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BOUNDS OF THE COEFFICIENTS FOR UNIFORMLY CLOSE-TO-CONVEX FUNCTIONS

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Abstract. In this paper we find bounds on the coefficients of series expansion for so called n - uniformly close-to-convex functions of order α which had previously been defined, as well as for the functions which can be obtained by applying a integral operator of Libera-Pascu type on the first ones. Thus there have been used the results given by N.N.Pascu, S.Kanas, T.Yaguchi, A.Wisniowska and C.I.Magdaş.

Key Words and Phrases: (n, α) - uniformly convex, n - uniformly close-to-convex of type α, integral operator of Libera-Pascu type.

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1 Introduction

Denote by A the set of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

analytic in the open disk U, and by S its subclass consisting of all univalent functions. Subclasses of S-type functions have been lately studied which have geometrical properties such as being uniformly convex UCV, uniformly starlike UST, (n, α) - uniformly convex etc. The coefficient problem and the relationship among them is frequently studied.

In this paper we find bounds on the coefficients of series expansion upon unit disk U for so called n - uniformly close-to-convex functions of type α previously introduced [2] as well as for the coefficients of the functions obtained from the latter ones by applying an integral operator of Libera-Pascu type.

2 Preliminary results

Let $D^n f$ be the differential operator defined by G.Sălăgean recurrently; $D^n : A \to A$, $n \in N \cup \{0\}$

$$\begin{gathered} D^0 f(z) = f(z) \\ D' f(z) = D f(z) = z f'(z) \\ D^n f(z) = D(D^{n-1} f(z)),\ z \in U. \end{gathered} \tag{2.1}$$

Definition 2.1. [7]. *Let $f \in S$, $\alpha \geq 0$, $n \in N \cup \{0\}$. We say that f is (α, n)–uniformly convex if*

$$Re\left(\frac{D^{n+1}f(z)}{D^n f(z)}\right) > \left|\frac{D^{n+1}f(z)}{D^n f(z)} - 1\right|, \quad z \in U. \tag{2.2}$$

We denote by $(\alpha, n) - UCV$ the set of these functions.

Independently, C.I.Magdaş [4] gives a similar definition but he named these functions n - uniformly starlike of type α and denote by $USn(\alpha)$ the set of these functions.

For cases $n = 0$ and $n = 1$ on obtain the classes $\alpha - ST$ respectively $\alpha - UCV$ defined and studied in detail by S.Kanas and A.Wisniowska [5], [6].

Observe that the family of (n, α)–uniformly convex functions describes the class of analytic functions f, since the expression $D^{n+1}f(z)/D^n f(z)$, $z \in U$, $n \in N \cup \{0\}$ lies in the conic region Ω_α which depends on the parameter α.

Denote by $\mathcal{P}(p_\alpha)$ [5], [6] $(0 \leq \alpha < \infty)$, the family of the functions p, such that $p \in \mathcal{P}$, and $p \prec p_\alpha$ in U, where the function p_α maps the unit disk conformally onto the region Ω_α, such that $1 \in \Omega_\alpha$ and

$$\partial\Omega_\alpha = \{u + iv : u^2 = \alpha^2(u-1)^2 + \alpha^2 v^2\} \tag{2.3}$$

The domain Ω_α is elliptic for $\alpha > 1$, hyperbolic when $0 < \alpha < 1$, parabolic when $\alpha = 1$, and a right half-plane when $\alpha = 0$ (for complete information see [5]).

Here $\mathcal{P}$ denotes the well known class of Caratheodory functions. These functions, which play the role of extremal functions of the classes $\mathcal{P}(p_\alpha)$, have been obtained in [5].

Obviously

$$p_0(z) = \frac{1+z}{1-z} \text{ and } p_1(z) = 1 + \frac{2}{\pi^2}\left(\log\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)^2, \quad z \in U$$

(see [8]) and, when $0 < \alpha < 1$, [5]

$$p_\alpha(z) = \frac{1}{1-\alpha^2}\cos\left\{Ai\log\frac{1+\sqrt{z}}{1-\sqrt{z}}\right\} - \frac{\alpha^2}{1-\alpha^2}, \quad z \in U$$

or equivalently

$$p_\alpha(z) = \frac{1}{2(1-\alpha^2)}\left[\left(\frac{1-\sqrt{z}}{1+\sqrt{z}}\right)^A + \left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)^A\right] - \frac{\alpha^2}{1-\alpha^2}$$

where $A = \dfrac{2}{\pi}\arccos\alpha$. Finally when $\alpha > 1$, the function p_α has the form (cf.[5], [6])

$$p_\alpha(z) = \frac{1}{\alpha^2-1}\sin\left(\frac{\pi}{2K(k)}\int_0^{\frac{u(z)}{\sqrt{k}}}\frac{dt}{\sqrt{1-t^2}\sqrt{1-k^2t^2}}\right) + \frac{\alpha^2}{\alpha^2-1}$$

with

$$u(z) = (z - \sqrt{k})\Big/(1 - \sqrt{k}z), \quad 0 < k < 1, \quad z \in U$$

and k is chosen such that $k = \cosh(\pi K'(k)) \Big/ (4K(k))$. $K(k)$ is Legendre's complete elliptic integral of the first kind, and $K'(k)$ is the complementary integral of $K(k)$. Denote the coefficients of p_α by q_n, $n \geq 1$. Then we have

$$p_\alpha(z) = 1 + q_1 z + q_2 z^2 + \dots, \quad z \in U$$

Remark. The domain $p_\alpha(U) = \Omega_\alpha$, $\alpha \in [0, \infty)$, is convex. Then, as an immediate consequence of the well known Rogosinski result for subordinate functions, we obtain for $p \in \mathcal{P}(p_\alpha)$ where $p(z) = 1 + p_1 z + p_2 z^2 + \dots$, $z \in U$

$$|p_n| \leq |q_1| \text{ where } q_1(\alpha) = \begin{cases} \dfrac{8(\arccos \alpha)^2}{\pi^2(1-\alpha^2)} & 0 \leq \alpha < 1 \\ \dfrac{8}{\pi^2} & \alpha = 1 \\ \dfrac{\pi^2}{4\sqrt{k}(\alpha^2-1)K^2(k)(1+k)} & \alpha > 1 \end{cases} \tag{2.4}$$

Remark [6]. Let the coefficient $q_1 := q_1(\alpha)$ be a function of the variable α. Then, the function $q_1(\alpha)$ is positive and strictly decreasing on the interval $[0, \infty]$ and its values are included in the interval $(0, 2]$ with $q_1(0) = 2$, $q_1(\sqrt{2}/2) = 1$, $q_1(1) = 8/\pi^2 = 0,81$. The values of $q_1(\alpha)$ tend to 0 when $\alpha \to \infty$ and so does p_n.

Definition 2.2. [2]. *Let be $f \in A$, $\alpha \geq 0$, $n \in N \cup \{0\}$. We say that f is n-uniformly close-to-convex of α type if there exists a function $g \in (\alpha, n) - UCV$ (or $US_n(\alpha)$) so that*

$$Re\left(\frac{D^{n+1}f(z)}{D^n g(z)}\right) > \alpha \left|\frac{D^{n+1}f(z)}{D^n g(z)} - 1\right|, \quad z \in U$$

We note the set of these functions by $UCC_n(\alpha)$.

Remark. $f \in UCC_n(\alpha)$ if and only if $h(z) = D^{n+1}f(z) \Big/ D^n g(z)$ takes all its values in Ω_α.

We denote by La the integral operator $La : A \to A$ defined by

$$f(z) = La(F(z)) = \frac{1+a}{z^a} \int_0^z F(t) t^{a-1} dt \tag{2.5}$$

where $a \in C$, $Re\, a \geq 0$, introduced by Libera for $a = 1$ and later generalized for the largest domain of parameter a ($a \in C, Re\, a \geq 0$) by N.N.Pascu.

THEOREM A. [1] *For every $a \in C$, $Re\, a \geq 0$, $\alpha \geq 0$, $n \in N \cup \{0\}$*

$$La[UCC_n(\alpha)] \subset UCC_n(\alpha)$$

In other words, by applying the operator La it is obtained a subclass of $UCC_n(\alpha)$. We denote by $UCCP_n(\alpha)$ this set.

3 Main results

In the following we give bounds for the coefficients of series expansions for these functions belonging to the sets $UCC_n(\alpha)$ and $UCCP_n(\alpha)$.

Theorem 3.1. *If* $F(z) = z + \sum_{j=2}^{\infty} a_j z^j$ *belong to the class* $UCC_n(\alpha)$ *then*

$$|a_j| \leq \frac{\prod_{K=0}^{j-2} (q_1 + K)}{j^n (j-1)!} \tag{3.1}$$

where $q_1 = q_1(\alpha)$ *is given by (2.4).*

Proof. Let $f \in UCC_n(\alpha)$, $f(z) = z + \sum_{j=2}^{\infty} a_j z^j$, and $g \in (\alpha, u) - UCV$, $g(z) = z + \sum_{j=2}^{\infty} b_j z^j$.

According to the results of [7], we have the estimation

$$|b_j| \leq \frac{q_1(q_1+1)\dots(q_1+j-2)}{(j-1)! j^n} \quad j \geq 2 \tag{3.2}$$

Also, $f \in UCC_n(\alpha) \iff h(z) = D^{n+1} f(z) / D^n g(z) \prec p_\alpha(z)$ where $p_\alpha(U) = \Omega_\alpha$.

Let $h(z) = 1 + c_1 z + c_2 z^2 + \dots$; then using the consequences of the subordination relationships as well as Rogosinski's theorem, from (2.4) we get

$$|c_j| \leq q_1(\alpha) \tag{3.3}$$

It is easy to observe that

$$D^{n+1} f(z) = z + \sum_{j=2}^{\infty} j^{n+1} c_j z^j$$

and

$$D^n g(z) = z + \sum_{j=2}^{\infty} j^n b_j z^j$$

Then by identifying the equal powers, we get

$$j^{n+1} a_j = c_j + 2^n b_2 c_{j-2} + 3^n b_3 c_{j-3} + \dots + c_1 (j-1)^n b_{j-1} + j^n b_j.$$

By using inequalities in (3.2) and (3.3), it follows

$$j^{n+1} |a_j| \leq q_1 \left[1 + 2^n |b_2| + 3^n |b_3| + \dots + (j-1)^n |b_{j-1}| \right] + j^n |b_j|$$

or

$$j^{n+1}|a_j| \leq q_1 \left[1 + \sum_{l=2}^{j-1} \frac{\prod_{K=0}^{l-2} (q_1 + K)}{(l-1)!} \right] + \frac{\prod_{K=0}^{j-2} (q_1 + K)}{(j-1)!}.$$

By mathematical induction we find that

$$1 + \sum_{l=2}^{j-1} \frac{\prod_{K=0}^{l-2} (q_1 + K)}{(l-1)!} = \frac{\prod_{K=1}^{j-2} (q_1 + K)}{(j-2)!}$$

Thus

$$j^{n+1}|a_j| \leq q_1 \frac{\prod_{K=1}^{j-2} (q_1 + K)}{(j-2)!} + \frac{\prod_{K=0}^{j-2} (q_1 + K)}{(j-1)!} =$$

$$= \frac{\prod_{K=1}^{j-2} (q_1 + K)}{(j-1)!} [q_1(j-1) + q_1] = \frac{\prod_{K=1}^{j-2} (q_1 + K)}{(j-1)!} q_1 j$$

or

$$|a_j| \leq \frac{\prod_{K=0}^{j-2} (q_1 + K)}{j^n (j-1)!}.$$

For special values of the parameters n, α bounds on the coefficients are obtained, as given by F.Ronning, W.Ma, D.Minda respectively S.Kanas ([4], [5], [6], [8]).

Theorem 3.2. *Let $a \in C$, $Re\ a \geq 0$, $n \in N \cup \{0\}$, $\alpha \geq 0$. If $F \in UCC_n(\alpha)$, $F(z) = z + \sum_{j=2}^{\infty} a_j z^j$, then for $f(z) = La\ F(z)$, with $f(z) = z + \sum_{j=2}^{\infty} b_j z^j$, where La is the Libera–Pascu integral operator (2.5), we get*

$$|b_j| \leq \left| \frac{a+1}{a+j} \right| \frac{\prod_{K=0}^{j-2} (q_1 + K)}{j^n (j-1)!}, \quad j = 2, 3, \ldots$$

where $q_1 = q_1(\alpha)$ is given by (2.4).

Proof. We observe that $f(z) = La\ F(z)$ is equivalent to

$$(1 + a)F(z) = a\ f(z) + z\ f'(z) \quad a \in C, Re\ a \geq 0 \tag{3.4}$$

If in (3.4) we put f and F with the above forms then, by identifying the coefficients of the terms in z^j, we get

$$b_j \cdot (a + j) = (1 + a)a_j$$

According to (3.1) it follows that

$$|b_j| \leq \left|\frac{a+1}{a+j}\right| \frac{\prod_{K=0}^{j-2}(q_1+K)}{j^n(j-1)!}$$

which completes the proof of Theorem 3.2.

For $a=1$ we have

$$|a_j| \leq \frac{2\prod_{K=0}^{j-2}(q_1+K)}{j^{n-1}(j+1)!}.$$

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