

## NONDISCRETE INDUCTION PRINCIPLE AND MOSER-ALTMAN EXISTENCE RESULTS

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**Abstract.** A lot of set comparison statements related to the Nondiscrete Induction Principle due to Potra and Ptak [4, ch. 1] are given. As a by-product of these, some Moser–Altman solvability results over scaled normed spaces are discussed.

**Keywords:** Nondiscrete Induction Principle, rate of convergence, scale of normed spaces, existence/solvability result.

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### 1 Introduction

Let  $\{(E_\xi, \|\cdot\|_\xi); 0 \leq \xi \leq 1\}$  be a family of (complex) normed spaces with

(1H1)  $E_0 \supseteq E_\xi \supseteq E_1$ , when  $0 \leq \xi \leq 1$ .

(We shall refer to it as a *scaled family* of normed spaces). Assume that

(1H2)  $\|\cdot\|_\zeta$  is complete (over  $E_\zeta$ ), for a certain index  $\zeta$  in  $]0, 1[$ ,

(1H3)  $\|x\|_\zeta \leq N \|x\|_1^\rho \cdot \|x\|_0^{1-\rho}$ ,  $\forall x \in E_1$ , for some  $\rho$  in  $]0, 1[$ ,  $N = N(\rho) > 0$ .

Further, let  $\{(H_0, \|\cdot\|_0), (H_1, \|\cdot\|_1)\}$  be a pair of (complex) normed spaces with

(1H4)  $H_0 \supseteq H_1$  (in the algebraic sense).

Given the map  $x \mapsto Tx$  from  $E_\zeta$  to  $H_0$ , we may ask under which differential supplementary conditions about our data is the operator equation

$$T(x) = 0 \tag{OE}$$

solvable over certain regions of the ambient space  $E_\zeta$ . A basic result in this direction was established by Moser [3, ch. 1], with the aid of certain sequential approximation methods. An essential refinement of the argument developed there was performed by Ptak [5]. Namely, for the given  $\Omega > 0$ , put

(1D1)  $D_1 = \{x \in E_1; \|x\|_\zeta \leq \Omega\}$ ,  $D_1^\zeta$  is the closure of  $D_1$  in  $(E_\zeta, \|\cdot\|_\zeta)$ .

The basic regularity conditions to be accepted by this map are

(1H5)  $T(D_1) \subseteq H_1$  and  $T$  is continuous on  $D_1^\zeta$

(1H6)  $\|Tx\|_1 \leq M \max\{S, \|x\|_1\}$ ,  $x \in D_1$ , for some  $M > 0$ ,  $S \geq 1$ .

In this case, the announced result may be phrased as

**Theorem 1.1.** *Suppose that, for a certain map  $L : D_1 \times E_1 \rightarrow H_1$ , a number  $\beta$  in  $]0, 1[$  with*

(1H7)  $\|T(x+u) - T(x) - L(x, u)\|_0 \leq M \|u\|_1^\beta \|u\|_0^{2-\beta}$ , whenever  $x, x+u \in D_1$

as well as a couple  $(\lambda, \mu)$  of strictly positive numbers with

(1H8)  $1 < \frac{\mu+1}{\mu-\lambda} < \min \left\{ 2 - \beta \frac{\lambda + (\lambda+1)(\mu+1)}{\lambda(\mu+\beta)}, \frac{\lambda(1-\rho)}{\rho} \right\}$

it is the case that: for each  $x \in D_1$  and  $w \in T(D_1)$  with

(1H9)  $\|w\|_0 \leq m^{-\lambda}$ , where  $m = \max\{\frac{1}{M} \|w\|_1, \|x\|_1, S\}$

and for each  $Q > 1$ , there exists  $u = u(x, w, Q)$  in  $E_1$  in such a way that

$$\|L(x, u) - w\|_0 \leq MmQ^{-\mu}, \quad \|u\|_1 \leq MmQ, \quad \|u\|_0 \leq M \|L(x, u)\|_0 \quad (1.1)$$

Then, one may find a sufficiently small  $\varepsilon > 0$  with the property: the operator equation (OE) has at least a solution in  $D_1^\varepsilon$  provided  $\|T(0)\|_0 < \varepsilon$ .

The basic tool of these investigations is the Nondiscrete Induction Principle obtained by the quoted author (see also Potra and Ptak [4, ch. 1]). It is the main aim of this exposition to show that further technical extensions of Theorem 1.1 are possible; we refer to Section 3 for details. The core of this device is a *set comparison statement* (related to the Nondiscrete Induction Principle) developed in Section 2, via *i-normal* functions. Further, Section 4 is devoted to the question of the conclusions in Section 2 being retainable beyond the *i-normality* setting. Loosely speaking, a positive answer to this is obtainable if the *ordinary induction* is substituted by the *transfinite* one. And, in Section 5, the obtained facts are applied to a class of *solvability results* considered by Altman [1]. Some other aspects will be discussed elsewhere.

## 2 Set comparison statements

Let in the following  $J$  stand for the (real) interval  $]0, \delta[$ , where  $0 < \delta \leq \infty$ . Further, let  $(X_1, d_1), (X, d)$  be a couple of metric spaces with

(2H1)  $X_1 \subseteq X$  and  $d$  is complete over  $X$ .

Finally, let  $(Y, \|\cdot\|)$  be a Banach space. Take a multivalued function  $t \mapsto F(t)$  from  $J$  to  $X_1 \times Y$  and put

(2D1)  $\left\{ \begin{array}{l} F(0) = \{(x, y) \in X \times Y; \text{ there exists a sequence } ((t_n, x_n, y_n)) \\ \text{ in } F \text{ with } t_n \rightarrow 0, x_n \xrightarrow{d} x, y_n \rightarrow y\}. \end{array} \right.$

This set is well defined; but it may be sometimes empty. For a number of both practical and theoretical reasons, it would be useful to determine under which requirements about our data is this removable. Clearly, a condition like

(2H2)  $F$  is proper (in the sense:  $\text{dom}(F) \neq \emptyset$ )

is indispensable for this. But it is not sufficient; because, another indispensable requirement like

(2H3)  $\text{dom}(F)$  is 0-accessible ( $0 \in \text{cl dom}(F)$ )

is not deductible from the above one. As we shall see below, this last condition is obtainable from the working hypotheses under consideration. To state them, we need some conventions. For each couple  $\{A, B\}$  of nonempty parts in  $X_1 \times Y$  and each  $\tau_1 = (\xi_1, \xi, \eta)$  in  $R_+^3$ , let us introduce the relation

(2D2)  $\left\{ \begin{array}{l} \Delta_1(A, B) \leq \tau_1 \text{ iff for each } (x, y) \in A \text{ there exists} \\ (x^*, y^*) \in B \text{ with } d_1(x, x^*) \leq \xi_1, d(x, x^*) \leq \xi, \|y - y^*\| \leq \eta. \end{array} \right.$

Likewise, for each couple  $\{A, B\}$  as before and each  $\tau = (\xi, \eta)$  in  $R_+^2$ , define the "truncated" relation

(2D3)  $\left\{ \begin{array}{l} \Delta(A, B) \leq \tau \text{ iff for each } (x, y) \in A \text{ there exists} \\ (x^*, y^*) \in B \text{ with } d(x, x^*) \leq \xi, \|y - y^*\| \leq \eta. \end{array} \right.$

The following implication is holding

$$\Delta_1(A, B) \leq \tau_1 \implies \Delta(A, B) \leq \tau \tag{2.1}$$

as well as the transitivity formula

$$\left\{ \begin{array}{l} \Delta_1(A, B) \leq \tau_1, \Delta_1(B, C) \leq \sigma_1 \implies \Delta_1(A, C) \leq \tau_1 + \sigma_1 \\ (\Delta(A, B) \leq \tau, \Delta(B, C) \leq \sigma \implies \Delta(A, C) \leq \tau + \sigma). \end{array} \right. \tag{2.2}$$

Since the verification is immediate, we do not give details.

Further, let  $\varphi$  be some function from  $J$  to itself. Call this object, *iteratively normal* (abbreviated *i-normal*) provided

(2D4)  $\varphi^n(t) \rightarrow 0$  as  $n \rightarrow \infty$ , for each  $t \in J$ .

(Here,  $\varphi^n$  stands for the  $n$ -th iterate of  $\varphi$ ). Note that, if in addition,

(2H4)  $\varphi$  is increasing (over  $J$ )

then, we necessarily have

$$\varphi(t) < t, \quad t \in J \quad (\text{i.e., } \varphi \text{ is regressive}); \tag{2.3}$$

see, for instance, Matkowski [2]. But, in the absence of (2H4), the conclusion above may be false, as the example below shows (with  $J = ]0, 1[$ )

(2D5)  $\left\{ \begin{array}{l} \varphi(1/(1 + 2^{k+1})) = 1/(1 + 2^k), \quad \text{for } k = 0, 1, \dots, \\ \varphi(t) = (1/2)t, \quad \text{in the remaining cases.} \end{array} \right.$

Finally, for each function  $\chi : J \rightarrow R$  and each  $r$  in  $J^* = J \cup \{0\}$  put

$$(2D6) \quad \chi(r_+) = \liminf_{t \rightarrow r_+} \chi(t), \quad \chi(r^+) = \limsup_{t \rightarrow r^+} \chi(t).$$

Note that the case of both these expressions being  $\mp\infty$  cannot be avoided. Likewise, for each function  $t \vdash \Phi_1(t) = (\chi_1(t), \chi(t), \psi(t))$  from  $J$  to  $R^3$ , and each  $r$  as before, denote

$$(2D7) \quad \Phi_1(r_+) = (\chi_1(r_+), \chi(r_+), \psi(r_+)), \quad \Phi_1(r^+) = (\chi_1(r^+), \chi(r^+), \psi(r^+)),$$

whenever the right member exists (as an element of  $R^3$ ). The corresponding form of this convention for the "truncated" function  $t \vdash \Phi(t) = (\chi(t), \psi(t))$  from  $J$  to  $R^2$  is introduced in an evident manner.

Throughout this section,  $t \vdash F(t)$  is a proper multivalued function from  $J$  to  $X_1 \times Y$ . The following set comparison result is our starting point.

**Theorem 2.1.** *Suppose that, an  $i$ -normal function  $\varphi : J \rightarrow J$  and a function  $t \vdash \Phi_1(t) = (\chi_1(t), \chi(t), \psi(t))$  from  $J$  to  $R \times R_+^2$  exist with*

$$(2H5) \quad \Delta_1(F(t), F(\varphi(t))) \leq \Phi_1(t) - \Phi_1(\varphi(t)), \quad t \in \text{dom}(F).$$

*Then,  $\text{dom}(F)$  is 0-accessible,  $F(0)$  is nonempty and*

$$\Delta(F(r), F(0)) \leq \Phi(r) - \Phi(0_+), \quad r \in \text{dom}(F). \quad (2.4)$$

**PROOF.** Let  $r \in \text{dom}(F)$  and  $(x_0, y_0) \in F(r)$  be arbitrary fixed. By (2H5), we may construct a sequence  $((t_n, x_n, y_n))$  in  $F$  with

$$t_0 = r \quad \text{and} \quad t_n \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty \quad (2.5)$$

$$\begin{cases} d_1(x_n, x_m) \leq \chi_1(t_n) - \chi_1(t_m) \\ d(x_n, x_m) \leq \chi(t_n) - \chi(t_m) \\ \|y_n - y_m\| \leq \psi(t_n) - \psi(t_m) \end{cases} \quad \text{for } n \leq m. \quad (2.6)$$

This firstly shows that  $\text{dom}(F)$  is 0-accessible. Secondly, as  $(\chi(t_n))$ ,  $(\psi(t_n))$  are descending and bounded from below (in  $\mathbb{R}$ ), these are Cauchy sequences. Hence, so are  $(x_n)$  (in  $X$ ) and  $(y_n)$  (in  $Y$ ); wherefrom (by completeness)

$$x_n \xrightarrow{d} x_\infty, \quad y_n \rightarrow y_\infty \quad \text{as } n \rightarrow \infty, \quad \text{for some } x_\infty \in X, \quad y_\infty \in Y.$$

This, coupled with (2.5) shows that  $(x_\infty, y_\infty)$  is an element of  $F(0)$ . Finally, passing to limit as  $m \rightarrow \infty$  in (2.6) one has

$$d(x_n, x_\infty) \leq \chi(t_n) - \lim_m \chi(t_m), \quad \|y_n - y_\infty\| \leq \psi(t_n) - \lim_m \psi(t_m), \quad \text{for all } n. \quad (2.7)$$

Adding the inequality  $\chi(0_+) \leq \lim_m \chi(t_m)$ ,  $\psi(0_+) \leq \lim_m \psi(t_m)$  yields (by the arbitrariness of  $(x_0, y_0)$  in  $F(r)$ ) the conclusion (2.4) we need.  $\square$

The following version of the obtained statement is particularly useful for applications. Take a certain function  $t \vdash \pi(t)$  from  $J$  to  $R_+$ . We say that the  $i$ -normal function  $\varphi : J \rightarrow J$  is a *rate of convergence* (modulo  $\pi$ ) in case

$$(2D8) \quad K[\varphi, \pi](t) = \sum_{n=0}^{\infty} \pi(\varphi^n(t)) < \infty, \quad t \in J.$$

Likewise, given the function  $t \vdash \Pi(t) = (\pi_X(t), \pi_Y(t))$  from  $J$  to  $R_+^2$ , we say that  $\varphi$  is a rate of convergence (modulo  $\Pi$ ) when

$$(2D9) \quad K[\varphi, \Pi](t) = (K[\varphi, \pi_X](t), K[\varphi, \pi_Y](t)) \text{ exists (in } R_+^2), \forall t \in J.$$

**Theorem 2.2.** *Suppose that, for a certain function  $\chi_1 : J \rightarrow R$ , some function  $t \vdash \Pi(t) = (\pi_X(t), \pi_Y(t))$  from  $J$  to  $R_+^2$  and a certain  $i$ -normal function  $\varphi : J \rightarrow J$  with*

(2H6)  $\varphi$  is a rate of convergence (modulo  $\Pi$ )

one has

$$(2H7) \quad \Delta_1(F(t), F(\varphi(t))) \leq (\chi_1(t) - \chi_1(\varphi(t)), \Pi(t)), \quad t \in \text{dom}(F).$$

Then,  $\text{dom}(F)$  is 0-accessible,  $F(0)$  is nonempty and

$$\Delta(F(r), F(0)) \leq K[\varphi, \Pi](r), \quad r \in \text{dom}(F). \quad (2.8)$$

It is not hard to see that this statement is logically reducible to Theorem 2.2 above. In fact, assume that conditions of Theorem 2.2 are accepted. Then, with  $\Phi = K[\varphi, \Pi]$ , it is clear that (2H7) implies (2H5), in view of

$$\Phi(t) = \pi(t) + \Phi(\varphi(t)), \quad \text{for all } t \in J. \quad (2.9)$$

Hence, Theorem 2.1 applies to these data; and as such, (2.4) is retainable with  $\Phi = K[\varphi, \Pi]$ . This finally combined with  $K[\varphi, \Pi](0_+) \geq 0$ , yields (2.8); hence the claim. Moreover, the converse inclusion is also true; i.e., Theorem 2.1 is logically reducible to the above statement. Indeed, assume that conditions of Theorem 2.1 are in force. (Note that, as a direct consequence of this,  $\text{dom}(F)$  is invariant under  $\varphi$ ). Let the function  $t \vdash \Pi(t)$  from  $J$  to  $R_+^2$  be introduced as

$$(2D10) \quad \left| \begin{array}{l} \Pi(t) = \Phi(t) - \Phi(\varphi(t)), \text{ if } t \in \text{dom}(F) \\ = 0, \text{ otherwise.} \end{array} \right.$$

We show that  $\varphi$  is a rate of convergence (modulo  $\Pi$ ) over  $J$ . To verify this, let the number  $t$  in  $J$  be arbitrary fixed. If the orbit  $\{\varphi^n(t); n = 0, 1, \dots\}$  intersects  $\text{dom}(F)$ , denote by  $k = k(t)$  the minimal rank with  $\varphi^k(t) \in \text{dom}(F)$ . Since, by the admitted condition, the sequence  $(\Phi(\varphi^n(t)))_{n \geq k}$  decreases, one has

$$K[\varphi, \Pi](t) = \Phi(\varphi^k(t)) - \lim_n \Phi(\varphi^n(t)) \leq \Phi(\varphi^k(t)) - \Phi(0_+). \quad (2.10)$$

And, if the orbit  $\{\varphi^n(t); n = 0, 1, \dots\}$  does not intersect  $\text{dom}(F)$ , one has  $K[\varphi, \Pi](t) = 0$ ; hence the assertion. As a consequence, Theorem 2.2 applies (to the precised context); wherefrom, its conclusion (2.8) is retainable. This finally combined with (2.10) yields (2.4); hence the claim. Summing up, we proved

**Proposition 2.1.** *Under the precised conventions, we have*

$$\text{Theorem 2.1} \iff \text{Theorem 2.2.} \quad (2.11)$$

*That is: these two statements are logically equivalent to each other.*

An interesting particular case of Theorem 2.2 corresponds to the choice

$$(2D11) \quad \pi = i \quad (= \text{the identity function over } J).$$

The corresponding to (2H6) property will be referred to as  $\varphi$  being a *standard rate of convergence*; i.e.,

$$(2D12) \quad K[\varphi](t) = \sum_{n=0}^{\infty} \varphi^n(t) < \infty, \quad t \in J.$$

We thus have (cf. Potra and Ptak [4, ch. 1, Sect. 7])

**Theorem 2.3.** *Suppose that, for a certain function  $t \mapsto \chi_1(t)$  from  $J$  to  $R$  and some standard rate of convergence  $\varphi : J \rightarrow J$ , it is the case that*

$$(2H8) \quad \Delta_1(F(t), F(\varphi(t))) \leq (\chi_1(t) - \chi_1(\varphi(t)), t, t), \quad t \in \text{dom}(F).$$

*Then,  $\text{dom}(F)$  is 0-accessible,  $F(0)$  is nonempty and*

$$\Delta(F(r), F(0)) \leq (K[\varphi](r), K[\varphi](r)), \quad r \in \text{dom}(F). \quad (2.12)$$

We close this section with a few remarks about the basic regularity condition (2H6). Let  $J$  stand for the open interval  $]0, 1[$ .

**Proposition 2.2.** *The function*

$$(2D13) \quad \varphi(t) = t^\alpha, \quad t \in J, \quad \text{for some } \alpha > 1$$

*is a rate of convergence with respect to each function of the form*

$$(2D14) \quad \pi(t) = t^\gamma, \quad t \in J, \quad \text{for some } \gamma > 0.$$

*Moreover, the evaluation below holds*

$$K[\varphi, \pi](t) \leq \frac{t^\gamma}{1 - t^{(\alpha-1)\gamma}}, \quad t \in J. \quad (2.13)$$

PROOF. Clearly,  $\varphi$  is  $i$ -normal and regressive. Moreover, by the Bernoulli inequality, one has

$$\pi(\varphi^n(t)) = t^{\gamma\alpha^n} \leq t^{\gamma(1+n(\alpha-1))}, \quad t \in J, \quad \text{for } n = 0, 1, \dots$$

So, by simply summing these inequalities, (2.13) follows.  $\square$

In particular, when  $\gamma = 1$ , this result says that each function  $\varphi$  like in (2D13) is a standard rate of convergence over  $J$ ; see also Ptak [5]. Some other examples may be found in Potra and Ptak [op.cit., ch. 1, Sect. 4].

### 3 Moser existence theorem

Let  $E_1, E_0$  and  $E$  be a triple of (complex) vector spaces with  $E_1 \subseteq E \subseteq E_0$ , equipped with the norms  $\|\cdot\|_1, \|\cdot\|_0$  and  $\|\cdot\|$ , respectively. Assume

(3H1)  $\|\cdot\|$  is (Banach) complete (over  $E$ );

as well as (for some exponent  $\rho$  in  $]0, 1[$  and some constant  $N = N(\rho) > 0$ )

(3H2)  $\|x\| \leq N \|x\|_1^\rho \cdot \|x\|_0^{1-\rho}$ , for all  $x \in E_1$ .

Further, let  $H$  be a (complex) vector space, endowed with a norm  $\|\cdot\|$  fulfilling (3H1) (with  $E = H$ ). Finally, take a map  $x \mapsto Tx$  from  $E$  to  $H$ . As in Section 1, we intend to solve the operator equation (OE) over certain regions of  $E$ . Precisely, if we denote (for a certain  $\Omega > 0$ )

(3D1)  $D_1 = \{x \in E_1; \|x\|_0 < \Omega, \|x\| < \Omega\}$

the regions in question are of the form

(3D2)  $D_1^E$  is the closure of  $D_1$  in  $(E, \|\cdot\|)$ .

To accomplish this device, we have to impose some general regularity conditions upon our data. The first one involves the *graph* of  $T$ :

(3H3)  $\text{gr}(T)$  is closed in  $E \times H$  (endowed the product norm).

The second one involves the *differentiability* properties of  $T$

(3H4)  $\left\{ \begin{array}{l} \text{the Gateaux derivative } T'(x)(u) = \lim_{\tau \rightarrow 0^+} (1/\tau)(T(x + \tau u) - T(x)) \\ \text{exists (as an element of } H) \text{ for each } x \in D_1, u \in E_1. \end{array} \right.$

(Note that this does not mean that  $u \mapsto T'(x)(u)$  is a linear continuous operator from  $E$  to  $H$ ). Likewise, we need some specific regularity conditions for our data. The formulation of these follows essentially the developments in Moser [3, ch. 1]. Namely, the first one is an *approximation* between the differential map  $u \mapsto T'(x)(u)$  and the difference operator  $u \mapsto (T(x + u) - T(x))$  (involved in its definition): there exist some exponent  $\beta$  in  $]0, 1[$  and some  $P > 0$  such that

(3H5)  $\|T(x + u) - T(x) - T'(x)(u)\| \leq P \|u\|_1^\beta \cdot \|u\|_0^{2-\beta}$ , whenever  $x, x + u \in D_1$ .

The second condition is a kind of *surjectivity* for the differential map  $u \mapsto T'(x)(u)$ : namely (for some triple of exponents  $\lambda, \mu, \nu > 0$  and some  $M > 0$ )

(3H6)  $\left\{ \begin{array}{l} \text{for each } x \in D_1 \text{ and } \Gamma > 1 \text{ satisfying } \|x\|_1 \leq \Gamma, \|Tx\| \leq \Gamma^{-\lambda} \\ \text{and each } \Theta > 1, \text{ there exists } u = u(x, \Gamma, \Theta) \text{ in } E_1 \text{ with} \\ \|T'(x)(u) + Tx\| \leq M\Gamma\Theta^{-\mu}, \|u\|_1 \leq M\Gamma^\nu\Theta, \|u\|_0 \leq M\Gamma\Theta^{-\mu} + \Gamma^{-\lambda}. \end{array} \right.$

The final assumption to be imposed is the one relating the exponents above:

(3H7)  $1 < \frac{\nu\mu + 1}{\mu - \lambda} < \min \left\{ \mu \left( \frac{1}{\rho} - 1 - \frac{\nu}{\lambda} - \frac{1}{\lambda\mu} \right), \frac{\beta\mu}{\beta + \mu} \left( \frac{2}{\beta} - 1 - \frac{\nu}{\lambda} - \frac{1}{\lambda\mu} \right) \right\}$ .

The following answer to the posed question is now available.

**Theorem 3.1.** *Let the conditions above be in force. Then, there exists some  $\varepsilon$  in  $]0, 1[$  with the property:*

$$\text{the operator equation (OE) has a solution in } D_1^E \text{ whenever } \|T(0)\| < \varepsilon. \quad (3.1)$$

PROOF. We intend to show that a certain version of Theorem 2.1 is applicable to our data. Precisely, put  $J = ]0, 1[$ ; and, with  $\alpha > 1, \gamma \in ]0, 1[$  arbitrary for the moment, let the functions  $\varphi, \pi$  be introduced as in Proposition 2.2. Further, put

$$(3D3) \quad \chi_0 = \chi = K[\varphi, \pi]; \quad \chi_1(t) = -t^{-1/\lambda}, \quad \psi(t) = 3t, \quad t \in J.$$

Also, with  $c$  in  $]0, 1[$  arbitrary for the moment, define a multivalued function  $t \mapsto F(t)$  from  $J_c = ]0, c[$  to  $E_1 \times H$  according to

$$(3D4) \quad \left\{ \begin{array}{l} F(t) = \{(x, y) \in \text{gr}(T); \quad x \in D_1, \quad \|x\|_1 \leq -\chi_1(t), \\ \|x\|_0 \leq \Omega - \chi_0(t), \quad \|x\| \leq \Omega - \chi(t), \quad \|y\| \leq t\}, \quad t \in J_c. \end{array} \right.$$

We claim that, one may determine a couple of exponents  $(\alpha, \gamma)$  and a number  $c \in ]0, 1[$  in such a way that conditions of the quoted statement be fulfilled. The verification of this is composed of three distinct steps.

(i) We firstly determine the pair of exponents  $(\alpha, \gamma)$  in such a way that the apriori implication be retainable:

$$\left\{ \begin{array}{l} \text{for each } (t, x, y) \text{ in } F \text{ there exists } (t', x', y') \text{ in } F \text{ with } t' = \varphi(t), \\ \|x - x'\|_1 \leq \chi_1(t) - \chi_1(t'), \\ \|x - x'\|_0 \leq \pi(t) (= \chi_0(t) - \chi_0(t')), \\ \|x - x'\| \leq \pi(t) (= \chi(t) - \chi(t')) \text{ and} \\ \|y - y'\| \leq \psi(t) - \psi(t'). \end{array} \right. \quad (3.2)$$

This, essentially, consists in the following. Let the number  $t$  in  $\text{dom}(F)$  be given and take some  $(x, y)$  in  $F(t)$ ; namely

$$(3H8) \quad \left\{ \begin{array}{l} (x, y) \in \text{gr}(T), \quad x \in D_1 \text{ and} \\ \|x\|_1 \leq -\chi_1(t), \quad \|x\|_0 \leq \Omega - \chi_0(t), \quad \|x\| \leq \Omega - \chi(t), \quad \|y\| \leq t. \end{array} \right.$$

Note that, with  $\Gamma = t^{-1/\lambda} (> 1)$ , one has  $\|x\|_1 \leq \Gamma, \|Tx\| = \|y\| \leq \Gamma^{-\lambda}$ . So, by (3H6), it follows that, for the arbitrary fixed  $\Theta > 1$  there must be some point  $u = u(x, \Gamma, \Theta)$  in  $E_1$  with the properties stated there. Denote

$$(3D5) \quad x' = x + u, \quad y' = Tx' \quad (\text{where } u \text{ is as before}).$$

We show that, under a suitable choice of  $\Theta$ , all requirements in (3.2) are satisfied.

(ia) The first couple of conditions involving  $x'$ , namely

$$\|x'\|_1 \leq -\chi_1(t'), \quad \|x - x'\|_1 \leq \chi_1(t) - \chi_1(t')$$

is verified as soon as the second half of it is true:  $t^{-1/\lambda} + \|u\|_1 \leq t^{-\alpha/\lambda}$ . But

$$t^{-1/\lambda} + \|u\|_1 \leq t^{-1/\lambda} + Mt^{-\nu/\lambda}\Theta \quad \text{cf. (3H6)}. \quad (3.3)$$

So, a sufficient couple of conditions for the expected relation to hold is

$$(3H9) \quad t^{-1/\lambda} < \frac{1}{2}t^{-\alpha/\lambda} \quad (\text{hence } t^{(\alpha-1)/\lambda} < \frac{1}{2})$$

$$(3H10) \quad Mt^{-\nu/\lambda}\Theta < \frac{1}{2}t^{-\alpha/\lambda} \quad (\text{hence } \Theta < \frac{1}{2M}t^{(\nu-\alpha)/\lambda}).$$

(ib) The second couple of conditions involving  $x'$ , namely

$$\|x'\|_0 \leq \Omega - \chi_0(t') = \Omega - \chi_0(t) + \pi(t), \quad \|x - x'\|_0 \leq \pi(t)$$

is fulfilled as soon as its second half is true:  $\|u\|_0 \leq t^\gamma$ . But, evidently,

$$\|u\|_0 \leq Mt^{-1/\lambda}\Theta^{-\mu} + t \quad (\text{in view of (3H6)}). \quad (3.4)$$

Now, an appropriate condition to be imposed further is

$$(3H11) \quad Mt^{-1/\lambda}\Theta^{-\mu} < \frac{1}{2}t^\alpha \quad (\text{hence } \Theta > 2M^{1/\mu}t^{-\alpha/\mu-1/\lambda\mu}).$$

In this case, (3.4) yields (via  $\alpha > 1$ )

$$\|u\|_0 \leq \frac{1}{2}t^\alpha + t \leq \frac{1}{2}t + t = \frac{3}{2}t;$$

so, a sufficient condition for the expected relation to hold is

$$(3H12) \quad \frac{3}{2}t < t^\gamma \quad (\text{hence } t^{1-\gamma} < \frac{2}{3}).$$

(ic) The third couple of conditions involving  $x'$ , namely

$$\|x'\| \leq \Omega - \chi(t') = \Omega - \chi(t) + \pi(t), \quad \|x - x'\| \leq \pi(t)$$

is fulfilled as soon as its second half is true:  $\|u\| \leq t^\gamma$ . But, evidently, (3D2), (3H6) and (3H11) give

$$\|u\| \leq N \|u\|_1^\rho \|u\|_0^{1-\rho} \leq N(Mt^{-\nu/\lambda}\Theta)^\rho \left(\frac{3}{2}t\right)^{1-\rho} = NM^\rho \left(\frac{3}{2}\right)^{1-\rho} t^{1-\rho-\nu\rho/\lambda}\Theta^\rho.$$

So, a sufficient condition for the precised relation to hold is

$$(3H13) \quad \frac{3N}{2} \left(\frac{2M}{3}\right)^\rho t^{1-\rho-\nu\rho/\lambda}\Theta^\rho < t^\gamma \quad (\text{hence } \Theta < \frac{3}{2M} \left(\frac{2}{3N}\right)^{1/\rho} t^{\gamma/\rho+1+\nu/\lambda-1/\rho}).$$

Note that, in combination with the developments in (ib), one has

$$\|x'\|_0 < \Omega, \quad \|x'\| < \Omega \quad (\text{hence } x' \in D_1). \quad (3.5)$$

(id) Finally, the couple of conditions related to  $y' = Tx'$  may be written as

$$\|y'\| \leq t', \quad \|y - y'\| \leq \psi(t) - \psi(t').$$

It may be treated according to the lines below. Clearly,

$$Tx' = (T(x+u) - T(x) - T'(x)(u)) + (T'(x)(u) + T(x))$$

So, by (3H5), (3H6), (3H11) and the remark (3.5) above,

$$\begin{aligned} \|y'\| &\leq P \|u\|_1^\beta \|u\|_0^{2-\beta} + Mt^{-1/\lambda} \Theta^{-\mu} \leq \\ &\leq PM^\beta \left(\frac{3}{2}\right)^{2-\beta} (t^{-\nu/\lambda} \Theta)^\beta t^{2-\beta} + \frac{1}{2} t^\alpha \leq \\ &\leq PM^\beta \left(\frac{3}{2}\right)^{2-\beta} t^{2-\beta-\nu\beta/\lambda} \Theta^\beta + \frac{1}{2} t^\alpha. \end{aligned} \quad (3.6)$$

It follows that, a sufficient condition for the first relation of this couple is

$$(3H14) \quad \frac{9P}{4} \left(\frac{2M}{3}\right)^\beta t^{2-\beta-\nu\beta/\lambda} \Theta^\beta < \frac{1}{2} t^\alpha \quad (\text{hence } \Theta < \frac{3}{2M} \left(\frac{2}{9P}\right)^{1/\beta} t^{\alpha/\beta+1+\nu/\lambda-2/\beta}).$$

Moreover, the second relation of the couple takes place whenever

$$(3H15) \quad t^{\alpha-1} < \frac{1}{2}.$$

This follows at once from the chain of inequalities

$$\|y - y'\| \leq t + t^\alpha = \frac{1 + t^{\alpha-1}}{1 - t^{\alpha-1}} (t - t^\alpha) \leq 3(t - t^\alpha) = \psi(t) - \psi(t').$$

Summing up, the sufficient conditions under which (3.2) be retainable are those described by (3H9)–(3H15). Let us analyze now the effectiveness of the ones involving the parameter  $\Theta$ ; namely, (3H10), (3H11), (3H13) and (3H14). Firstly, by (3H11), a sufficient condition for  $\Theta > 1$  is

$$(3H16) \quad t^{\alpha+1/\lambda} < 2M.$$

Secondly, the existence of a solution ( $\Theta$ ) for the inequalities above is a direct consequence of the conditions below

$$(3H17) \quad \begin{cases} (2M)^{1/\mu} t^{-\alpha/\mu-1/\lambda\mu} < \frac{1}{2M} t^{\nu/\lambda-\alpha/\lambda} \\ (2M)^{1/\mu} t^{-\alpha/\mu-1/\lambda\mu} < \frac{3}{2M} \left(\frac{2}{3N}\right)^{1/\rho} t^{\gamma/\rho+1+\nu/\lambda-1/\rho} \\ (2M)^{1/\mu} t^{-\alpha/\mu-1/\lambda\mu} < \frac{3}{2M} \left(\frac{2}{9P}\right)^{1/\beta} t^{\alpha/\beta+1+\nu/\lambda-2/\beta}. \end{cases}$$

or, equivalently,

$$(3H18) \quad \begin{cases} t^{\alpha/\lambda-\alpha/\mu-\nu/\lambda-1/\lambda\mu} < \left(\frac{1}{2M}\right)^{1-1/\mu} \\ t^{1/\rho-\alpha/\mu-\gamma/\rho-1-\nu/\lambda-1/\lambda\mu} < 3 \left(\frac{1}{2M}\right)^{1-1/\mu} \left(\frac{2}{3N}\right)^{1/\rho} \\ t^{2/\beta-\alpha/\beta-\alpha/\mu-1-\nu/\lambda-1/\lambda\mu} < 3 \left(\frac{1}{2M}\right)^{1-1/\mu} \left(\frac{2}{9P}\right)^{1/\beta}. \end{cases}$$

The existence of a solution for such inequalities requires the exponents in the left members be strictly positive; that is, conditions of the form below

$$(3H19) \quad \alpha \left( \frac{1}{\lambda} - \frac{1}{\mu} \right) > \frac{\nu}{\lambda} + \frac{1}{\lambda\mu}, \quad \frac{\alpha}{\mu} + \frac{\gamma}{\rho} < \frac{1}{\rho} - 1 - \frac{\nu}{\lambda} - \frac{1}{\lambda\mu}, \quad \alpha \left( \frac{1}{\beta} + \frac{1}{\mu} \right) < \frac{2}{\beta} - 1 - \frac{\nu}{\lambda} - \frac{1}{\lambda\mu}.$$

A sufficient condition for these inequalities to produce one solution  $(\alpha, \gamma)$  as desired is that their associate "limit" relations (for  $\gamma = 0$ )

$$(3H20) \quad \alpha \left( \frac{1}{\lambda} - \frac{1}{\mu} \right) > \frac{\nu}{\lambda} + \frac{1}{\lambda\mu}, \quad \frac{\alpha}{\mu} < \frac{1}{\rho} - 1 - \frac{\nu}{\lambda} - \frac{1}{\lambda\mu}, \quad \alpha \left( \frac{1}{\beta} + \frac{1}{\mu} \right) < \frac{2}{\beta} - 1 - \frac{\nu}{\lambda} - \frac{1}{\lambda\mu}$$

have a solution  $\alpha > 1$ . To discuss it, note that, by (3H7) (the first half) one gets  $\mu > \lambda$ ; and then,  $\alpha > \frac{\nu\mu + 1}{\mu - \lambda}$  (by the first part of (3H20)).

So, combining again with (3H7) (the first half) yields  $\alpha > 1$ . But then, the existence of a solution  $(\alpha)$  to the inequalities in (3H20) is clear, as soon as

$$\frac{\nu\mu + 1}{\mu - \lambda} \cdot \frac{1}{\mu} < \frac{1}{\rho} - 1 - \frac{\nu}{\lambda} - \frac{1}{\lambda\mu}, \quad \frac{\nu\mu}{\mu - \lambda} \left( \frac{1}{\beta} + \frac{1}{\mu} \right) < \frac{2}{\beta} - 1 - \frac{\nu}{\lambda} - \frac{1}{\lambda\mu}.$$

Since these are just (3H7) (the second half) we are done.

(ii) It remains now to discuss the effectiveness over  $J_c$  of the inequalities in the variable  $(t)$ . Put

$$(3D6) \quad \left\{ \begin{array}{l} c \text{ is the supremum of all points } t \in J \text{ fulfilling} \\ \text{the requirements (3H9), (3H12), (3H15), (3H16) and (3H18).} \end{array} \right.$$

It is clear, by the above discussion, that under this choice, the apriori implication (3.2) is retainable.

(iii) Finally, we intend to assure the properness of the multivalued function  $F$ , in the sense

$$(0, T(0)) \in F(\eta), \quad \text{for some } \eta \text{ in } J_c. \quad (3.7)$$

To this end, put

$$(3D7) \quad \varepsilon = \sup\{t \in J_c; t^\gamma / (1 - t^{(n-1)\gamma}) \leq \Omega\}.$$

In view of Proposition 2.2 above

$$\chi_0(t) = \chi(t) < \Omega, \quad \text{when } 0 < \eta < \varepsilon. \quad (3.8)$$

So, if  $\|T(0)\| < \varepsilon$ , it is clear that  $(0, T(0)) \in F(\eta)$ , whenever  $\|T(0)\| \leq \eta < \varepsilon$ ; and, as such,  $F$  is proper.

As a consequence of all these developments, Theorem 2.1 is applicable to our data. This, combined with

$$Tx = 0, \quad \text{for each } x \in F(0), \quad (3.9)$$

yields the conclusion we need.  $\square$

In particular, when  $\nu = 1$ , the obtained statement is comparable with the one (subsumed to Theorem 1.1) due to Ptak [5]. But, as above said, his result refines Moser's developments in this area which, in particular, are applicable to partial differential equation on the torus like

$$F_k \left( (u_i); (x_j); \left( \frac{\partial u_i}{\partial x_j}; 1 \leq i \leq m, 1 \leq j \leq n \right) \right) = 0, \quad k = 1, \dots, p, \quad (PDT)$$

where,  $(F_1, \dots, F_p)$  are  $2\pi$ -periodic in  $(x_1, \dots, x_n)$  and admit sufficiently many derivatives in

$$\| (x_1, \dots, x_n) \| + \left\| \left( \frac{\partial u_i}{\partial x_j}; 1 \leq i \leq m, 1 \leq j \leq n \right) \right\| \leq 1. \quad (Dom)$$

(See Moser [op.cit., ch. 2] for details). Hence, Theorem 3.1 is also working in such a context. And, its effective application yields an improvement of the quoted results; we do not give details.

#### 4 Some transfinite versions

A simple inspection of the developments in Section 2 shows that the  $i$ -normality property of the function  $t \vdash \varphi(t)$  is the essential one to solve the posed question by an *ordinary induction* argument. It is therefore natural to ask of what happens when a condition of this type is no longer fulfilled. As we shall see below, an appropriate answer to this is still available; but, in such a case, the appropriate instrument to be used is the *transfinite induction*.

Let the real interval  $J$  be taken as in Section 2. Further, let  $(X_1, d_1), (X, d)$  be a couple of metric spaces fulfilling (2H1), as well as

(4H1)  $e = \max\{d_1, d\}$  is complete over  $X_1$ .

Finally, let  $(Y, \|\cdot\|)$  be a Banach space. Take a multivalued function  $t \vdash F(t)$  from  $J$  to  $X_1 \times Y$  and introduce the limit set  $F(0)$  exactly as in Section 2. We are interested to determine sufficient conditions under which  $F(0)$  be nonempty as well as global evaluations of this set, in terms of our data. To this end, put

$$(4D1) \quad \left\{ \begin{array}{l} F(t+) = \{(x, y) \in X_1 \times Y; \text{there exists a sequence } ((t_n, x_n, y_n)) \\ \text{in } F \text{ with } t_n \downarrow t, x_n \xrightarrow{e} x, y_n \rightarrow y\}, \text{ for each } t \in J. \end{array} \right.$$

The basic assumption about  $F$  (to be used further) is

(4H2)  $F$  is closed at the right ( $F(t+) \subseteq F(t)$ , for all  $t \in J$ ).

Further, call the function  $\chi : J \rightarrow R$ , *lower semicontinuous at the right*, when

(4D2)  $\chi(t) \leq \chi(t+)$ , for all  $t \in J$ .

This allows us to introduce a similar property for each function  $t \vdash \Phi_1(t) = (\chi_1(t), \chi(t), \psi(t))$  from  $J$  to  $R^3$ ; namely, by means of (4D2), where  $(\leq)$  is the standard order of  $R^3$  (induced by the cone  $R_+^3$ ). The corresponding form of this convention for its "truncated" function  $t \vdash \bar{\Phi}(t) = (\chi(t), \psi(t))$  from  $J$  to  $R^2$  is clear; we do not give details.

The following transfinite version of Theorem 2.1 is now available.

**Theorem 4.1.** Suppose that a lower semicontinuous at the right function  $t \mapsto \Phi_1(t) = (\chi_1(t), \chi(t), \psi(t))$  from  $J$  to  $R \times R_+^2$  exists with

$$(4H3) \quad \left\{ \begin{array}{l} \text{for each } t \in \text{dom}(F) \text{ there exists } t' < t \text{ with} \\ \Delta_1(F(t), F(t')) \leq \Phi_1(t) - \Phi_1(t'). \end{array} \right.$$

Then,  $\text{dom}(F)$  is 0-accessible,  $F(0)$  is nonempty and

$$\Delta(F(r), F(0)) \leq \Phi(r) - \Phi(0_+), \quad r \in \text{dom}(F). \quad (4.1)$$

PROOF. Let the point  $(t_0, x_0, y_0)$  in  $F$  (with  $t_0 = r$ ) be arbitrary fixed. By (4H3), there exists  $(t_1, x_1, y_1) \in F$  with  $t_0 > t_1, d_1(x_0, x_1) \leq \chi_1(t_0) - \chi_1(t_1), d(x_0, x_1) \leq \chi(t_0) - \chi(t_1), \|y_0 - y_1\| \leq \psi(t_0) - \psi(t_1)$ . Further, again by this condition, there exists, for the obtained element, some  $(t_2, x_2, y_2) \in F$  with  $t_1 > t_2, d_1(x_1, x_2) \leq \chi_1(t_1) - \chi_1(t_2), d(x_1, x_2) \leq \chi(t_1) - \chi(t_2), \|y_1 - y_2\| \leq \psi(t_1) - \psi(t_2)$ , and so on. Generally, assume that, for the ordinal  $\eta$ , we constructed a net  $((t_\nu, x_\nu, y_\nu))_{\nu < \eta}$  in  $F$  with the property: for each  $\zeta < \eta$

$$\left\{ \begin{array}{l} t_\lambda > t_\mu, \\ d_1(x_\lambda, x_\mu) \leq \chi_1(t_\lambda) - \chi_1(t_\mu), \\ d(x_\lambda, x_\mu) \leq \chi(t_\lambda) - \chi(t_\mu), \\ \|y_\lambda - y_\mu\| \leq \psi(t_\lambda) - \psi(t_\mu), \end{array} \right. \quad \text{whenever } \lambda < \mu \leq \zeta. \quad (4.2)$$

If  $\eta$  is a first kind ordinal, put  $\eta - 1 = \zeta$ . By (4H3), there exists, for  $(t_\zeta, x_\zeta, y_\zeta)$  in  $F$ , some  $(t_\eta, x_\eta, y_\eta)$  in  $F$  with

$$\left\{ \begin{array}{l} t_\zeta > t_\eta, \\ d_1(x_\zeta, x_\eta) \leq \chi_1(t_\zeta) - \chi_1(t_\eta), \\ d(x_\zeta, x_\eta) \leq \chi(t_\zeta) - \chi(t_\eta), \\ \|y_\zeta - y_\eta\| \leq \psi(t_\zeta) - \psi(t_\eta); \end{array} \right.$$

hence (4.2) holds with  $\zeta = \eta$  and the process is continuable beyond  $\eta$ . Assume  $\eta$  is a second kind ordinal and

$$(4H4) \quad 0 < t_\eta = \lim_{\nu} t_\nu (= \inf t_\nu).$$

Since  $(\chi_1(t_\nu)), (\chi(t_\nu)), (\psi(t_\nu))$  are descending and bounded below nets, these are Cauchy ones. So, by (4.2),  $(x_\nu)$  is a Cauchy net (modulo  $\varepsilon$ ) in  $X_1$  and  $(y_\nu)$  is a Cauchy net in  $Y$ ; wherefrom, by completeness, the limits

$$x_\eta = \lim_{\nu} x_\nu \text{ (in } (X_1, \varepsilon)), \quad y_\eta = \lim_{\nu} y_\nu \text{ (in } Y) \quad (4.3)$$

exist. This finally combined with (4H2) shows that  $(t_\eta, x_\eta, y_\eta) \in F$ . Moreover, passing to limit over  $\mu$  in (4.2), one deduces that this relation holds with  $\zeta = \eta$ ; so, the process is continuable beyond  $\eta$ . Now, by a simple cardinality argument involving the net  $(t_\nu)$ , one has

$$t_\varepsilon = 0, \text{ for some (second kind) ordinal } \varepsilon = \varepsilon(t_0, x_0, y_0). \quad (4.4)$$

Moreover, we must have, by the above

$$\varepsilon \geq \omega \quad (= \text{the first transfinite ordinal}). \quad (4.5)$$

On the other hand, the chain of relations

$$0 = t_\varepsilon = \lim_{\nu} t_\nu \quad (= \inf_{\nu} t_\nu) \quad (4.6)$$

shows that there must be a strictly increasing and cofinal in  $]0, \varepsilon[$  sequence of ordinals  $(\nu_n)$  with

$$0 = \inf_n t_{\nu_n} \quad (\text{hence } t_{\nu_n} \rightarrow 0 \text{ as } n \rightarrow \infty). \quad (4.7)$$

As a direct consequence of this

$$\varepsilon < \Omega \quad (= \text{the first uncountable ordinal}); \quad (4.8)$$

see, for instance, Sierpinski [6, ch. 15, Sect. 2]. Since, by (4.2),  $(\chi(t_{\nu_n}))$ ,  $(\psi(t_{\nu_n}))$  are descending and bounded from below, these are Cauchy sequences. Hence, so are  $(x_{\nu_n})$  (modulo  $d$ ) and  $(y_{\nu_n})$ ; wherefrom, by completeness,

$$x_{\nu_n} \xrightarrow{d} x_\infty, \quad y_{\nu_n} \rightarrow y_\infty \text{ as } n \rightarrow \infty, \quad \text{for some } x_\infty \in X, \quad y_\infty \in Y.$$

This, along with (4.7), shows that  $(x_\infty, y_\infty) \in F(0)$ . Moreover, passing to limit in (4.2) yields

$$d(x_\nu, x_\infty) \leq \chi(t_\nu) - \lim_n \chi(t_{\nu_n}), \quad \|y_\nu - y_\infty\| \leq \psi(t_\nu) - \lim_n \psi(t_{\nu_n})$$

for all such  $\nu$ . Adding the inequalities  $\chi(0_+) \leq \lim_n \chi(t_{\nu_n})$ ,  $\psi(0_+) \leq \lim_n \psi(t_{\nu_n})$  one gets, by the arbitrariness of  $(x_0, y_0)$  in  $F(r)$ , the conclusion (4.1) we need. The proof is thereby complete.  $\square$

The formulation of our set comparison statement allows us clarifying its relationships with Theorem 2.1 above. Precisely, whenever

$$(4H5) \quad \varepsilon(t_0, x_0, y_0) = \omega, \quad \text{for each } (t_0, x_0, y_0) \in F,$$

Theorem 4.1 above reduces to the quoted statement. This, however, cannot be taken in a formal sense only. Some further aspects will be delineated elsewhere.

## 5 Altman solvability results

As we had already occasion to show, the (sequential) type set comparison statements in Section 2 were used in deriving the Moser solvability results (in Section 3). So, it is natural to presume that a similar device is valid for their transfinite versions we just exposed. It is our aim in what follows to verify this claim; as we shall see, the appropriate techniques to be followed are comparable with the ones in Altman [1].

Let  $E_1$ ,  $E_0$  and  $E$  be a triple of (complex) vector spaces with  $E_1 \subseteq E \subseteq E_0$ , equipped with the norms  $\|\cdot\|_1$ ,  $\|\cdot\|_0$  and  $\|\cdot\|$  respectively. These are assumed to satisfy (3H1)+(3H2), as well as

(5H1)  $\|\cdot\|_* = \max(\|\cdot\|_1, \|\cdot\|_0, \|\cdot\|)$  is complete over  $E_1$ .

Further, let  $H$  be a (complex) vector space, endowed with a norm  $\|\cdot\|$  fulfilling (3H1) (with  $E = H$ ). Finally, take a map  $x \mapsto Tx$  from  $E$  to  $H$ . Under the model of Section 3, we intend to solve the operator equation (OE) over certain regions of  $E$ , described as in (3D1)+(3D2). To this end, we have to impose some general/specific regularity assumptions upon our data. The general ones are (3H3) and (3H4). The specific ones are (3H6) and

$$(5H2) \quad 1 > \max \left\{ \rho \left( 1 + \frac{1}{\lambda} \right), \frac{\lambda + 1}{\mu} + \nu \right\}.$$

In this case, the following answer to the posed problem is available.

**Theorem 5.1.** *Let the conditions above be admitted. Then, there exists a number  $\varepsilon$  in  $]0, 1[$  with the property*

$$\text{the operator equation (OE) has a solution in } D_1^E, \text{ whenever } \|T(0)\| < \varepsilon. \quad (5.1)$$

PROOF. We intend to show that a certain version of Theorem 4.1 is applicable to our data. Precisely, put  $J = ]0, 1[$  and fix some  $\delta$  in  $]0, 1[$ . With  $K > 0$  arbitrary for the moment, let us introduce the functions

$$(5D1) \quad \chi_1(t) = -t^{-1/\lambda}, \quad \chi_0(t) = \psi(t) = \frac{1 + \delta}{1 - \delta}t, \quad \chi(t) = Kt^{1 - \rho(1 + 1/\lambda)}, \quad t \in J.$$

Also, with  $c$  in  $]0, 1[$  arbitrary for the moment, define a multivalued function  $t \mapsto F(t)$  from  $J_c = ]0, c[$  to  $E_1 \times H$  according to

$$(5D2) \quad \left\{ \begin{array}{l} F(t) = \{(x, y) \in \text{gr}(T); x \in D_1, \|x\|_1 \leq -\chi_1(t), \\ \|x\|_0 \leq \Omega - \chi_0(t), \|x\| \leq \Omega - \chi(t), \|y\| \leq t\}, \quad t \in J_c. \end{array} \right.$$

We claim that, one may determine the couple of constants  $(\delta, K)$  and the number  $c \in ]0, 1[$  in such a way that conditions of the quoted statement be fulfilled. The verification of this is composed of four distinct steps.

(i) We firstly determine the couple of constants  $(\delta, K)$  in order that the apriori implication be retainable:

$$\left\{ \begin{array}{l} \text{for each } (t, x, y) \text{ in } F \text{ there exists } (t', x', y') \text{ in } F \text{ with } t' < t, \\ \|x - x'\|_1 \leq \chi_1(t) - \chi_1(t'), \\ \|x - x'\|_0 \leq \chi_0(t) - \chi_0(t'), \\ \|x - x'\| \leq \chi(t) - \chi(t'), \\ \|y - y'\| \leq \psi(t) - \psi(t'). \end{array} \right. \quad (5.2)$$

This, essentially, consists in the following. Let the number  $t$  in  $\text{dom}(F)$  be given and take some  $(x, y)$  in  $F(t)$ ; namely,

$$(5H3) \quad \left\{ \begin{array}{l} (x, y) \in \text{gr}(T), \quad x \in D_1, \text{ and} \\ \|x\|_1 \leq -\chi_1(t), \quad \|x\|_0 \leq \Omega - \chi_0(t), \quad \|x\| \leq \Omega - \chi(t), \quad \|y\| \leq t. \end{array} \right.$$

Note that, with  $\Gamma = t^{-1/\lambda} (> 1)$ , one has  $\|x\|_1 \leq \Gamma$ ,  $\|Tx\| = \|y\| \leq \Gamma^{-\lambda}$ . Put also  $L = M + 1$ . By (3H6) it follows that, for the arbitrary fixed  $\Theta > 1$ , there must be some point  $u = u(x, \Gamma, \Theta)$  in  $E_1$  with the properties

$$\|T'(x)(u) + Tx\| < L\Gamma\Theta^{-\mu}, \quad \|u\|_1 < L\Gamma^\nu\Theta, \quad \|u\|_0 < L\Gamma\Theta^{-\mu} + \Gamma^{-\lambda}. \quad (5.3)$$

By the former of these (cf. the definition of the Gateaux derivative)

$$\|(1/\tau)(T(x + \tau u) - T(x)) + T(x)\| < L\Gamma\Theta^{-\mu}, \quad (5.4)$$

where  $\tau = \tau(\Theta)$  is small enough (in  $J$ ).

Fix in the following such a number  $\tau$  and put

$$(5D3) \quad t' = (1 - \tau)t + \tau\delta t, \quad x' = x + \tau u.$$

We show that, under a suitable choice of  $\Theta$ , all requirements in (3.2) are satisfied.

(ia) A first condition to be posed upon  $\Theta$  may be phrased as

$$(5H4) \quad Lt^{-1/\lambda}\Theta^{-\mu} < \delta t \quad (\text{hence } \Theta > (\frac{1}{\delta})^{1/\mu} t^{-1/\mu - 1/\lambda\mu}).$$

It then follows from (5.4) that

$$\|x' - (1 - \tau)Tx\| < \tau\delta t \quad (\text{with } \tau \text{ as before}). \quad (5.5)$$

As a direct consequence of this, plus the choice of  $(x, y) = (x, Tx)$

$$\|Tx'\| \leq (1 - \tau)\|Tx\| + \tau\delta t \leq (1 - \tau)t + \tau\delta t = t'.$$

On the other hand,

$$\|Tx' - Tx\| \leq \tau(\|Tx\| + \delta t) \leq \tau(1 + \delta)t;$$

wherefrom, combining with

$$\tau = \frac{t - t'}{(1 - \delta)t} \quad (\text{deduced from (5D3)(the first half)}) \quad (5.6)$$

one gets

$$\|Tx' - Tx\| \leq \frac{1 + \delta}{1 - \delta}(t - t') = \psi(t) - \psi(t').$$

Hence, the requirements in (5.2) involving  $y' = Tx'$  are fulfilled.

(ib) Further, by (5.3)(the second part), (5D3)(the second part) and (5.6),

$$\|x - x'\|_1 < \tau Lt^{-\nu/\lambda}\Theta < \frac{L}{1 - \delta} t^{-1-\nu/\lambda}\Theta(t - t'). \quad (5.7)$$

So, if we impose the condition

$$(5H5) \quad \frac{L}{1 - \delta} t^{-1-\nu/\lambda}\Theta < \frac{1}{\lambda} t^{-1-1/\lambda} \quad (\text{hence } \Theta < \frac{1 - \delta}{L\lambda} t^{(\nu-1)/\lambda})$$

it will follow from this, by a mean value argument, that

$$\|x - x'\|_1 \leq \frac{1}{\lambda} t^{-1-1/\lambda} (t - t') \leq \chi_1(t) - \chi_1(t')$$

(wherefrom  $\|x'\|_1 \leq -\chi_1(t')$ ).

So, the requirements in (5.2) (related to  $\|\cdot\|_1$ ) involving  $x'$  are satisfied.

(ic) On the other hand, by (5.3)(the third part), (5D3)(the second part), (5H4) and (5H6),

$$dst \|x - x'\|_0 \leq \tau(1 + \delta)t = \chi_0(t) - \chi_0(t') \quad (5.8)$$

(wherefrom  $\|x'\|_0 \leq \Omega - \chi_0(t')$ ).

Hence, the requirements in (5.2) (related to  $\|\cdot\|_0$ ) involving  $x'$  are fulfilled too.

(id) Finally, by (3H2) and (5.6)–(5.8),

$$\begin{aligned} \|x - x'\| &\leq N \|x - x'\|_1^\rho \|x - x'\|_0^{1-\rho} \leq \\ &\leq N (Lt^{-\nu/\lambda} \Theta)^\rho [(1 + \delta)t]^{1-\rho} \tau = \\ &= NL^\rho (1 + \delta)^{1-\rho} t^{1-\rho-\nu\rho/\lambda} \Theta^\rho \tau = \\ &= \frac{N}{1-\delta} L^\rho (1 + \delta)^{1-\rho} t^{-\rho-\nu\rho/\lambda} \Theta^\rho (t - t'). \end{aligned}$$

So, if we take (5H5) into account, as well as (5H2)(the first half)

$$\|x - x'\| \leq (N/\lambda^\rho) \left( \frac{1 + \delta}{1 - \delta} \right)^{1-\rho} t^{-\rho-\rho/\lambda} (t - t') = \frac{K}{1 - \rho(1 + 1/\lambda)} t^{-\rho-\rho/\lambda} (t - t');$$

where, by definition

$$(5D4) \quad K = (N/\lambda^\rho) \left[ 1 - \rho \left( 1 + \frac{1}{\lambda} \right) \right] \left( \frac{1 + \delta}{1 - \delta} \right)^{1-\rho}.$$

And this, by a mean value argument, yields

$$\|x - x'\| \leq \chi(t) - \chi(t') \quad (\text{hence } \|x\| \leq \Omega - \chi(t')).$$

In other words, the requirements in (5.2) (related to  $\|\cdot\|$ ) involving  $x'$  are satisfied.

Summing up, the sufficient conditions under which (5.2) be retainable are (5H4)+(5H5). Note that, by the former of these,  $\Theta > 1$  (since  $L > 1$  and  $0 < \delta < 1$ ). So, the existence of a solution  $\Theta > 1$  for the inequalities above is a direct consequence of

$$(5H6) \quad \left( \frac{L}{\delta} \right)^{1/\mu} t^{-1/\mu-1/\lambda\mu} < \frac{1-\delta}{L\lambda} t^{\nu/\lambda-1/\lambda},$$

or, equivalently,

$$(5H7) \quad t^{1/\lambda-\nu/\lambda-1/\mu-1/\lambda\mu} < \frac{1-\delta}{L\lambda} \left( \frac{\delta}{L} \right)^{1/\mu}.$$

The existence of a solution for such an inequality requires the exponent in the left member be strictly positive; that is

$$(5H8) \quad \frac{1}{\lambda} - \frac{\nu}{\lambda} - \frac{1}{\mu} - \frac{1}{\lambda\mu} > 0 \quad \left( \text{wherefrom } 1 - \nu > \frac{\lambda + 1}{\mu} \right).$$

Since this is just (5H2) (the second half) we are done.

(ii) It remains now to discuss the effectiveness over  $J_c$  of the inequality in the variable  $(t)$ . Put

(5D5)  $c$  is the supremum of all  $t \in J$  fulfilling (5H7).

It is clear, by the above discussion that, under this choice, the apriori implication (5.2) is retainable.

(iii) Further, we claim that the multivalued function  $t \vdash F(t)$  we already introduced is closed at the right, in the sense

$$\left\{ \begin{array}{l} \text{for each sequence } (t_n, x_n, y_n) \text{ in } F \text{ and each } (t, x, y) \text{ in } J_c \times E_1 \times H \text{ with } t_n \downarrow t \\ \|x_n - x\|_1 \rightarrow 0, \quad \|x_n - x\|_0 \rightarrow 0, \\ \|x_n - x\| \rightarrow 0, \quad \|y_n - y\| \rightarrow 0 \end{array} \right. \quad \text{we have } (t, x, y) \in F. \quad (5.9)$$

In fact, let the sequence  $((t_n, x_n, y_n))$  in  $F$  and the element  $(t, x, y)$  in  $J_c \times E_1 \times H$  be as in the premise of (5.9). By the definition of  $F$ , one has

$$\|x_n\|_0 \leq \Omega - \chi_0(t_n), \quad \|x_n\| \leq \Omega - \chi(t_n), \quad \text{for all } n.$$

And from this, by a limit process,

$$\|x\|_0 \leq \Omega - \chi(t) < \Omega, \quad \|x\| \leq \Omega - \chi(t) < \Omega \quad (\text{hence, in particular, } x \in D_1).$$

On the other hand, the same definition says that

$$\|x_n\|_1 \leq -\chi_1(t_n), \quad \|y_n\| \leq t_n, \quad \text{for all } n;$$

so, necessarily,

$$\|x\|_1 \leq -\chi_1(t), \quad \|y\| \leq t \quad (\text{by a limit process}).$$

Finally, the relations

$$(x_n, y_n) \in \text{gr}(T) \quad (\text{i.e., } y_n = Tx_n) \text{ for all } n$$

and the convergence properties (5.9) give  $(x, y) \in \text{gr}(T)$ ; hence the claim.

(iv) It remains now to assure the properness of the multivalued function  $F$ :

$$(0, T(0)) \in F(\eta), \quad \text{for some } \eta \text{ in } J_c. \quad (5.10)$$

To this end, put

$$(5D6) \quad \varepsilon = \sup\{t \in J_c; \max(\chi_0(t), \chi(t)) \leq \Omega\}.$$

By the very definition of these functions,

$$\max(\chi_0(\eta), \chi(\eta)) < \Omega, \quad \text{when } 0 < \eta < \varepsilon. \quad (5.11)$$

So, if  $\|T(0)\| < \varepsilon$ , one has  $(0, T(0)) \in F(\eta)$  whenever  $\|T(0)\| \leq \eta < \varepsilon$ .

As a consequence of all these developments, Theorem 4.1 is applicable to our data. This, combined with a remark like in (3.9), ends the argument.  $\square$

As already precised in a previous place, this result is comparable with a similar one due to Altman [op.cit.] which, among others, is applicable to quasilinear Dirichlet problems of the form

$$\begin{cases} \sum_{|\alpha|=2m} a_\alpha(x, D^\gamma u) D^\alpha u + B(x, D^\beta u) = h(x), & \text{on } \Omega \\ D^j u|_\Gamma = \varphi_j \quad (\text{for all } j), & \text{on } \Gamma = \text{bd}(\Omega). \end{cases} \quad (QDP)$$

Namely, Theorem 5.1 is also working in such a context; we do not give details. For different aspects of technical nature we refer to the paper by Turinici [7].

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