) \leq \alpha(A), \text{ where } A(u) = \{f(u) : f \in A\}$$

b) *If A is also equicontinuous, then*

$$\alpha(A) = \sup_{u\in J} \alpha(A(u)) = \alpha(A(J))$$

(iv) a) *If* $\{x_n\} \subseteq C(J, X)$ *is an equicontinuous family, then*

$$\alpha(\{\int_{u_0}^{u} x_n(s)\})ds \leq \int_{u_0}^{u} \alpha(\{x_n(s)\})ds$$

b) *If* $\{\epsilon_n\} \subseteq R$ *with* $\epsilon_n \to 0$, *then* $\alpha(\{x_n(u - \epsilon_n)\}) = \alpha(\{x_n(u)\})$ *for every* $u > u_0$.
(v) *If* $f(u, s, x(s))$ *is integrable for each* $x \in A \subseteq C(J, X)$, *then*

$$\alpha\left(\left\{\int_{u_0}^{u} f(u, s, x(s))ds : x \in A\right\}\right) \leq |u - u_0|\alpha(clco(f(u, [u, u_0], A[u, u_0]))).$$

We use the following result which is a generalization of Nohel [9].

Lemma 2.2. *Let* $H_1 \in C(J \times R^+, R^+), H_2 \in C(J \times J \times R^+, R^+)$. *Let* $H_1(u, x), H_2(u, s, x)$ *be monotone nondecreasing in x for each* $u \in J, (u, s) \in J \times J$ *respectively and*

$$m(u) \leq H_1(u, m(u)) + \int_{u_0}^{u} H_2(u, s, m(s))ds, \qquad \text{for} \qquad u \geq u_0,$$

where $m \in C(J, R^+)$. *Then* $m(u) \leq k(u)$, *where* $k(u)$ *is the maximal solution of*

$$\nu(u) = H_1(u, \nu(u)) + \int_{u_0}^{u} H_2(u, s, \nu(s))ds.$$

Also,we need the following fixed point theorem due to Darbo [2].

Theorem 2.1 (Darbo). *If A is a bounded closed convex subset of a Banach space X and* $T : A \to A$ *is a continuous mapping such that for any bounded subset B of A we have*

$$\alpha(T(B)) \leq k\alpha(B),$$

where k is a constant, $0 \leq k < 1$, *then T has a fixed point.*

3 Main theorems

Theorem 3.1. *Let X be a reflexive Banach space and $F : J \times T(X) \to T(X)$, $G : J \times J \times T(X) \to T(X)$ be continuous. Suppose that the following assumptions are satisfied:*

(i) $\|F_t(u, \phi(u))\| \leq M$ *for* $(u, \phi(u)) \in J \times T(X)$ *and* $\|G_t(u, s, \phi(s))\| \leq N$ *for* $(u, s, \phi(s)) \in J \times J \times T(X)$, and $t \in (0, 1]$, *where* $M > 0, N > 0$;

(ii)
$$\lim_{u \to v} \sup\{\|F_t(u, \phi(u)) - F_t(v, \phi(v))\| : \phi(u) \in B\} = 0;$$

$$\lim_{u \to v} \sup\left\{ \int_I \|G_t(u, s, \phi(s)) - G_t(v, s, \phi(s))\| ds : \phi(s) \in B \right\} = 0;$$

for every bounded set $B \subseteq T(X)$ *and every interval* $I \subseteq J$;

(iii) $\alpha(F_t(J \times B)) \leq \beta\alpha(B(t))$ *and* $\alpha(G_t(J \times J \times B)) \leq \beta^*\alpha(B(t))$ *for each bounded set* $B \subseteq T(X)$ *and* $t \in (0, 1]$ *where* $0 < \beta < \frac{1}{2}, \beta^* > 0$.

Then there exists a local solution $\phi(u) \in T(X)$ *for the problem* $(\mathcal{FIE})$ *on* $[u_0, u_0 + d^*]$ *for some* $d^* > 0$.

PROOF. Let $J = [u_0, u_0 + d]$, take $r > 0$ such that $r > M$ and put

$$B = \{w \in T(X) : \|w(t)\| \leq r \quad \text{for all} \;\; t \in (0, 1]\}.$$

Then $J \times B, J \times J \times B$ are bounded sets in $J \times T(X), J \times J \times T(X)$ respectively. Choose $d^* > 0$ such that $d^* = \min\{d, \frac{r-M}{N}, \frac{1-2\beta}{2\beta^*}\}$ and put $J^* = [u_0, u_0 + d^*]$, $\Omega = \{\phi : \phi : J^* \to \mathcal{R}(\mathcal{X})$ is continuous$\}$ then according to uniform convergence topology on J^*, Ω becomes a locally convex topological vector space. Consider the subset Φ of Ω defined by

$$\Phi = \{\phi \in \Omega : \|\phi(s)(t)\| \leq M + Nd^* \quad \text{for all} \quad (s, t) \in J^* \times (0, 1]\}.$$

Clearly, Φ is closed, bounded and convex. Define a mapping $T : \Phi \to \Omega$ as follows:

$$(T\phi)(u) = F(u, \phi(u)) + \int_{u_0}^{u} G(u, s, \phi(s))ds, u \in J^*$$

If $\phi \in \Phi$, then

$$\begin{aligned} \|(T\phi)(u)(t)\| &\leq \|F_t(u, \phi(u))\| + \int_{u_0}^{u} \|G_t(u, s, \phi(s))\| ds \\ &\leq M + N|u - u_0| \\ &\leq M + Nd^* \end{aligned}$$

for all $(u, t) \in J^* \times (0, 1]$. Then $T\Phi \subseteq \Phi$ and T is bounded.

To prove T is continuous, let $\phi_\alpha \in \Phi$ be a net convergence to ϕ in Φ. This means, by definition, that $\phi_\alpha(s) \to \phi(s)$ on J^*. By the continuity of F and G we have

$$F(u, \phi_\alpha(u)) \to F(u, \phi(u)) \quad \text{and} \quad G(u, s, \phi_\alpha(s)) \to G(u, s, \phi(s)).$$

Using (i) and by applying bounded convergent theorem, it follows that

$$\int_{u_0}^{u} G(u,s,\phi_\alpha(s))ds \to \int_{u_0}^{u} G(u,s,\phi(s))ds$$

as $\phi_\alpha(s) \to \phi(s)$. Hence $T\phi_\alpha(s) \to T\phi(s)$.

If $u, v \in J^*, u > v$ and $\phi \in \Phi$ then we get

$$\begin{aligned} \|(T\phi)(u)(t) - (T\phi)(v)(t)\| &\le \|F_t(u,\phi(u)) - F_t(v,\phi(v))\| + \int_u^v \|G_t(u,s,\phi(s))\|ds \\ &\quad + \int_{u_0}^{u} \|G_t(u,s,\phi(s)) - G_t(v,s,\phi(s))\|ds \\ &\le N|u-v| + \|F_t(u,\phi(u)) - F_t(v,\phi(v))\| \\ &\quad + \int_{u_0}^{u} \|G_t(u,s,\phi(s)) - G_t(v,s,\phi(s))\|ds \end{aligned}$$

Let $\delta_1 < \epsilon/3N$, then $|u-v| < \delta_1$ implies $N(u-v) < \epsilon/3$. By assumption (ii), there exist $\delta_2 > 0, \delta_3 > 0$ such that

$$\begin{aligned} |u-v| < \delta_2 &\Rightarrow \|F_t(u,\phi(u)) - F_t(v,\phi(v))\| < \epsilon/3, \\ |u-v| < \delta_3 &\Rightarrow \textstyle\int_{u_0}^{u} \|G_t(u,s,\phi(s)) - G_t(v,s,\phi(s))\|ds < \epsilon/3. \end{aligned}$$

Thus for every basis neighborhood of the origin 0 in $\mathcal{R}(\mathcal{X})$, $U = U(0; t_1 \cdots t_m; \epsilon)$ we take $\delta = \min\{\delta_1, \delta_2, \delta_3\}$ such that $\|(T\phi)(u)(t_i) - (T\phi)(v)(t_i)\| < \epsilon$, for $i = 1, \cdots, m$ and $|u-v| < \delta$. Hence $(T\phi)(u) - (T\phi)(v) \in U$. This shows that $T(\Phi)$ is a set of equicontinuous mappings.

Now let $\Psi \subset \Phi$, then by Lemma 1 we get

$$\begin{aligned} \alpha(T\Psi(u)(t)) &= \alpha(\{F_t(u,\phi(u)) + \int_{u_0}^{u} G_t(u,s,\phi(s))ds : \phi \in \Psi\}) \\ &\le \alpha(clcoF_t(u,\Psi(u))) + |u-u_0|\alpha(clcoG_t(u,[u,u_0],\Psi([u,u_0]))) \\ &\le \alpha(F_t(J^* \times \Psi(J^*))) + |u-u_0|\alpha(G_t(J^* \times J^* \times \Psi(J^*))) \\ &\le \alpha(F_t(J \times \Psi(J^*))) + |u-u_0|\alpha(G_t(J \times J \times \Psi(J^*))) \\ &\le \beta\alpha(\Psi(J^*))(t) + |u-u_0|\beta^*\alpha(\Psi(J^*))(t) \\ &\le (\beta + \beta^* d^*)\alpha(\Psi(J^*))(t) \\ &\le (1/2)\alpha(\Psi(J^*))(t), \quad \text{for every} \quad t \in (0,1]. \end{aligned}$$

Thus,

$$\alpha(T\Psi) = \sup_{u \in J^*} \alpha(T\Psi(u)) \le \frac{1}{2}\alpha(\Psi).$$

So by Darbo's fixed point theorem, T has a fixed point ϕ in Φ. Clearly such a fixed point is a solution of $(\mathcal{FIE})$. □

Theorem 3.2. *Let $F : J \times T(X) \to T(X), G : J \times J \times T(X) \to T(X)$ be uniformly continuous. Assume the following conditions are satisfied:*

(*i*) (*i*) and (*ii*) of Theorem 3.1 hold;
(*ii*) $\alpha(F_t(J \times B)) \leq g(u, \alpha(B(t)), \alpha(G_t(J \times J \times B)) \leq h(u, s, \alpha(B(t))$ for every bounded set $B \subseteq T(X)$, where $g \in C(J \times R^+, R^+), h \in C(J \times J \times R^+, R^+), g(u, x)$ and $h(u, s, x)$ are monotone nondecreasing in x for each $u \in J, (u, s) \in J \times J$ and the equation

$$x(u) = g(u, x(u)) + \int_{u_0}^{u} h(u, s, x(s))ds$$

has a unique solution $x(u) \equiv 0$.

Then there exists $\phi(u)$ to $(\mathcal{FIE})$ on $[u_0, u_0 + d^*]$ for some $d^* > 0$.

PROOF. Let r, d^* and J^* be as in Theorem 3.1. Define a sequence $\{\phi_n\}$ in Ω by

$$\phi_n(u) = F(u, \phi_n(u)) + \int_{u_0}^{u} G(u, s, \phi_n(s - d^*/n))ds, \qquad u_0 < u < u_0 + d^*.$$

The equicontinuity and the uniform boundedness of $\{\phi_n\}$ follows as in Theorem 3.1. Since F and G are uniformly continuous and $\{\phi_n\}$ is equicontinuous, the sequence $\{F(u, \phi_n(u))\}$ and $\{G(u, s, \phi_n(s - d^*/n))\}$ are equicontinuous. Using Lemma 2.1 and assumption (*ii*), we get

$$\begin{aligned}
\alpha(\{\phi_n(u)(t)\}) &= \alpha(\{F_t(u, \phi_n(u)) + \int_{u_0}^{u} G_t(u, s, \phi_n(s - d^*/n))ds\}) \\
&\leq \alpha(\{F_t(u, \phi_n(u))\}) + \int_{u_0}^{u} \alpha(\{G_t(u, s, \phi_n(s - d^*/n))ds\}) \\
&= \alpha(F_t(u, \{\phi_n(u)\})) + \int_{u_0}^{u} \alpha(G_t(u, s, \{\phi_n(s - d^*/n)\}))ds \\
&\leq g(u, \alpha(\{\phi_n(u)\})) + \int_{u_0}^{u} h(u, s, \alpha(\{\phi_n(s - d^*/n)\}))ds \\
&= g(u, \alpha(\{\phi_n(u)(t)\})) + \int_{u_0}^{u} h(u, s, \alpha(\{\phi_n(s)(t)\}))ds.
\end{aligned}$$

Consequently,

$$\alpha(\{\phi_n(u)(t)\}) \leq k(u)$$

where $k(u)$ is the maximal solution of

$$\nu(u) = g(u, \nu(u)) + \int_{u_0}^{u} h(u, s, \nu(s))ds.$$

Hence, $\alpha(\{\phi_n(u)(t)\}) \equiv 0$.

Thus $\{\phi_n\}$ contains a uniformly convergent subsequence $\{\phi_{n_i}\}$. If $\phi_{n_i} \to \phi$, then the bounded convergence theorem implies

$$F(u, \phi_{n_i}(u)) \to F(u, \phi(u))$$

and

$$\int_{u_0}^{u} G(u, s, \phi_{n_i}(s))ds \to \int_{u_0}^{u} G(u, s, \phi(s))ds.$$

Thus ϕ is a solution of $(\mathcal{FIE})$. □

Remark. Using the above technique we can establish the local existence theorem and approximate solutions of the following general fuzzy integral equation

$$\phi(u) = F(u, \phi(u), \int_{u_0}^{u} k(u, s, \phi(s))ds)$$

where $k : J \times J \times T(X) \to T(X)$ and $F : J \times T(X) \times T(X) \to T(X)$, are continuous, $J = [u_0, u_0 + d]$ and $T(X)$ is a regular fuzzy set.

References

1. K. Balachandran and J.P. Dauer, *Existence of solution of perturbed fuzzy integral equations in Banach Spaces*, Indian J.Pure Appl.Math., 28(1997), pp. 1461–1468.
2. G. Darbo, *Punti uniti in transformaziani a codomino noncompatto*, Rend.Sem.Mat. Univ.Padova, 24(1955), pp. 84–92.
3. Z. Ding and A. Kandel, *Existence and stability of fuzzy differential equations*, J.Fuzzy Math., 5(1997), pp. 681–697.
4. Z. Ding, M. Ma and A. Kandel, *Existence of solutions of fuzzy differential equations with parameters*, Inform.Sci., 99(1997), pp. 205–217.
5. O. Kaleva, *Fuzzy differential equations, Fuzzy Sets and Systems*, 24(1987), pp. 301–317.
6. O. Kaleva, *The Cauchy problem for fuzzy differential equations, Fuzzy Sets and Systems*, 35(1990), pp. 389–396.
7. R.H. Martin, *Nonlinear Operators and Differential Equations in Banach Spaces*, John Wiley, New York, 1976.
8. J. Mordeson and W. Newman, *Fuzzy integral equations*, Inform.Sci., 87(1995), pp. 215–229.
9. J.A. Nohel, *Some problems in nonlinear Volterra integral equations*, Bull.Amer.Math.Soc., 68(1962), pp. 323–329.
10. H. Ouyang, *Topological properties of the spaces of regular fuzzy sets*, J.Math.Anal.Appl., 129(1988), pp. 346–361.
11. H. Ouyang and Y. Wu, *On fuzzy differential equations, Fuzzy Sets and Systems*, 32(1989), pp. 321–325.
12. J.Y. Park, H.K. Han and J.U. Jeong, *Asymptotic behaviour of solutions of fuzzy differential equations*, Fuzzy Sets and Systems, 91(1997), pp. 361–364.
13. J.Y. Park, Y.C. Kwun and J.U. Jeong, *Existence of solutions of fuzzy integral equations in Banach spaces*, Fuzzy Sets and Systems, 72(1995), pp. 373–378.
14. D.W. Pearson, *A property of linear fuzzy differential equations*, Appl.Math.Lett., 10(1997), pp. 99–103.
15. H. Radstrom, *An embedding theorem for spaces of convex sets*, Proc.Amer.Math.Soc., 3(1952), pp. 165–169.
16. P.V. Subrahmanyam and S.K. Sudarsanam, *A note on fuzzy Volterra integral equations*, Fuzzy Sets and Systems, 81(1996), pp. 237–240.
17. P.V. Subrahmanyam and S.K. Sudarsanam, *An existence theorem for a fuzzy functional integral equation*, J.Fuzzy Math., 5(1997), pp. 723–732.
18. H. Wang, S. Song and Q. Liu, *On the basic solutions to (DG)fuzzy integral equation*, J.Fuzzy Math., 6(1998), pp. 75–82.
19. C. Wu, S. Song and E. Lee, *Approximate solutions,existence and uniqueness of the Cauchy problem of fuzzy differential equations*, J.Math.Anal.Appl., 202(1996), pp. 629–644.
20. C. Wu, S. Song and E. Lee, *On the basic solutions to the generalized fuzzy integral equation*, Fuzzy Sets and Systems, 95(1998), pp. 255–260.
21. Y. Zhang, Z. Qiao and G. Wang, *Solving processes for a system of first order fuzzy differential equations*, Fuzzy Sets and Systems, 95(1998), pp. 333–347.