

ON RELATIONSHIPS BETWEEN DIFFERENT KINDS OF STABILITY OF FUNCTIONAL DIFFERENTIAL EQUATIONS

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1 Introduction

The study of stability and asymptotic behaviour of solutions of ordinary differential equations has made considerable use of the variation of constants formula, both the classical linear version and the nonlinear version of Alekseev. In [7] G.A. Shanholdt has developed an analogue of Alekseev's formula for use with perturbed nonlinear functional equation. In this paper, the Shanholdt's representation will be used to obtain some results concerning the relationships between different kinds of stability in variation and classical types of stability for the unperturbed differential difference equation. The problem of the integral (Lipschitz) stability will be studied directly and, also, using a comparison ordinary differential equation. The imposed conditions will assure the existence of a Liapunov function whose properties will be also used.

2 Preliminaries

In the sequel we will use the standard notations introduced by J.K. Hale [3]. Suppose the real numbers r and τ are fixed with $r \geq 0$ and $\tau \in \mathbb{R}$, E^n is an n -dimensional linear vector space with norm $|\cdot|$ and $C([a, b], E^n)$ is the Banach space of continuous functions mapping the interval $[a, b]$ into E^n with the topology of uniform convergence. When $[a, b] = [-r, 0]$, we let $C = C([-r, 0], E^n)$, and denote the norm of $\varphi \in C$ by $\|\varphi\| = \sup\{|\varphi(\theta)|; -r \leq \theta \leq 0\}$.

If $x \in C([a - r, b], E^n)$ for any $a \leq b$, then for each $t \in [a, b]$, the symbol x_t will denote an element of the space C defined by $x_t(\theta) = x(t + \theta)$, $-r \leq \theta \leq 0$.

Let A be an open subset of C , $\Gamma = (\tau, \infty) \times A$, and $f : \Gamma \rightarrow E^n$ be a given function. A functional differential equation is a relation of the form

$$x'(t) = f(t, x_t), \tag{2.1}$$

where $x'(t)$ is the right-hand derivative of $x(u)$ at $u = t$. For any $\sigma \in (\tau, \infty)$, $\varphi \in C$, we say that $x = x_t(\sigma, \varphi)$ is a *solution of (2.1) with initial function φ at σ* , or a *solution of (2.1) textitthrough (σ, φ)* , if there exists an $a > 0$ such that $x \in C[\sigma - r, \sigma + a], E^n$, $x_\sigma = \varphi$ and x satisfies (2.1) for $t \in [\sigma, \sigma + a)$. The existence of at least one solution of (2.1) is discussed in [3; pp. 13–15] and the existence of an exactly one solution in [3; pp. 21–23].

Besides the study of the equation (2.1) we will be also interested in the study of the perturbed differential difference equation

$$y'(t) = f(t, y_t) + g(t, y_t). \quad (2.2)$$

It is assumed that $f, g : \Gamma \rightarrow E^n$ are continuous in Γ , and that $f(t, \varphi)$ has a continuous Fréchet derivative $f'(t, \varphi)$ with respect to φ in Γ .

Corresponding to each solution $x_t(\sigma, \varphi)$ of (2.1), one defines a linear functional equation

$$z'(t) = f'(t, x_t(\sigma, \varphi))z_t, \quad t \in J(\sigma, \varphi) \quad (2.3)$$

where $J(\sigma, \varphi)$ is the maximal interval of existence of $x_t(\sigma, \varphi)$.

Equation (2.3) is called the *linear variational equation of (2.1) with respect to $x_t(\sigma, \varphi)$* . In what follows, the family of linear operators associated with (2.3) will be denoted by $T(t, \sigma : \varphi)$, $t \geq \sigma$. For more details see [7].

If the $n \times n$ matrix function Y_0 is defined by: $Y_0(\theta) = O$, $-\tau \leq \theta < 0$, $Y_0(0) = I$ - the unity matrix, then by Theorem 3 [7], for any $(\sigma, \varphi) \in (\tau, \infty) \times A$, if $J(\sigma, \varphi) = [\sigma, \infty)$ and solutions of (2.2) are unique, we have

$$y_t(\sigma, \varphi) = x_t(\sigma, \varphi) + \int_{\sigma}^t T(t, s : y_s(\sigma, \varphi)) Y_0 g(s, y_s(\sigma, \varphi)) ds, \quad (2.4)$$

so long as $(\sigma, \varphi) \in (\tau, \infty) \times A$ and t is in an interval in which $y_t(\sigma, \varphi)$ exists.

We also note that by Theorem 2 [7], if $A_0 \subset A$ is a convex set and for every $(\sigma, \varphi) \in (\tau, \infty) \times A_0$, $J(\sigma, \varphi) \equiv [\sigma, \infty)$, then

$$x_t(\sigma, \varphi_2) - x_t(\sigma, \varphi_1) = \int_0^1 T(t, \sigma : \varphi_1 + \xi(\varphi_2 - \varphi_1)) d\xi (\varphi_2 - \varphi_1), \quad (2.5)$$

provided $(\sigma, \varphi_1), (\sigma, \varphi_2) \in (\tau, \infty) \times A_0$, $0 \leq \xi \leq 1$. A noteworthy particular case of (2.5) is obtained for $f(t, 0) \equiv 0$, namely

$$x_t(\sigma, \varphi) = \int_0^1 T(t, \sigma : \xi\varphi) d\xi \varphi. \quad (2.6)$$

We now also give the definitions of different kinds of stability in terms of the behaviour of solutions of (2.1), (2.2) and of the variational equation (2.3). In what follows we shall assume that $f(t, 0) = g(t, 0) = 0$.

Definition 2.1. *The solution $x = 0$ of (2.1) is said to be:*

- a) *Uniformly Lipschitz stable (ULS) if there exists a constant $\alpha > 0$, which is independent of σ , and another constant $M = M(\alpha) \geq 1$, such that $\|x_t(\sigma, \varphi)\| \leq M(\alpha)\|\varphi\|$, for all $t \geq \sigma > \tau$ provided that $\|\varphi\| \leq \alpha$;*
- b) *Globally uniformly Lipschitz stable (GULS) if there exists a constant $M \geq 1$ such that $\|x_t(\sigma, \varphi)\| \leq M\|\varphi\|$ holds, for all $t \geq \sigma > \tau$ and $\varphi \in C$;*

- c) Globally exponentially asymptotically Lipschitz stable (GEALS) if there exists two constants $L \geq 1$ and $\lambda > 0$ such that

$$\|x_t(\sigma, \tau)\| \leq L \exp[-\lambda(t - \sigma)] \cdot \|\varphi\|, \text{ for all } t \geq \sigma > \tau \text{ and } \varphi \in C.$$

When the above estimate holds with $L = L(\sigma)$, $L(t) \geq 1$ being a continuous function for $t \geq \sigma > \tau$, then we shall say that it is a matter of the generalized (GEALS).

Definition 2.2. The solution $x = 0$ of (2.1) is said to be:

- a) Integrally uniformly stable (IUS) if for every $\alpha > 0$ there exists $\beta(\alpha) > 0$ such that

$$\|y_t(\sigma, \varphi)\| \leq \beta(\alpha), \quad t \geq \sigma > \tau \text{ provided that } \|\varphi\| \leq \alpha \text{ and } \int_{\sigma}^{\infty} \sup\{\|g(s, y_s)\|; \|y_s\| \leq \alpha\} ds < \infty, \text{ for any solution } y_t(\sigma, \varphi) \text{ of (2.2);}$$

- b) Integrally uniformly Lipschitz stable (IULS) if for every $\alpha > 0$ there exists $\beta(\alpha) \geq 1$ such

$$\text{that } \|y_t(\sigma, \varphi)\| \leq \beta(\alpha)\|\varphi\|, \quad t \geq \sigma > \tau, \text{ provided that } \|\varphi\| \leq \alpha \text{ and } \int_{\sigma}^{\infty} \sup\{\|g(s, y_s)\|; \|y_s\| \leq \alpha\} ds < \infty, \text{ for any solution } y_t(\sigma, \varphi) \text{ of (2.2).}$$

Remark. The above definitions are improvements, in a certain sense, of Definitions 1.3 and 1.5 from [2]. At the same time they are more closely from the classical integral stability concept.

Remark. In the case when α is allowed be ∞ it will a matter of the global (IUS), respectively, of the global (IULS), i.e., (GIUS) and (GIULS).

Definition 2.3. The solution $x = 0$ of (2.1) is said to be:

- a) Uniformly stable in variation (USV) if for each $\alpha > 0$ there exists $M(\alpha) \geq 1$ such that the operator $T(t, \sigma : \varphi)$ of the variational equation (2.3) satisfies $\|T(t, \sigma : \varphi)\| \leq M(\alpha)$, for all $t \geq \sigma > \tau$ and $\|\varphi\| \leq \alpha$;
- b) Globally uniformly stable in variation (GUSV) if there is a constant $M \geq 1$ such that $\|T(t, \sigma : \varphi)\| \leq M$, for all $t \geq \sigma > \tau$ and $\varphi \in C$;
- c) Generalized globally exponentially asymptotically stable in variation (GEASV) if $\|T(t, \sigma : \varphi)\| \leq L(\sigma) \exp[\alpha(t) - \alpha(\sigma)]$, for all $t \geq \sigma > \tau$ and $\varphi \in C$, where $L(t) \geq 1$ is a continuous function for $t \geq \sigma > \tau$ and $\alpha(t)$ is a continuous function, possessing a continuous derivative for $t \geq \sigma > \tau$;
- d) Generalized uniformly globally exponentially asymptotically stable in variation (GUEASV) if in c) we take $L(\sigma) \equiv L \geq 1$;
- e) Exponentially asymptotically stable in variation (EASV) if there exists two constants $L \geq 1$ and $\lambda > 0$ such that $\|T(t, \sigma : \varphi)\| \leq L \exp[-\lambda(t - \sigma)]$, for all $t \geq \sigma > \tau$ and $\varphi \in C$.

The (EASV) is also called the global (EASV).

Remark. Definition 2.3 - e) is a particular case of 2.3 - d) for $L(\sigma) \equiv L = \text{const.}$, $\alpha(t) = -\lambda t$, $\lambda > 0$.

Remark. Sometimes ([5], [6]), in Definition 2.3 - c) one asks for $\alpha(t)$ to belong to the class \mathcal{K} , i.e., $\alpha \in C([0, \rho], R^+)$, $\alpha(0) = 0$ and $\alpha(t)$ is strictly increasing.

3 Stability in variation and Lipschitz stability

We now give an extension of Theorem 1 [8] to the case of nonlinear functional equation (2.1) which is, in fact, a consequence of Theorem 5 [7], i.e., of the relation (2.6).

Theorem 3.1.

- (i) If the solution $x = 0$ of equation (2.1) is (USV), then it is (ULS);
- (ii) If the solution $x = 0$ of equation (2.1) is (GUSV), then it is (GULS).

PROOF. (an outline) If $x_t(\sigma, \varphi)$ is a solution of (2.1), then, taking into account the (USV) of the solution $x = 0$, it follows that for each $\alpha > 0$ there exists $M(\alpha) \geq 1$ such that $\|T(t, \sigma : \varphi)\| \leq M(\alpha)$ for all $t \geq \sigma > \tau$ and $\|\varphi\| \leq \alpha$ (Def. 2.3 - a)). Now, formula (2.6) implies that the solution $x = 0$ of (2.1) is (ULS). \square

Remark. We note that the following implications hold: (GUSV) \rightarrow (GULS) \rightarrow (US).

For the linear functional differential equation

$$z'(t) = A(t)z_t, \quad (3.1)$$

where $A(\cdot)$ is a continuous function from (τ, ∞) into the space of continuous linear operators from C into E^n , we have ([7]; [3, pp. 80-82]) the family of continuous linear operators $T(t, \sigma) : C \rightarrow C$, $t \geq \sigma > \tau$, given, for any $\varphi \in C$, by

$$T(t, \sigma)\varphi = z_t(\sigma, \varphi). \quad (3.2)$$

From here one obtains the equivalence (GUSV) \leftrightarrow (GULS) \leftrightarrow (US). The equivalence (GULS) \leftrightarrow (US) follows from Theorem 1 [8], and the implication (US) \rightarrow (GUSV) follows from the fact that for the linear equation (3.1), from [4; p. 163], there exists a linear operator $U(t, \sigma)$ such that the solution of (3.1) through (σ, φ) can be represented by $z_t(\sigma, \varphi) = U(t, \sigma)\varphi$ and there exists a constant $M \geq 1$ such that $\|U(t, \sigma)\| \leq M$ for $t \geq \sigma > \tau$. On the other hand, by Theorem 1 [8], the Fréchet derivative with respect to φ of $x_t(\sigma, \varphi)$ is equals $T(t, \sigma : \varphi)$. Therefore, in the linear case $T(t, \sigma : \varphi) = U(t, \sigma)$ and $\|T(t, \sigma : \varphi)\| = \|U(t, \sigma)\| \leq M$ for $t \geq \sigma > \tau$ and $\varphi \in C$, what implies the (GUSV).

Remark. Theorem 3.1 which is an extension to the nonlinear case of Theorem 1 [8] is also closed to Theorem 4 [6].

Remark. The proofs of Theorem 3.1 can be also obtained by using the method of Liapunov function. Thus, from the (GUSV) of the $x = 0$, by a particular case of Theorem 1 [6], an analogue for the functional differential equations of Theorem 2.2 [1], it follows the existence of a continuous functional $V(t, \psi) \geq 0$, $(t, \psi) \in (\sigma, \infty) \times C$ such that

- i) $\|x_t\| \leq V(t, x_t) \leq M\|x_t\|$, for $x_t \in C$, $t \geq \sigma > \tau$, $M = \text{const.} \geq 1$;
- ii) $V'(t, x_t) \leq 0$, for $t \geq \sigma > \tau$.

For the solution $x_t(\sigma, \varphi)$, from ii), we have $V(t, x_t(\sigma, \varphi)) \leq V(\sigma, \varphi)$, $t \geq \sigma > \tau$. From i) we obtain $\|x_t\| \leq V(t, x_t) \leq V(\sigma, \varphi) \leq M\|\varphi\|$, $t \geq \sigma > \tau$, from where it follows that $\|x_t(\sigma, \varphi)\| \leq M\|\varphi\|$ holds for $t \geq \sigma > \tau$ and $\varphi \in C$. This completes the proof.

Further, we intend to establish relationships between the various defined types of stability to (2.1) (see Section 2) in connection with stability properties of a comparison ordinary differential equation.

Theorem 3.2. Assume that the following hypotheses hold:

- (i) the solution $x = 0$ of equation (2.1) is (GUEASV);
(ii) there exists a continuous function $\omega : [\sigma, \infty) \times R^+ \rightarrow R^+$, $\omega(t, 0) = 0$, such that

$$\|g(t, \varphi)\| \leq \omega(t, \|\varphi\|), \text{ for } t \geq \sigma > \tau, \varphi \in C \quad (3.3)$$

- (iii) the zero solution of the comparison ordinary differential equation

$$u'(t) = \alpha'(t)u + L\omega(t, u) \quad (3.4)$$

is (US).

Then the zero solution of (2.2) is (ULS).

PROOF. From the (GUEASV) of the solution $x = 0$, according to Theorem 1 [6], there exists a Liapunov function $V(t, \varphi)$ such that $\|\varphi\| \leq V(t, \varphi) \leq L\|\varphi\|$, for $t \geq \sigma > \tau$, $\varphi \in C$. At the same time, from (ii) and Theorem 7 [6], we have $\|y_t(\sigma, \varphi)\| \leq u(t; \sigma, u_0)$, $t \geq \sigma > \tau$, $\|\varphi\| \leq u_0$, where $u(t, \sigma, u_0) = V(t, \varphi)$, $u_0 = u(\sigma, \sigma; u_0) \geq V(\sigma, \varphi)$. By the (US) of the zero solution of (3.4), for any $\varepsilon > 0$ there exists $\delta > 0$ such that $|u(t)| \equiv |u(t; \sigma, u_0)| \leq \varepsilon$, for $|u_0| < \delta$. Taking $u_0 = L\|\varphi\|$, where $\varepsilon \leq \|\varphi\| \leq \delta/L$, we find, on the one hand, that $\|y_t(\sigma, \varphi)\| \leq u(t; \sigma, u_0) \leq \varepsilon \leq L\varepsilon \leq L\|\varphi\|$, $t \geq \sigma > \tau$, $\varepsilon \leq \|\varphi\| \leq \delta/L$.

On the other hand, $V(\sigma, \varphi) \leq L\|\varphi\| = u_0$ which is an asking of Theorem 7 [6]. Thus, the proof is complete. \square

Remark. Obviously, Lipschitz stability implies Liapunov stability. Theorem 3.2 give conditions under which Liapunov stability of the zero solution of the comparison equation (3.4) and a certain estimate on $T(t, \sigma; \varphi)$ imply Lipschitz stability of the zero solution of (2.2).

Remark. In particular case where $\alpha(t) = \alpha \equiv \text{const.}$, the (GUEASV) means, in fact, the (GUSV) of the zero solution of (2.1) and (3.4) becomes $u' = L\omega(t, u)$. Thus Theorem 3.2 is related to Theorem 2.2 [1] and Theorem 4 [8].

Remark. If we take $a(t) = -\beta(t)$, $\beta(t) \in \mathcal{K}$, then the (GUEASV) and the (GEASV) of the zero solution of (2.1) will be understood in the sense of papers [5] and [6].

Remark. If $\alpha(t) = -\lambda t$, $\lambda > 0$ and $w(t, u) \equiv \gamma(t)$ where $\gamma(t)$ belongs to the space M , i.e.,

$\int_t^{t+1} \gamma(s)ds < \infty$ for $t \geq \sigma$, then because of $\lim_{t \rightarrow \infty} e^{-\alpha t} \int_{\sigma}^t e^{\alpha s} \gamma(s)ds = 0$, it follows that

$$u(t, \sigma; u_0) = u_0 e^{-\lambda(t-\sigma)} + L e^{-\alpha t} \int_{\sigma}^t e^{\alpha s} \gamma(s)ds \rightarrow 0 \text{ as } t \rightarrow \infty.$$

Therefore, we have $\|x_t(\sigma, \varphi)\| \rightarrow 0$ as $t \rightarrow \infty$.

The above results can also be formulated in the case of the (GEASV). In this case, according to Theorems 1 and 7 [6] the equation (3.4) becomes

$$u'(t) = \alpha'(t)u + L(t)\omega(t, u). \quad (3.5)$$

Thus, we have

Theorem 3.3. *Assume that:*

- (i) *the solution $x = 0$ of equation (2.1) is (GEASV);*
- (ii) *the estimate (3.3) holds;*
- (iii) *the zero solution of equation (3.5) is (EASV).*

Then the zero solution of (2.2) is generalized (GEALS).

PROOF. (an outline) As above, we have $\|y_t(\sigma, \varphi)\| \leq u(t; \sigma, u_0)$, where $u(t; \sigma, u_0)$ is the maximal solution of (3.5) existing for $t \geq \sigma > \tau$, with $u_0 = L(\sigma)\|\varphi\|$, $\varphi \in C$. This time we have $V(\sigma, \varphi) \leq L(\sigma)\|\varphi\| = u_0$. If $\Phi^u(t, \sigma; u_0)$ is the fundamental solution of the variational equation associated to (3.5) and corresponding to the maximal solution $u(t; \sigma, u_0)$, then by the (EASV) of the solution $x = 0$, it follows

$$\begin{aligned} \|y_t(\sigma, \varphi)\| &\leq u(t, \sigma; u_0) \leq \int_0^1 |\Phi^u(t, \sigma; su_0)| ds |u_0| \leq \\ &\leq Ke^{-\lambda(t-\sigma)} |u_0| \leq KL(\sigma)e^{-\lambda(t-\sigma)}\|\varphi\|, \end{aligned}$$

for all $t \geq \sigma > \tau$ and $\varphi \in C$. □

4 Integral stability, Lipschitz stability and stability in variation

In this section we shall give some results concerning the relationships between the two types of integral stability, on the one hand, and their relations with the Lipschitz stability and the stability in variation.

Theorem 4.1. *If the zero solution of (2.1) is (IULS) then it is also (IUS).*

PROOF. Let the zero solution of (2.1) be (IULS). Then, for every $\alpha > 0$ there exists $\beta(\alpha) \geq 1$ such that for any solution $y_t = y_t(\sigma, \varphi)$ of (2.2) with $t \geq \sigma > \tau$, $\|y_t(\sigma, \varphi)\| \leq \beta(\alpha)\|\varphi\|$ holds, provided that $\|\varphi\| \leq \alpha$ and $\int_{\sigma}^{\infty} \sup\{\|g(s, y_s)\|; \|y_s\| \leq \alpha\} ds < \infty$. For $\alpha > 0$ we choose, every time, $\beta_1 \geq \beta(\alpha)$ such that if $\|\varphi\| \leq \alpha$ we have $\|y_t(\sigma, \varphi)\| \leq \beta(\alpha)\|\varphi\| \leq \beta_1$. Therefore, $\|y_t(\sigma, \varphi)\| \leq \beta_1$ whenever $\|\varphi\| \leq \alpha$ and $\int_{\sigma}^{\infty} \sup\{\|g(s, y_s)\|; \|y_s\| \leq \alpha\} ds < \infty$. From here it follows the (IUS) of the zero solution of (2.1). □

Further we shall denote by $a(t, \sigma) \equiv \|T(t, \sigma; \varphi)\|$ the norm of the linear operator $T(t, \sigma; \varphi) \equiv T^{-1}(\sigma, t; \varphi)$ and we shall give an extension to the nonlinear case of Theorem 2.3 [2].

Theorem 4.2. *Assume that the zero solution of (2.1) is (USL) and that $\|T(t, \sigma; \varphi)g(t, \varphi)\| \leq \lambda(t)\|\varphi\|$ for $t \geq \sigma > \tau$, $\varphi \in C$, where $\lambda(t)$ is a nondecreasing positive function such that $\int_{\sigma}^{\infty} \lambda(s) ds < \infty$ and $\int_{\sigma}^{\infty} a(\sigma, s)\lambda(s) ds < \infty$. Then the zero solution of (2.1) is (IULS).*

PROOF. Let $x_t(\sigma, \varphi)$ and $y_t(\sigma, \varphi)$ be solutions of the equations (2.1) and (2.2), respectively, with the same initial function φ . By using the nonlinear variation of constants formula (2.4), the solutions of (2.1) and (2.2) are related by

$$y_t(\sigma, \varphi) = x_t(\sigma, \varphi) + \int_{\sigma}^t T(t, s : y_s(\sigma, \varphi)) Y_0 g(s, y_s(\sigma, \varphi)) ds \quad (4.1)$$

so long as $(\sigma, \varphi) \in (\tau, \infty) \times A$ and t is an open interval in which $y_t(\sigma, \varphi)$ exists.

Since the zero solution of (2.1) is (ULS), for each $\alpha > 0$ there exists $M(\alpha) \geq 1$ such that $\|x_t(\sigma, \varphi)\| \leq M(\alpha)\|\varphi\|$ for all $t \geq \sigma > \tau$ provided that $\|\varphi\| \leq \alpha$. Then, from (4.1) we have

$$\begin{aligned} \|y_t(\sigma, \varphi)\| &\leq M(\alpha)\|\varphi\| + \int_{\sigma}^t \|T(t, s : y_s(\sigma, \varphi)) Y_0 g(s, y_s(\sigma, \varphi))\| ds \leq \\ &\leq M(\alpha)\|\varphi\| + \int_{\sigma}^t \lambda(s) \|y_s(\sigma, \varphi)\| ds, \end{aligned}$$

and an application of the Bellman inequality yields $\|y_t(\sigma, \varphi)\| \leq M(\alpha)K\|\varphi\|$, where $K = \exp \left[\int_{\sigma}^{\infty} \lambda(s) ds \right] < \infty$, provided that $\|\varphi\| \leq \alpha$ and

$$\begin{aligned} &\int_{\sigma}^{\infty} \{ \|g(s, y_s(\sigma, \varphi))\|; \|y_s(\sigma, \varphi)\| \leq \alpha \} ds = \\ &= \int_{\sigma}^{\infty} \sup \{ \|T(\sigma, s : y_s) T(s, \sigma : y_s) g(s, y_s(\sigma, \varphi))\|; \|y_s(\sigma, \varphi)\| \leq \alpha \} ds \leq \\ &\leq \alpha \int_{\sigma}^{\infty} a(\sigma, s) \lambda(s) ds < \infty. \end{aligned}$$

Therefore, the zero solution of (2.1) is (IULS). \square

Corollary 4.1. *Under the above hypotheses, taking into account Theorem 3.1, i), if the zero solution of (2.1) is (USV) it follows that it is (IULS).*

Corollary 4.2. *By the same way one can be shown that, under the above hypotheses, the (GUSV) of the solution $x = 0$ implies its (GIULS).*

Further we intend to establish a relationship between the (GEASV) of the zero solution of (2.1) and its (IUS) using, to this end, the comparison scalar differential equation (3.5).

Theorem 4.3. *Assume that the zero solution of (2.1) is (GEASV) and that (3.3) holds. In addition, we assume that the zero solution of equation*

$$u'(t) = \alpha'(t)u \quad (4.2)$$

is (IUS). Then the zero solution of (2.1) is also (IUS).

PROOF. Since the zero solution of (4.2) is (IUS), for every $\alpha > 0$ there exists $\beta(\alpha) \geq 1$ such that, the inequality $|v(t; \sigma, v_0)| \leq \beta(\alpha)$ for all $t \geq \sigma > \tau$ holds provided that $|v_0| \leq \alpha$ and $\int_{\sigma}^t \sup\{L(s)\omega(s, v(s; \sigma, v_0)); |v(s; \sigma, v_0)| \leq \alpha\} ds < \infty$, for any solution $v(t) \equiv v(t; \sigma, v_0)$ of

$$v'(t) = \alpha'(t)v + L(t)\omega(t, v). \quad (4.3)$$

Taking again into account Theorems 1 and 7 [6] one obtains

$$\|y_t(\sigma, \varphi)\| \leq v(t; \sigma, v_0) \leq \beta(\alpha), \quad v_0 = v(\sigma; \sigma, v_0) \geq V(\sigma, \varphi),$$

for all $t \geq \sigma > \tau$.

Moreover, we have

$$\begin{aligned} \|\varphi\| \leq V(\sigma, \varphi) \leq v_0 \leq \alpha \text{ and } \int_{\sigma}^{\infty} \sup\{\|g(s, y_s)\|; \|y_s\| \leq \alpha\} ds \leq \\ \leq \int_{\sigma}^{\infty} \sup\{L(s)\omega(s, v(s)); |v(s)| \leq \alpha\} ds < \infty. \end{aligned}$$

Thus, $\|y_t(\sigma, \varphi)\| \leq \beta(\alpha)$ provided that $\|\varphi\| \leq \alpha$ and $\int_{\sigma}^{\infty} \sup\{\|g(s, y_s)\|; \|y_s\| \leq \alpha\} ds < \infty$, i.e., the zero solution of (2.1) is (IUS). \square

Remark. A similar result can be formulated in the case of the (IULS).

Remark. The proof of Theorem 4.3 asks for $\omega(t, r)$ to be nondecreasing in the second argument.

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