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ON RIESZ' SPLITTING PROPERTY

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Introduction

The essential role of the Riesz' splitting property:

$$\forall s, s_1, s_2 \in S, s \leq s_1 + s_2, \exists s_1', s_2' \in S \text{ such that } s = s_1' + s_2', s_1 \leq s_1', s_2 \leq s_2'$$

in potential theory was recognized by G. Mokobodzki [8].

This axiom is also the main ingredient in the definition of a H–cone [2].

It was also proved that this property holds in the cone of excessive functions with respect to a (sub–markovian) resolvent of kernels. Hence, the same is true for the cone of excessive functions with respect to a (sub–markovian) semi–group of kernels.

However, a direct proof in the semi–group case (i.e. without using the associated resolvent) is complicated by the fact that there is no monotony of the reduced function $t \mapsto R^t f := \inf\{s \in \mathcal{F} | s \geq f, P_t s \leq s\}$.

In [10] a general proof in the abstract case (such as the one considered by [5] for the case of a single kernel or of a resolvent), was presented.

Such a proof covers hence not only the case of cones of functions, but also the general, abstract case, including, but not limited to, for example cones of classes of functions or cones of measures.

Even for the case of a semi–group of kernels on (measurable) functions, we obtain a direct proof for the fact that the supramedian or excessive functions form a potential cone.

The purpose of this paper is to consider some explicit examples of (abstract) cones of potentials of excessive elements with respect to a semi–dynamical system. More precisely, X will be one of the following convex cones, with the usual operations and order:

- measurable (positive) functions on a measurable space;
- l.s.c. positive functions on a locally compact space;
- classes with respect to a family of "negligible" sets of measurable (positive) functions on a measurable space;
- continuous positive real functions on a compact stonean space;
- positive measures on a measurable space.

In each case, modifications of the general proof under specific hypotheses are given.

1 Preliminaries

(1.1) Let us recall the following definitions [9]:

(X, Φ) is a *semi-dynamical system on algebraic structure* (s.d.s.a.s. for short) when X is a lattice and a convex cone, such that $x_1 \leq x_2 \leq \ldots \leq x_n \leq \ldots \Longrightarrow \exists \bigvee_n x_n$ în X.

For such a situation, we use the notation $x_n \nearrow y$, where $y = \bigvee_n x_n$.

$\Phi : [0, +\infty) \times X \to X$ has the properties:

$$\Phi(s, \Phi(t, x)) = \Phi(s + t, x)$$

and the other natural compatibility requirements (with the ordered convex cone structure on X).

$x \in X$ is called *supramedian* if it is cancellable, i.e.

$$y + x \leq y' + x \Longrightarrow y \leq y'$$

and $\Phi(t, x) \leq x$, $\forall t \geq 0$.

X_s will denote the set of supramedian elements. Let us remark that, for $x \in X_s$, the map $t \mapsto \Phi(t, x)$ is decreasing. Hence, the following notation makes sense:

$$\tilde{x} := \bigvee_{t \geq 0} \Phi(t, x)$$

$x \in X$ is called *excessive* if $x \in X_s$ and $x \leq \tilde{x}$.

X_e will denote the set of excessive elements.

X_s and X_e are convex sub-cones, in which there exists the countable, dominated increasing supremum. Moreover, they are *absorbent* parts with respect to the relation $\leq_\Phi$ (which means that $x \in X_s$ resp. $X_e \Longrightarrow \Phi(t, x) \in X_s$ resp. X_e).

We say that two s.d.s.a.s. (X, Φ) and (Y, Ψ) are in duality through:

$$[\cdot, \cdot] : X \times Y \to [0, +\infty]$$

if the following properties hold:

$$[x + x', y] = [x, y] + [x', y]; \quad [x, y + y'] = [x, y] + [x, y']$$
$$x \leq x' \Longrightarrow [x, y] \leq [x', y]; \quad y \leq y' \Longrightarrow [x, y] \leq [x, y']$$
$$[\Phi(t, x), y] = [x, \Psi(t, y)], \quad \forall t > 0$$
$$x_n \nearrow x \Longrightarrow [x_n, y] \nearrow [x, y]; \quad y_n \nearrow y \Longrightarrow [x, y_n] \nearrow [x, y]$$

For each system (X, Φ), let us consider the set X^* of all applications

$$\mu : X \to [0, +\infty]$$

which have the properties:

$$\mu(x+x') = \mu(x) + \mu(x')$$

$$\mu(a.x) = a.\mu(x),\ \forall a \geq 0$$

$$x \leq x' \Longrightarrow \mu(x) \leq \mu(x')$$

$$x_n \nearrow x \Longrightarrow \mu(x_n) \nearrow \mu(x)$$

The set X^* is organized with pointwise addition, scalar multiplication and order. A s.d.s.a.s. is defined on X^* as:

$$\Phi^*(t,\mu)(x) := \mu\left[\Phi(t,x)\right]$$

(X,Φ) and (X^*,Φ^*) are in a canonical duality, through: $[x,\mu] := \mu(x)$.

In [5], the condition

$$y \leq z \Longrightarrow \exists x \text{ with } x+y=z$$

is imposed from the beginning. Adopting such a condition reduces essentially the flexibility; for instance, the case when X contains only excessive elements is excluded.

(1.2) $\mathcal{G}$ will denote the set of all parts $G \subseteq [0,+\infty)$, which are countable and:

$$t,t' \in G \Longrightarrow t+t' \in G$$

$$\forall t \in [0,+\infty) \exists t_n \in G \text{ such that } t_n \searrow t$$

$\mathcal{G}$ will be ordered by inclusion and will serve as an index set.

$\mathcal{G}$ is clearly non–void, containing for example the set of positive, rational numbers.

$\mathcal{G}$ is increasing, since $G_1 + G_2$ contains $G_1 \cup G_2$ (we may accept that $0 \in G$). For each $t \in [0,+\infty)$ and each $G \in \mathcal{G}$, we denote $G^t := \{s+n.t | s \in G, n \in \mathbb{N}\}$. Clearly $G^t \in \mathcal{G}$ and $G \subseteq G^t$ while $t \in G^t$. Let us consider the set $\tilde{G} := \{t'-t | t,t' \in G, t' \geq t\}$. We have $\tilde{G} \in \mathcal{G}$ and $t,t' \in \tilde{G}$, $t' \geq t \Longrightarrow t'-t \in \tilde{G}$.

(1.3) We are going to prove, for various particular cones, the existence of the reduced function for the supramedian elements, meaning that, for each $u,v \in X_s$, $v \leq u$, the set $\{s \in X_s |\ s+v \geq u\}$ has the smallest element, which will be denoted as $R^s(u-v)$. Moreover, there exists $v' \in X_s$ such that $u = v' + R^s(u-v)$. The proof in the abstract case, but under additional hypothesis, is given in [10]. In the next chapters, we will provide only the additional arguments needed to show that much weaker or no assumptions are necessary in each particular case.

Let us remark that, from this fact, the similar result obtained by replacing X_s with X_e holds. Indeed: let us consider $u,v \in X_e$, $v \leq u$. There exists $w := R^s(u-v) \in X_s$. Hence $\tilde{w} \in X_e$ and, from $w+v \geq u$ we get, by excessive regularization: $\tilde{w}+v \geq u$. Conversely, let $s \in X_e$ be such that $s+v \geq u$. We get $s \geq w$, hence $s \geq \tilde{w}$. So, the reduced $R^e(u-v)$ exists and equals $\tilde{w}$. Moreover, let $v' \in X_s$ be such that $u = v'+w$. Again by regularization we get $u = \tilde{v'} + \tilde{w}$, hence $\tilde{v'} \in X_e$.

Finally, from the above property follows the usual form of Riesz' splitting property. Indeed, let $s, s_1, s_2 \in X_s$ be such that $s \leq s_1 + s_2$. Let us denote $s_2' := R^s(s - s \wedge s_1)$ and let s_1' be the element from X_s for which $s = s_1' + s_2'$. Since s_2' is the smallest element, we get: $s_2 + s \wedge s_1 = (s_2+s) \wedge (s_2+s_1) \geq s$, hence $s_2' \leq s_2$. But s_2' also verifies $s_2' + s \wedge s_1 = (s_2'+s) \wedge (s_2'+s_1) \geq s$, hence $s_2' + s_1 \geq s = s_1' + s_2'$, so the conclusion: $s_1 \geq s_1'$.

In order to obtain the same property in X_e, one has to suppose that $x, y \in X_e \Longrightarrow x \wedge y \in X_e$.

2 Measurable functions

(2.1) Let $(E, \mathcal{E})$ be a measurable space. We denote by $\mathcal{F}$ the set of all $\mathcal{E}$–measurable functions $E \to [0, +\infty]$, organized as an ordered convex cone with the pointwise algebraic operations (where $0 \cdot \infty = 0$) and order. $\mathcal{F}$ is a σ–complete lattice, in which $\vee$ and $\wedge$ are computed pointwise.

Cancellable in $\mathcal{F}$ means finite. When $f \leq g$, there exists $h \in \mathcal{F}$ such that $f + h = g$ (h is arbitrarily measurable defined on the set $[f = \infty]$).

Clearly $\mathcal{F}^*$ is exactly the set of all (positive) measures on $(E, \mathcal{E})$.

Each s.d.s.a.s. on $\mathcal{F}$ is simply a semi-group $\mathcal{P} = (P_t)_{t>0}$ of kernels on $(E, \mathcal{E})$. We shall keep the usual notations $\mathcal{S}_\mathcal{P}$, resp. $\mathcal{E}_\mathcal{P}$ for the sets of $\mathcal{P}$–supramedian, resp. excessive functions.

Proposition 2.2. *Let $s \in \mathcal{E}_\mathcal{P}$, $s' \in \mathcal{S}_\mathcal{P}$ and let $f \in \mathcal{F}$ be such that $f + s' = s$. Then each of the sets: $\{g \in \mathcal{S}_\mathcal{P} | g \geq f\}$ and $\{g \in \mathcal{E}_\mathcal{P} | g \geq f\}$ has a smallest element, which are in fact equal and independent of the choice of f.*

As usual, these reduced functions will be denoted by $R^s(s - s')$, resp. $R^e(s - s')$.

PROOF: (G. Mokobodzki). Let us denote the elements of $G \in \mathcal{G}$ as: $G = \{\alpha_1, \alpha_2, \ldots, \alpha_n, \ldots\}$. We define inductively the sequence $(f_n)_n$ as follows:

$$f_1 = f; \quad f_{n+1} = \max(f_n, P_{\alpha_1} f_n, \ldots, P_{\alpha_n} f_n)$$

The sequence $(f_n)_n$ is clearly increasing. Let us denote $f^G := \sup_n f_n$. For each fixed $k \in \mathrm{N}$ and each $n \geq k$, we have:

$$P_{\alpha_k}(f^G) = P_{\alpha_k}\left(\sup_n f_n\right) = \sup_n P_{\alpha_k} f_n \leq \sup_n f_{n+1} = f^G$$

which means that $P_t f^G \leq f^G, \forall t \in G$. Clearly $f^G \geq f$.

The problem is to obtain the same property for any $t > 0$.

Let us denote by $\mathcal{A}$ the set of all functions $f \in \mathcal{F}$, for which there exist $s \in \mathcal{E}_\mathcal{P}$ and $s' \in \mathcal{S}_\mathcal{P}$ such that $f + s' = s$. It is clear that $f \in \mathcal{A} \Longrightarrow P_t f \in \mathcal{A}, \forall t > 0$, since $s \in \mathcal{S}_\mathcal{P}$, resp. $\mathcal{E}_\mathcal{P} \Longrightarrow P_t s \in \mathcal{S}_\mathcal{P}$, resp. $\mathcal{E}_\mathcal{P}$. Moreover, taking the limit for $t \searrow 0$, it results that $\lim_{t \searrow 0} P_t f$ exists and is $\geq f$, except for a $\mathcal{P}$–negligible set (contained in $[\hat{s}' = \infty]$, where by $\hat{s}'$ we denote the excessive regularization of s').

$\mathcal{A}$ is also closed for max. Indeed: $f + s' = s$ and $g + s_1' = s_1 \Longrightarrow s + s_1 = \max(f, g) + \min(s + s_1', s_1 + s')$ where $s + s_1 \in \mathcal{E}_\mathcal{P}$, $\min(s + s_1', s_1 + s') \in \mathcal{S}_\mathcal{P}$.

Using these remarks, we obtain that $f \in \mathcal{A} \Longrightarrow f_n \in \mathcal{A}, \forall n \in \mathrm{N}$. Let us consider now $t > 0$; let $t_n \in G$, $t_n \searrow t$. If we write $t_n = t + h_n$, it follows that $h_n \searrow 0$; moreover, we have $P_{t_n} f^G \leq f^G$, $\forall n$. Hence: $f^G \geq P_{t_n} f^G = P_t(P_{h_n} f^G)$; by Fatou's lemma:

$$f^G \geq \liminf_n P_t\left[P_{h_n} f^G\right] \geq P_t\left[\liminf_n P_{h_n} f^G\right] \geq P_t\left[\liminf_n P_{h_n} f_m\right], \forall m$$

while $\liminf_n P_{h_n} f_m \geq f_m$, except for a $\mathcal{P}$-negligible set. So, we may continue with:

$$\geq P_t \left[\liminf_n P_{h_n} f_m\right] \geq P_t f_m$$

everywhere; finally, we get $f^G \geq P_t f^G$.

Hence, for each $f \in \mathcal{A}$, the function f^G has the properties: $f^G \geq f$, $f^G \in \mathcal{S}_\mathcal{P}$. Conversely, for $g \in \mathcal{S}_\mathcal{P}$ such that $g \geq f$ we get $g \geq P_t g \geq P_t f$, $\forall t > 0$, hence $g \geq f_n$, $\forall n$, which gives finally $g \geq f^G$.

Let us denote now:

$$R^s f = \inf\{g \in \mathcal{S}_\mathcal{P} | g \geq f\}$$

It was proved that $R^s f = f^G$, meaning that $R^s f \in \mathcal{S}_\mathcal{P}$. In other words, $R^s f \geq f$ exists and is the smallest element of the set $\{g \in \mathcal{S}_\mathcal{P} | g \geq f\}$. Moreover, $R^s f$ is the same, for any $G \in \mathcal{G}$ and any f (of course, such that $f + s' = s$). □

Remark 2.3. *a*) Let us consider the set $\mathcal{A}'$ of all functions $h \in \mathcal{F}$ for which $\liminf_{t\to 0} P_t h \geq h$, or equivalently: $\limsup_{n\to+\infty} P_{s_n} h \geq h$, for any sequence $s_n \searrow 0$. Again with Fatou's lemma, we get $h \in \mathcal{A}' \Longrightarrow P_t h \in \mathcal{A}'$. It follows that starting with $f \in \mathcal{A}'$ we have $P_{t_n} f \leq f$, $\forall t_n \in G$, and, writing successively, for each $t > 0$: $t_n = t + h_n$ we get:

$$P_{h_n}(P_t f) = P_{t_n} f \leq f$$

hence $\limsup_n P_{h_n}(P_t f) \leq f$; since $P_t f \in \mathcal{A}'$, we obtain as a conclusion

$$P_t f \leq \limsup_n P_{h_n}(P_t f) \leq f$$

The set of all functions with the property $\limsup_{t\to 0} P_t f \geq f$ is **not** stable for P_t, but is usefull when dealing with the reduced functions in excessives (see also condition (a) from [10]). Indeed, for such functions we have:

$$\hat{f^G} = \lim_{t\to 0} P_t f^G \geq \limsup_{t\to 0} P_t f \geq f$$

while $\hat{f^G} \in \mathcal{E}_\mathcal{P}$.

b) There exist other cases in which the reduced function also exists.

In the special case $\mathcal{E} = \mathcal{P}(E)$ (for example, when E is at most countable) $\mathcal{F}$ is a complete lattice and $\vee$, $\wedge$ are computed pointwise. In this case, for arbitrary (numerical, positive) function f on E, dominated by an element fo $\mathcal{S}_\mathcal{P}$, one consider the increasing family (f^G) and define further $\overline{f} = \sup f^G$. If we suppose moreover that

$$P_t\left(\sup_i f_i\right) = \sup_i P_t f_i$$

we get then obviously the smallest element of the set: $\{g \in \mathcal{S}_\mathcal{P} | g \geq f\}$, which means that the reduced always exists.

A remarkable example is the translations semi–group on $\mathbb{R}$: it extends naturally to all numerical, positive functions. The supramedian functions are necessarily measurable, since they are monotone.

The same example shows the necessity of a hypothesis in (2.2) (as well as the fact that we have to consider the whole family $(f^G)_G$). Indeed, the Dirichlet function u is invariant for any rational translation, hence $u^G = u$. However, the reduced function is identically 1.

As another example, let us consider $g := 1 + \chi_{(-\infty,0]}$ (which is supramedian, but not excessive for the translation semi–group) and $f \equiv 1$ (which is excessive and continuous), then there is no reduced function for $g - f$ in the cone of excessives.

c) L. Beznea pointed out the usefullness of the very general framework of "maisons de jeu" [7]: the existence (in fact, the measurability) of the reduced function is proved, using an analiticity argument. In order to use this result in our situation, let us call a semi–group $\mathcal{P}$ analytic if the set:

$$\{(x, \varepsilon_x P_t) | t \geq 0\} \subseteq X \times \mathcal{M}_1^+$$

(where we accept that $P_0 := I$) is analytic (for example, when X is a compact metrisable space, and $\mathcal{M}_1^+$ is endowed with the "etroite" topology).

Particularly, when $(E, \mathcal{E})$ is a Suslin space, and the semi–group is measurable, then it results analytic [1][prop. 5.2.2., pg. 402] or [7][ch. X, pg. 69].

In such a situation, it follows that, for any $f \in \mathcal{F}$, dominated by an element of $\mathcal{S}_\mathcal{P}$, $R^s f$ exists and is also from $\mathcal{F}$.

Proposition 2.4. *Using the hypotheses and the notations from (2.2), there exists* $s_1 \in \mathcal{S}_\mathcal{P}$ *such that* $R^s(s - s') + s_1 = s$

PROOF: As in the general proof [10], we show by induction that, for each fixed $G \in \mathcal{G}$:

$$P_t s + f_n \leq s + P_t f^G, \ \forall t \in G$$

which gives:

$$P_t s + f^G \leq s + P_t f^G$$

Since clearly $R^s(s - s') \leq s$, there exists $g \in \mathcal{F}$ for which $g + R^s(s - s') = s$. Let us choose $g = \infty$ on the set $[s = \infty]$.

From $P_t g + P_t f^G = P_t s$ we get:

$$P_t g + P_t f^G + s = P_t s + g + f^G$$

$$g + f^G + P_t s \geq P_t g + P_t s + f^G$$

hence $g \geq P_t g$ except for the set $[s = \infty]$; in fact $g \in \mathcal{S}_\mathcal{P}$ by the choice of g. This relation holds for any $t > 0$, by the independence of G of the construction. ☐

Remark 2.5. When we construct the family (f^G) and work with $\overline{f}$ (as in (2. 3) a), let us select a function $g_G \in \mathcal{F}$ such that $g_G + f^G = s$. At the points where $f^G(x) < +\infty$ we have $P_t f^G(x) < +\infty$: hence $P_t g_G(x) \leq g_G(x)$. When $f^G(x) = +\infty$, we shall define $g_G(x) = +\infty$; then $P_t g_G \leq g_G$, $\forall t \in G$ holds everywhere.

We prove that the family $(g_G)_G$ is decreasing, considering again the two cases. Let us denote $\underline{g} \in \mathcal{F}$ a function for which $\overline{f} + \underline{g} = s$ and defined as $\underline{g}(x) = +\infty$ at each point such that $\overline{f}(x) = +\infty$. Again: $f^G + g_G = s = \overline{f} + \underline{g} \geq f^G + \underline{g}$ proves that $g_G \geq \underline{g}$ everywhere and for each $G \in \mathcal{G}$.

We get $\underline{g}(x) = \inf\limits_{G \in \mathcal{G}} g_G(x)$, $\forall x$. Indeed, let us denote $h(x) := \inf\limits_{G \in \mathcal{G}} g_G(x)$. It follows: $h + f^G \leq g_G + f^G = s = \overline{f} + \underline{g}$, hence $h + \overline{f} \leq \underline{g} + \overline{f}$ and finally $h \leq \underline{g}$ everywhere.

Finally, since for any $t > 0$ and $G \in \mathcal{G}$ there exists $G' \in \mathcal{G}$, $G \subseteq G'$ such that $t \in G'$ we conclude that: $P_t\underline{g} \leq P_t g_{G'} \leq g_{G'}$ hence $P_t\underline{g} \leq \underline{g}$, $\forall t > 0$, obtaining just the conclusion: $\underline{g} \in \mathcal{S}_{\mathcal{P}}$ and $\underline{g} + R^s(s - t) = s$.

This proof covers the cases considered in Remark (2.3).

3 Classes of functions

(3.1) Let (X, τ) be a topological space. $\mathcal{I}$ will denote the set of all l.s.c. functions $f : X \to [0, +\infty)$ such that the set $[f = +\infty]$ is of the first category. Clearly, $\mathcal{I}$ is an ordered convex cone with respect to the pointwise operations and order (where $0 \cdot +\infty = 0$).

For $f, g \in \mathcal{I}$, $f \leq g$, there does not exists necessarily $h \in \mathcal{I}$ such that $f + h = g$ (hence the specific order is different from the pointwise order on $\mathcal{I}$).

For any increasing and dominated family (f_i) from $\mathcal{I}$, $\sup f_i \in \mathcal{I}$, hence $\bigvee_i f_i$ exists and moreover $f + \bigvee_i f_i = \bigvee_i (f + f_i)$.

For each positive, Borel measure μ on $(X, \mathcal{B})$ and any $f \in \mathcal{I}$, $\mu(f)$ makes sense and one obtains in this way a map $\mu : \mathcal{I} \to [0, +\infty]$ with the properties:

(1) $\mu(f + g) = \mu(f) + \mu(g)$

(2) $f \leq g \Longrightarrow \mu(f) \leq \mu(g)$

(3) $f_n \nearrow f \Longrightarrow \mu(f_n) \nearrow \mu(f)$

If (X, τ) is locally Lindelöf (LL) [7], or if (X, τ) is locally compact and μ is a Radon measure, then the stronger property holds:

$$f_i \nearrow f \Longrightarrow \mu(f_i) \nearrow \mu(f).$$

For a σ-finite measure μ:

$$\forall f \in \mathcal{I} \exists f_n \nearrow f \text{ such that } \mu(f_n) < +\infty \; \forall n.$$

Conversely, let $\mu : \mathcal{I} \to [0, +\infty]$ be a map with the above properties (1) – (3). If, moreover (X, τ) is compact and $\mu(1) < +\infty$, then $\mu|_{\mathcal{C}^+}$ is a Radon measure and its extension to $\mathcal{I}$ is unique.

For a kernel V, we have the following equivalent properties:

(a) $V : \mathcal{I} \to \mathcal{I}$ and $V1$ is continuous.

(b) $V : \mathcal{C}^+ \to \mathcal{C}^+$.

Verification. $(a) \Longrightarrow (b)$ Let f be a continuous and bounded function. Writing $M = f + (M - f)$, we get $MV1 = Vf + V(M - f)$, since $V1$ is continuous and Vf, $V(M - f)$ are l.s.c. functions, they result necessarily continuous.

In fact, such a kernel maps l.s.c. bounded functions into continuous functions.

$(b) \Longrightarrow (a)$ Let us consider $f \in \mathcal{I}$; there exists $g_i \nearrow f$, such that $g_i \in \mathcal{C}^+$, hence $Vg_i \nearrow Vf$, while the fact that Vg_i are continuous imply that Vf is a l.s.c. function.

We consider now a s.d.s.a.s. $(\mathcal{I}, \Phi)$. Then Φ induces a unique semi-group of kernels on the separable, compact space (X, τ).

The measurability follows from [11], using $\mathcal{H} := \{f \in \mathcal{F} | P_t f \text{ measurable }\}$ and as $\mathcal{C}$ the set of (positive) continuous functions.

Conversely, each semi-group of (borelian) kernels, for which $f \in \mathcal{I} \Longrightarrow P_t f \in \mathcal{I}$ will define a s.d.s.a.s. on $\mathcal{I}$.

Changing the base cone may result in dramatic loses of properties. The next example (replacing $\mathcal{F}$ with $\mathcal{I}$) shows that one cannot drop the hypotheses from (2. 2) even when $s - s'$ has a clear meaning.

In fact, the reduced function of a difference (as the smallest supramedian element, dominating the difference) may well not exists.

Let us consider the cone of l.s.c. functions on $\mathbb{R}$, with the translations semi-group (it clearly acts on this cone and do posses all good qualities). Let us choose $s :\equiv 1$ and $s' := \chi_{(0,+\infty)}$

Then the difference $s - s'$ cannot have a reduced: suppose that a smallest l. s. c. decreasing function f would exists, such that $f \geq s - s'$. Then necessarily $f = \chi_{(-\infty,0]}$, which is not l.s.c. This example shows that a reduced function may exists in the cone of supramedian elements, but may not exists in the cone of excessive elements (even if such a reduced exists in the larger cone $\mathcal{F}$ for differences of excessive functions).

However, when the semi-group is associated with a strongly Feller resolvent then the excessive functions are l.s.c., hence the final result (i. e. Riesz' splitting property) still holds.

Let us mention also two general situations in which the Riesz' splitting proprty is preserved.

(i) If $X_1 \subseteq X$ is a sub-cone with the properties:

$$s \in X_1, s = s_1 + s_2 \Longrightarrow s_1, s_2 \in X_1$$

$$s \in X_1 \Longrightarrow \Phi(t, s) \in X_1, \forall t > 0$$

Then, if X_s posseses Riesz' splitting property, then X_s^1 has the same property.

(ii) Let $\varphi : X \to X_1$ be a morphism [12] with the properties:
$\varphi : X_s \to X_s^1$ is surjective;

$$u, v \in X \Longrightarrow \varphi(u \wedge v) = \varphi(u) \wedge \varphi(v)$$

Then, if X_s posseses Riesz' splitting property, then X_s^1 has the same property.

(3.2) Let us replace the cone $\mathcal{I}$ with the following one. $\mathcal{D}$ will denote the set of all functions $f : X \to [0, +\infty]$ for which there exist a sequence (f_n) of u.s.c. functions, such that $f = \sup_n f_n$ and $[f = +\infty]$ is a set of the first category.

Of course, we may suppose the sequence increasing and the functions f_n may be supposed bounded.

For a uniformisable and with countable base space (X, τ), if $f, g \in \mathcal{I}$, $f \leq g$ and $f < +\infty$, then $g - f \in \mathcal{D}$.

Clearly, $\mathcal{D}$ is a convex cone and a lattice; moreover for any increasing sequence (f_n) from $\mathcal{D}$ we have also $\sup_n f_n \in \mathcal{D}$.

The interest of this cone may be seen from the next:

Lemma 3.3. *For each $f \in \mathcal{D}$ let us denote by $\hat{f}$ the l.s.c. regularization of f. Then the set $[f \neq \hat{f}]$ is of the first category.*

PROOF: Indeed, let us consider $f \in \mathcal{D}$ and f_n u.s.c functions, such that $f = \sup_n f_n$. The set $[f > \hat{f}]$ may be written as

$$\bigcup_{m,n=1}^{\infty} \left[f_n \geq \frac{1}{m} + \hat{f}\right]$$

since: $f(x) \geq f_n(x) \geq \frac{1}{m} + \hat{f}(x) > \hat{f}(x)$.

Conversely, for a fixed x, let us choose m for which $\frac{2}{m} \leq f(x) - \hat{f}(x)$. Choose next n such that $f(x) \leq f_n(x) + \frac{1}{m}$; now: $f_n(x) \geq f(x) - \frac{1}{m} \geq 2/m + \hat{f}(x) - 1/m \geq 1/m + \hat{f}(x)$.

Next, each set $[f_n \geq \frac{1}{m} + \hat{f}]$ is closed. Let us suppose that there exists an open set $D \neq \emptyset$ for which:

$$D \subseteq [f_n \geq \frac{1}{m} + \hat{f}] \subseteq \left[f \geq \frac{1}{m} + \hat{f}\right].$$

Then:

$$\inf_{x \in D} f(x) \geq \frac{1}{m} + \inf_{x \in D} \hat{f}(x)$$

hence, for any $y \in D$ we get: $\hat{f}(y) \geq \frac{1}{m} + \inf_{x \in D} \hat{f}(x)$.

This is a contradiction, since we may choose $y \in D$ such that

$$\frac{1}{m} + \inf_{x \in D} \hat{f}(x) > \hat{f}(y)$$

□

From this property, we obtain:

$f, g \in \mathcal{D} \Longrightarrow \widehat{f+g} = \hat{f} + \hat{g}$, except a set of the first category

$f, g \in \mathcal{D}, f \leq g$ except a set of the first category $\Longrightarrow \hat{f} \leq \hat{g}$ except a set of the first category.

$f_n \in \mathcal{D}, f_n \nearrow f \Longrightarrow \hat{f}_n \nearrow \hat{f}$ except a set of the first category.

We remark also that $f \in \mathcal{D} \Longrightarrow \hat{f} \in \mathcal{I}$ and of course $f = \hat{f}$, except a set of the first category.

Hence, defining an equivalence relation as $f \sim g \overset{def}{\Longleftrightarrow} \hat{f} = \hat{g}$, the quotient sets $\mathcal{D}/\sim$ and $\mathcal{I}/\sim$ do coincide. Moreover, if (X, τ) is a Baire space, then $\mathcal{C}^+$ is canonically embedded in $\mathcal{I}/\sim$.

As a conclusion, $\mathcal{I}/\sim$ is an ordered convex cone and a conditional $\sigma-$ complete lattice.

(3.4) Let us consider (see also [1] for the case of a so called Dirichlet functional space) a general situation. E resp. $(E, \mathcal{E})$ will denote a set, resp. a measurable space. $\mathcal{N}$ will be a collection of "negligible" subsets of E:

$$A \subseteq B, B \in \mathcal{N} \Longrightarrow A \in \mathcal{N}$$

$$\emptyset \in \mathcal{N}$$

$$A_n \in \mathcal{N}, \forall n \Longrightarrow \bigcup_{n=1}^{\infty} A_n \in \mathcal{N}$$

Example. (i) The subsets which are at most countable.

(ii) The subsets which are μ–negligible, for each μ from a fixed family of measures. As a particular case: the $\mathcal{P}$–negligible parts (where $\mathcal{P}$ stands for a semi–group of kernels).

(iii) In the case of a "harmonic structure" (such as: a harmonic space [4], an H–cone [2], a standard space of balayage [3]) the polar or the semi–polar sets.

(iv) In the case of a topological space, the sets of the first category (meager).

In such a situation, we will consider a s.d.s.a.s. on one of the cones: classes of numerical, positive ($\mathcal{E}$–measurable) functions, with respect to the equality $\mathcal{N}$–a.e., with the usual ("pointwise") operations and order; (positive) measures, which neglects any set from $\mathcal{N}$.

There is a canonical embedding (a morphism) of the dual of the quotient cone $\mathcal{F}/\mathcal{N}$ into the dual of the cone $\mathcal{F}$; the image $\mathcal{M}_{\mathcal{N}}$ consists of those elements $\mu \in \mathcal{F}^*$ for which: $[f \neq g] \in \mathcal{N} \Longrightarrow \mu(f) = \mu(g)$. Indeed, if $m : \mathcal{F}/\mathcal{N} \to [0, +\infty]$, then defining $\tilde{m}(f) := m(\hat{f})$, one obtains $\tilde{m} \in \mathcal{F}^*$ with the desired property. Conversely, let $\mu \in \mathcal{F}^*$ with the above property. The definition $\hat{\mu}(\hat{f}) := \mu(f)$ is correct and moreover $\hat{\mu} \in \mathcal{F}/\mathcal{N}$. Finally, the two correspondences are inverse one another: $\tilde{\hat{\mu}}(f) = \hat{\mu}(\hat{f}) = \mu(f)$ while $\hat{\tilde{m}}(\hat{f}) = \tilde{m}(f) = m(\hat{f})$.

Let us remark that the dual of $\mathcal{F}_{\mathcal{N}}$ separates iff $\mathcal{N}$ contains each $A \in \mathcal{E}$ such that $\mu(A) = 0$, $\forall \mu \in \mathcal{M}_{\mathcal{N}}$.

(3.5) A semi–group of kernels $\mathcal{P}$ induces such a s.d.s.a.s. on $\mathcal{F}/\mathcal{N}$ if we suppose the following compatibility property:

$$f \leq g, \mathcal{N}\text{–a.e.} \Longrightarrow P_t f \leq P_t g, \mathcal{N}\text{–a.e.}\ , \forall t > 0$$

Such a property holds, for instance, if we suppose that:

$$f = 0, \mathcal{N}\text{–a.e.} \Longrightarrow P_t f = 0, \mathcal{N}\text{–a.e.}\ , \forall t > 0$$

($\mathcal{P}$ is "$\mathcal{N}$–absolutely continuous").

We shall impose the following "separability" condition:

$\forall$ increasing family $(f_i)_i$, there exists an increasing sequence $(f_{i_n})_n$, for which:

$$\sup_n f_{i_n} \geq f_i, \mathcal{N}\text{– a. e.}, \forall i$$

$(\mathcal{F}/\mathcal{N})_s$ will denote the set of elements $s \in \mathcal{F}/\mathcal{N}$, which are finite $\mathcal{N}$–a.e. and $P_t s \leq s$, $\forall t > 0$.

Let us discuss the Riesz' splitting property in this situation.

Proposition. *Let $f \in \mathcal{F}/\mathcal{N}$ be dominated by an element from $(\mathcal{F}/\mathcal{N})_s$. Then the set $\{g | g \in (\mathcal{F}/\mathcal{N})_s, g \geq f\}$ has the smallest element, denoted as $R^s f$.*

PROOF: Since max for finite parts and sup on increasing sequences exist and are pointwise computed, there is nothing new in the construction of f^G, as well as the extension of $P_t f^G \leq f^G$, $\forall t > 0$.

For the increasing family $(f^G)_G$ there exists the increasing sequence $(f^{G_n})_n$ such that:

$$\overline{f} := \sup_G f^G = \sup_n f^{G_n}$$

Now:

$$P_t f^G \leq P_t f^{G'} \leq f^{G'} \leq \overline{f}, \forall G$$

$$P_t \overline{f} = P_t(\sup_G f^G) = P_t(\sup_n f^{G_n}) = \sup_n P_t f^{G_n} \leq \overline{f}$$

hence $P_t \overline{f} \leq \overline{f}$, $\forall t > 0$. The conclusion is that $R^s f = \overline{f}$. □

(3.6) Let us return now to the Riesz' splitting property in the cone $\mathcal{I}/\sim$.

We suppose either that X is locally Lindelöff, or that: each kernel P_t commute with increasing arbitrary sup and that

A set of the first category $\Longrightarrow P_t(\chi_A) \equiv 0$

(of course, this second hypothesis is strictly stronger than the preceding one, as the example of the translation semi-group shows).

Moreover, we suppose that $P_t : \mathcal{I} \to \mathcal{I}$ has the next property:

$\forall$ A set of the first category $\Longrightarrow P_t(\chi_A) = 0$ except for a set of the first category. We still denote by $\mathcal{S}_\mathcal{P}$ the set of elements $s_1 \in \mathcal{I}/\sim$ such that $P_t s \leq s$, $\forall t > 0$.

Proposition. *Under the hypotheses and with the above notations, there exists $s_1 \in \mathcal{S}_\mathcal{P}$ such that $R^s(f - g) + s_1 = f$.*

PROOF: We may repeat the proofs as in Section 2, adding "except for a set of the first category" and providing separate arguments where non-countable procedures appear.

Starting with $f, g \in \mathcal{S}_\mathcal{P}$ choose $h \in \mathcal{D}$ such that $f = g + h$ and construct successively h_n and $h^G \in \mathcal{D}$ for each G; we have $P_t(h^G) \leq h^G$, $\forall t \in G$. Next, we consider the family $\hat{h}^G$ from $\mathcal{I}$ and denote $\overline{h} := \sup_G \hat{h}^G \in \mathcal{I}$.

Under each assumption, we obtain that $\overline{h} \in \mathcal{S}_\mathcal{P}$. Following the construction, we get next $g + \hat{h}^G \geq f$, except for a set of the first category, hence $g + \overline{h} \geq f$ except for a set of the first category. Conversely, let $u \in \mathcal{S}_\mathcal{P}$ be such that $g + u \geq f$ except for a set of the first category; we obtain successively: $u \geq h^G$ except for a set of the first category, hence $u \geq \overline{h}$ except for a set of the first category.

In conclusion, under the specified hypotheses, $\overline{h}$ is the smallest element from $\mathcal{S}_\mathcal{P}$ for which $g + \overline{h} \geq f$ except for a set of the first category. In other words, it equals the reduced function in $\mathcal{I}/\sim$ for $f - g$.

Let us consider now the second part of the proof. For each $t \in G$ we have:

$$P_t f + h_n \leq f + P_t(h^G)$$

except for a set of the first category. It follows that:

$$P_t f + h^{\hat{G}} \leq f + P_t(h^{\hat{G}})$$

except for a set of the first category. Since $h^{\hat{G}} \leq f$, there exists a function g_G (which may be supposed to belong to $\mathcal{I}$), such that $f = h^{\hat{G}} + g_G$ except for a set of the first category.

Hence we have proved that, except for a set of the first category, for each $t \in G$ we have $P_t g_G \leq g_G$. The family $(h^{\hat{G}})$ being increasing, it results that the family (g_G) is decreasing (meaning that $G \subseteq G' \Longrightarrow g_{G'} \leq g_G$ except for a set of the first category). There exists then a countable family (G_n) such that $\overline{h} = \sup_n \widehat{h^{G_n}}$. Accordingly: $\overline{h} + \underline{g} \geq h^{\hat{G}_m} + \inf_n g_{G_n}$, hence $\overline{h} + \underline{g} \geq \sup_m \widehat{h^{G_m}} + \inf_n g_{G_n}$, and we get $\underline{g} \geq \inf_n g_{G_n}$, of course, except for a set of the first category. The converse inequality being clear, it follows that $\underline{g} = \inf_n g_{G_n}$ except for a set of the first category (however, it could happen that $\underline{g} > \inf_G g_G$, since there is an uncountable family of sets of the first category).

On the other hand, there is a countable family such that $\widehat{\inf_G g_G} = \widehat{\inf_n g_{G_n}}$. Taking the union of the two countable sets, we get: $\underline{g} = \hat{\underline{g}} \geq \widehat{\inf_n g_{G_n}} = \widehat{\inf_G g_G}$, hence we still have $\underline{g} \leq \widehat{\inf_n g_{G_n}}$, except for a set of the first category, so we even have $\underline{g} = \widehat{\inf_n g_{G_n}}$, except for a set of the first category.

Now $P_t \underline{g} \leq P_t g_{G'} \leq g_{G'}$, except for a set of the first category, for each $t > 0$. Hence $P_t \underline{g} \leq \underline{g}$. The second part is thus proved: $R^s(f - g) + \underline{g} = f$, with $\underline{g} \in \mathcal{S_P}$. $\square$

4 The stonean case

(4.1) We present next a consistent example, where we are forced to consider sup for increasing uncountable families, without the possibilty to reduce to a countable subfamily.

Let (X, τ) be a stonean compact space. The set $\mathcal{C}^+$ of continuous, positive functions on X, finite on a dense set, is an ordered convex cone and (see [6] , [2]) a conditionally complete lattice. $\bigvee_i f_i = (\sup_i f_i)^\vee$ is the u. s. c. regularization of the (l.s.c.) function $\sup_i f_i$. Each kernel $V : \mathcal{C}^+ \to \mathcal{C}^+$ is in fact a true kernel on $(X, \mathcal{B})$, with the additional property: $f \in \mathcal{C}^+ \Longrightarrow Vf \in \mathcal{C}^+$. Particularly, each $\varepsilon_x V$ being a Radon measure, we have: $V(\sup_i f_i) = \sup_i Vf_i$, for any increasing and dominated family $(f_i)_i$.

We have to ask moreover $V\left(\bigvee_i f_i\right) = \bigvee_i V(f_i)$. This property is equivalent to the fact that each $\varepsilon_x V$ is a *normal* measure (meaning that it neglects any set of the first category).

We will consider semi-groups (i.e. s.d.s.a.s. on $\mathcal{C}^+$) $\mathcal{P} = (P_t)_{t>0}$, each kernel having this additional property; and will denote: $\mathcal{S_P} := \{f \in \mathcal{C}^+ | P_t f \leq f, \forall t > 0\}$.

(4.2) We prove next the existence of the reduced function and the Riesz' property in the frame defined above:

Proposition. *For each $f \in \mathcal{C}^+$ dominated by an element from $\mathcal{S_P}$, the set $\{g \in \mathcal{S_P} | g \geq f\}$ has the smallest element, denoted as $R^s f$.*

For $f, g \in \mathcal{S_P}$, $f \geq g$, there exists $\underline{g} \in \mathcal{S_P}$ such that $R^s(f - g) + \underline{g} = f$.

PROOF: Starting with $f \in \mathcal{C}^+$, we construct, as in §2, the increasing sequence f_n and obtain $f^G \in \mathcal{C}^+$. The family $(f^G)_G$ is increasing. Let us denote $\overline{f} := \bigvee_G f^G = (\sup_G f^G)^{\vee}$. Clearly $\overline{f} \geq f$; $P_t\overline{f} = P_t\left(\bigvee_G f^G\right) = \bigvee_G P_t f^G \leq \bigvee_G P_t f^{G'} \leq \bigvee_G f^{G'} = \overline{f}$, $\forall t > 0$. Conversely, let us consider $u \in \mathcal{S}_\mathcal{P}$ such that $u \geq f$. We obtain $u \geq P_t u \geq P_t f$, hence $u \geq f_n$ and $u \geq f^G$, so $u \geq \overline{f}$, which proves that $R^s f$ exists and equals $\overline{f}$.

It makes sense to define $h := f - g$ as an element of $\mathcal{C}^+$: since we define a l.s.c. function extended by 0 at all points at which g is infinite, and next u. s. c. regularized. We get in this way a continuous function, which still verifies $h + g = f$.

For the second part, let us consider $f, g \in \mathcal{S}_\mathcal{P}$, $f \geq g$ and let us denote $h := f - g \in \mathcal{C}^+$.

As already proved, for any $t \in G$: $P_t f + h^G \leq f + P_t(h^G)$. Let us fix a $t > 0$ and consider all the parts G', containing t. Since $\bigvee\limits_G h^G = \bigvee\limits_G h^{G'}$, we deduce $P_t f + h^G \leq f + \bigvee\limits_G P_t(h^G) = f + P_t(\overline{h})$ for each $t \in G$. Moreover, we get: $P_t f + \overline{h} \leq f + P_t(\overline{h})$, for any $t > 0$. If $\underline{g} \in \mathcal{C}^+$ satisfies $\underline{g} + \overline{h} = f$, then $P_t\underline{g} + P_t\overline{h} + \overline{h} \leq \underline{g} + \overline{h} + P_t\overline{h}$, hence $P_t\underline{g} \leq \underline{g}$, $\forall t > 0$. □

5 Measures

(5.1) Let $(E, \mathcal{E})$ be a measurable space. $\mathcal{M}$ will denote the set of all positive measures on $(E, \mathcal{E})$, with the pointwise structure of an ordered convex cone (see [1] for the lattice properties).

Proposition. *Suppose that $\mathcal{E}$ is countable generated. Then, for each increasing family $(\mu_i)_i$ of measures, dominated by a σ-finite measure, there exists an increasing sequence $(\mu_{i_n})_n$ for which $\bigvee_i \mu_i = \bigvee_n \mu_{i_n}$.*

$\mathcal{F}$ may be canonically embedded into the dual $\mathcal{M}^*$. As it will be seen, more generally for kernels, the elements from the image are characterized by additional properties of measurability and (vague) continuity.

We will consider kernels on $\mathcal{M}$, meaning maps $V : \mathcal{M} \to \mathcal{M}$ with the properties:

$$V(\mu + \nu) = V\mu + V\nu$$

$$\mu \leq \nu \Longrightarrow V\mu \leq V\nu$$

$$\mu_n \nearrow \mu \Longrightarrow V\mu_n \nearrow V\mu$$

This last property is equivalent with the next one:

$$V\left(\sum_{n=1}^{\infty} \mu_n\right) = \sum_{n=1}^{\infty} \mu_n$$

If $\mathcal{E}$ is countable generated, it is also equivalent with:

$$\mu_i \nearrow \mu \Longrightarrow V\mu_i \nearrow V\mu$$

As a specific property, we will consider:

$$(M) \quad x \mapsto V(\varepsilon_x)(A) \text{ is measurable } \forall A \in \mathcal{E}$$

(i) the kernels V on functions, with the property:

$$f \text{ continuous} \Longrightarrow Vf \text{ continuous}$$

and

(ii) the kernels V on measures, with the property:

$$\mu_i \to \mu \text{ vaguely } \Longrightarrow V\mu_i \to V\mu \text{ vaguely}$$

(5.7) Let us consider next a semi–group $(P_t)_{t>0}$ of kernels on measures (i.e. a s.d.s.a.s. on $\mathcal{M}$).

We have to suppose here that each P_t commutes with increasing arbitrary sup; such a property holds, for example, when for each increasing family (μ_i), there is an increasing sequence (μ_{i_n}), such that $\bigvee_i \mu_i = \bigvee_n \mu_{i_n}$ (when $\mathcal{E}$ is countably generated, cf. (5. 1)).

Let us denote by $\mathcal{M}_{\mathcal{P}}$ the cone of $\mathcal{P}$–supramedian measures, that is μ is archimedian and $P_t\mu \leq \mu$, $\forall t > 0$.

Proposition. *For each $\mu \in \mathcal{M}$ such that there exists $\mu_0 \in \mathcal{M}_{\mathcal{P}}$ with $\mu_0 \geq \mu$, the set $\{\nu \in \mathcal{M}_{\mathcal{P}} | \nu \geq \mu\}$ has the smallest element (denoted by $R^s\mu$).*

For each $\mu, \nu \in \mathcal{M}_{\mathcal{P}}$ with $\nu \leq \mu$, there exists $\nu' \in \mathcal{M}_{\mathcal{P}}$ such that $R^s(\mu - \nu) + \nu' = \mu$

PROOF: We define a sequence (depending also on G): $\mu_1 := \mu$; $\mu_{n+1} := \mu_n \vee P_{\alpha_1}\mu_n \vee \ldots \vee P_{\alpha_n}\mu_n$. This sequence is clearly increasing. Let us denote $\mu^G := \bigvee_n \mu_n$. For a fixed $k \in \mathbf{N}$, we may write, for any $n \geq k$:

$$P_{\alpha_k}\mu^G = P_{\alpha_k}\left(\sup_n \mu_n\right) = \sup_n P_{\alpha_k}\mu_n \leq \sup_n \mu_{n+1} = \mu^G$$

hence $P_t\mu^G \leq \mu^G$, $\forall t \in G$. Clearly $\mu^G \leq \mu_0$.

$(\mu^G)_{G\in\mathcal{G}}$ is an increasing family; hence, there exists $\overline{\mu} := \sup_{G\in\mathcal{G}} \mu^G$. For each $t > 0$ we may write:

$$P_t\overline{\mu} = P_t\left(\bigvee_G \mu^G\right) = \sup_G P_t\mu^G \leq \sup_G P_t\mu^{G'} \leq \sup_G \mu^{G'} \leq \overline{\mu}$$

Hence $R^s\mu$ exists and equals $\overline{\mu}$.

As for the second part: let $\mu, \nu \in \mathcal{M}_{\mathcal{P}}$, $\nu \leq \mu$. There exists a (not unique) $\varphi \in \mathcal{M}$ such that $\varphi + \nu = \mu$. We use the method above for φ (which is dominated by $\mu \in \mathcal{M}_{\mathcal{P}}$). That is, we consider the (increasing) sequence (φ_n); further $\varphi^G := \bigvee_n \varphi_n$ and finally $\overline{\varphi} := \bigvee_G \varphi^G$.

For each G fixed, we obtain by induction that

$$P_t\mu + \varphi_n \leq \mu + P_t\varphi^G,\ \forall t \in G$$

Let us denote $\overline{\varphi} := \bigvee_G \varphi^G$; we prove that $\overline{\varphi} = R^s(\mu - \nu)$, that is: $\overline{\varphi}$ is the smallest element of the set $\{\psi \in \mathcal{M}_{\mathcal{P}} | \psi + \nu \geq \mu\}$. Indeed, $\overline{\varphi}$ belongs clearly to the set; conversely, for any element ψ from the set then $\psi \geq \varphi$; by induction we get $\psi \geq \varphi_n$, $\forall n$, since $\psi \geq \varphi_n \Longrightarrow \psi \geq P_{\alpha_k}\psi \geq P_{\alpha_k}\varphi_n$. It results $\psi \geq \varphi^G, \forall G$, so that finally $\psi \geq \overline{\varphi}$. Especially, $\overline{\varphi}$ results independent of the choice of φ.

Since $\overline{\varphi} \leq \mu$ it follows that there exists $\nu' \in \mathcal{M}$(not unique!) such that $\overline{\varphi} + \nu' = \mu$. Let us prove that some version of ν' belongs to $\mathcal{M}_{\mathcal{P}}$.

For each G, from $\varphi^G \leq \mu$ we obtain φ_G with $\varphi^G + \varphi_G = \mu$. The property proved by induction gives now: $P_t\varphi^G + P_t\varphi_G = P_t\mu$ hence:

$$P_t\varphi^G + P_t\varphi_G + \varphi^G = P_t\mu + \varphi^G \leq \mu + P_t\varphi^G$$

so we get $P_t\varphi_G \leq \varphi_G$, $\forall t \in G$.

It follows that the family (φ_G) is decreasing. Moreover, we get

$$\overline{\varphi} + \bigwedge_G \varphi_G = \mu$$

Now $\nu' := \bigwedge\limits_G \varphi_G$ is the correct version, since for each $t > 0$:

$$P_t\nu' = P_t\left(\bigwedge_G \varphi_G\right) \leq P_t\varphi_{G_0}$$

$\forall G_0 \in \mathcal{G}$ with $t \in G_0$. Hence the conclusion. □

Remark 5.8. (*a*) Let us consider the case of a semi–group of true kernels $\mathcal{P}$; suppose moreover that there exists $g \leq 1$ such that $V_0 g > 0$ and $\mu_0(V_0 g) < +\infty$, ($\mathcal{V} = (V_\alpha)_{\alpha \geq 0}$ being the resolvent associated with the semi–group). Under such circumstances, it suffices to consider a fixed $G \in \mathcal{G}$, for example $G = \mathbf{Q}^+$.

We show first that, for any $h_n \searrow 0$, the sequence $P_{h_n}\mu^G$ is increasing. Let $h_n < h_m$, hence we may write $h_m = h_n + h$, with $h \in \mathbf{Q}^+$. We have:

$$P_{h_m}\mu^G = P_{h_n}(P_h\mu^G) \leq P_{h_n}\mu^G$$

Let us denote now

$$\nu := \bigvee_n P_{h_n}\mu^G \leq \mu^G$$

For any $f \in \mathcal{S}_{\mathcal{P}}$ we have:

$$\mu^G(f) \geq \nu(f) = \left(\bigvee_n P_{h_n}\mu^G\right)(f) = \sup_n \mu^G\left(P_{h_n}f\right) = \mu^G(\hat{f})$$

If $f = V_0 g$, then $f = \hat{f}$, hence $\mu^G(f) = \nu(f)$, so that $\nu = \mu^G$.

Let $t > 0$ and $t_n \searrow t$ be a sequence of rational numbers. Now $(P_{t_n}\mu^G)_n$ is increasing and moreover:

$$P_{t_n}\mu^G = P_t\left(P_{h_n}\mu^G\right) \nearrow P_t\mu^G$$

In this case $P_t\mu^G \leq \mu^G$, $\forall t > 0$. Hence μ^G is $\mathcal{P}$–supramedian (and even $\mathcal{P}$–excessive).

Defining now:

$$R^s\mu = \bigwedge\{\nu \in \mathcal{M}_\mathcal{P} | \nu \geq \mu\}$$

we have proved that $R^s\mu = \mu^G$, hence $R\mu \in \mathcal{M}_\mathcal{P}$, $R\mu \geq \mu$, meaning that $R^s\mu$ is the smallest element of the considered set.
Indeed, μ^G is $\mathcal{P}$–supramedian and $\mathcal{P}$–excessive, while $\mu^G \geq \mu$ by construction. For any measure ν which is $\mathcal{P}$–supramedian and $\nu \geq \mu$, we have $\nu \geq P_t\nu \geq P_t\mu$, hence further $\nu \geq \nu_n$; finally $\nu \geq \mu^G$.

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