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NONLINEAR ORDINARY DIFFERENTIAL EQUATIONS WITH DISCONTINUITIES

Classroom Notes

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1 Introduction

A differential equation is described as having a classical solution if for the equation, $x' = f(t, x, y)$ where $x(t_0) = \alpha$, there exists a solution $x_0(t)$ which is an absolutely continuous function, $x_0'(t) = f(t, x_0, y)$, and $x_0(t_0) = \alpha$.

However, not all differential equations have such classical solutions. One class of differential equations which do not have such solutions are those which are discontinuous due to a signum function in the right hand side. These functions have the advantage of being piece wise continuous, though the set of points at which the function is discontinuous might be uncountable.

A.F. Filippov established a new concept of existence of a solution to a differential equation which works especially well with this class of differential equations. In his work, he redefined the right hand side of the differential equation as a closed, convex set and changed the differential equation into a differential inclusion. Under certain conditions, existence of a Filippov solution can be established. The conditions considered in this paper are a modification of the Caratheodory conditions and are established in the main theorem as follows:

Theorem. *Let, in an open domain G, a vector-valued function $f(t, x)$ be measurable and almost everywhere satisfy the inequality:*

$$|f(t, x)| \leq m(t) \qquad \text{with } m(t) \text{ summable.}$$

Then for any point $(t_0, x_0) \in G$ there exists a Filippov solution of

$$x' = f(t, x), x(t_0) = x_0.$$

The solution is defined at least on the interval $[t_0 - d, t_0 + d]$, where d is such that the whole of a cylinder Z: $|t - t_0| \leq d$, $|x - x_0| \leq r$ is contained within the domain G, where $r = \max\left\{\int_{t_0-d}^{t_0} m(t)dt, \int_{t_0}^{t_0+d} m(t)dt\right\}$.

2 An Example

Consider the following differential equation

$$x'(t) = g(y) - \operatorname{sgn} x(t) = f(t,x,y), \quad x(0) = 0, \qquad t \in [0,t_0] \tag{1}$$

where $|g(y)| \leq 1$, and $g(y)$ is non constant. Notice that in this example, $g(y)$ is any function which is bounded by (1), such as $\sin(y)$ or $\cos(y)$. It can be shown, by contradiction, that this differential equation does not have a solution in the classical sense. That is, there does not exist a function $x_0(t)$ which is absolutely continuous and $x_0'(t) = f(t,x_0,y)$. Here, y stands for a known function $y = y(t)$.

Proof of claim: Suppose there does exist a solution x_0 of (1) such that $x_0 \in AC$ and $x_0'(t) = g(y) - \operatorname{sgn} x_0(t)$. Then $x_0(t) = \int_0^t g(y(s))\,ds - \int_0^t \operatorname{sgn} x_0(s)\,ds$. Since $x_0(0) = 0$ and $x_0 \in AC$, there exists a $t_1 \in (0,t]$ such that:

case 1: $x_0(s) > 0$, $s \in (0,t_1]$,
case 2: $x_0(s) < 0$, $s \in (0,t_1]$,
case 3: $x_0(s) = 0$, $s \in (0,t_1]$

If case 1 is true, it follows that

$$0 < x_0(t_1) = \int_0^{t_1} g(y(s))ds - \int_0^{t_1} \operatorname{sgn} x_0(s)ds \leq \int_0^{t_1} 1ds - \int_0^{t_1} 1ds = 0.$$

This is impossible.

If case 2 is true, it follows that

$$0 > x_0(t_1) = \int_0^{t_1} g(y(s))ds - \int_0^{t_1} \operatorname{sgn} x_0(s)ds \geq \int_0^{t_1} (-1)ds - \int_0^{t_1} (-1)ds = 0$$

This is also a contradiction.

If case 3 is true, it follows that

$$0 = x_0(t_1) = \int_0^{t_1} g(y(s))ds - \int_0^{t_1} \operatorname{sgn} x_0(s)ds \leq \int_0^{t_1} 1 - \int_0^{t_1} 0 = t_1.$$

This is possible, so that for $t \in (0,t_1]$, it must be true that $x_0(t) = 0$.

Now, suppose $t_1 < t_0$. Then there exists a t_2 such that

case 1: $x_0(s) > 0$, $s \in (t_1,t_2]$
case 2: $x_0(s) < 0$, $s \in (t_1,t_2]$

If case 1 is true, it follows that

$$\begin{aligned} 0 < x_0(t_2) &= \int_0^{t_2} \Big(g(y(s)) - \operatorname{sgn} x_0(s)\Big)ds = \\ &= \int_0^{t_1} \Big(g(y(s)) - \operatorname{sgn} x_0(s)\Big)ds + \int_{t_1}^{t_2} \Big(g(y(s)) - \operatorname{sgn} x_0(s)\Big)ds = \\ &= x_0(t_1) + \int_{t_1}^{t_2} g(y(s))ds - \int_{t_1}^{t_2} \operatorname{sgn} x_0(s)ds \leq \\ &\leq 0 + \int_{t_1}^{t_2} 1ds - \int_{t_1}^{t_2} 1ds \leq (t_2 - t_1) - (t_2 - t_1) = 0. \end{aligned}$$

Again, this is a contradiction and in the same manner, case 2 will lead to a contradiction. Therefore,

$$x_0(s) \equiv 0, \quad s \in [0, t_0]$$

and so,

$$x_0'(s) \equiv 0, \quad s \in [0, t_0]$$

and

$$g(s) = \operatorname{sgn} x_0(t) = 0, \quad s \in [0, t_0]$$

This contradicts the hypothesis that g is non-constant.

Therefore, there is no solution x_0 of (1) such that $x_0 \in AC$ and

$$x_0' = g(y(t)) - \operatorname{sgn} x_0(t).$$

□

3 Filippov Solutions

In order to find a solution to this differential equation, this paper will introduce the Filippov integral and the Filippov sense of solution of a differential equation. First, consider differential equations of the form:

$$x' = f(t, x), \quad x(t_0) = x_0. \tag{2}$$

where the function f is discontinuous. Specifically, consider the situation where the function f is piece-wise continuous, as in functions which contain a signum function. Thus, the set of points for which f is discontinuous is a set of measure 0. As in the example above, this differential equation does not always have a classical solution. In order to establish solutions for such differential equations, A.F. Filippov redefined the discontinuous function in the following way:

$$F(t, x) = \bigcap_{\delta > 0} \bigcap_{\mu N = 0} \overline{\text{co}} f(t, x^{\delta} \backslash N). \tag{3}$$

In other words, $F(t, x)$ is the smallest convex set which contains all the limit values of the function $f(t, x^*)$, where x^* tends toward x in a δ-neighborhood, excluding a set of measure zero. In this way, F is a multi-valued map. The map is single-valued at each point of continuity, but at the points, (t_0, x_0), of discontinuity of the function f, the map F "connects" the function f with a vertical line segment (in the case of $\mathbb{R}^2$), each point of which is in the set $F(t_0, x_0)$. Now, the function x is defined to be a solution of the differential equation (2) if $x \in AC$ and $x'(t) \in F\Big(t, x(t)\Big)$ a.e.

Note, however, that a Filippov solution is not unique. That is, the Filippov solution of a differential equation is a set of functions which satisfy the criteria. For example, given the following differential equation,

$$x' = \operatorname{sgn} x = f(t, x) \quad x(0) = 0$$

many solutions can be found. One solution for this equation is the function $x(t) = t$. This function is a Filippov solution as shown below:

$$x(0) = 0 \; x \in AC; \quad x'(t) = 1 \in F(t,x) \qquad \text{for } t \in [0, t_1].$$

Another possible solution is the function $x(t) = -t$, which also satisfies the differential equation: $x(0) = 0$; $x \in AC$

$$x'(t) = -1 \in F(t, x(t)) \qquad \text{for } t \in [0, t_1].$$

With this new definition of a solution, the solution of the differential equation is a set of functions which is the solution of the differential inclusion $x'(t) \in F(t,x)$.

Using the Filippov's redefinition of a solution of a differential equation, it is possible to extend the work of Caratheodory to establish criteria for existence of a Filippov solution. First, recall the theory of Caratheodory differential equations, specifically the Caratheodory conditions for a function f with domain D:

(1) The function $f(t,x)$ is defined and continuous in x for almost all t.
(2) The function $f(t,x)$ is measurable in t for each x.
(3) There exists a summable function $m(t)$ such that $|f(t,x)| \leq m(t)$.

The Caratheodory conditions will now be weakened to remove the requirement that the function f must be continuous in x almost everywhere. These are the Modified Caratheodory Conditions:

(1) The function $f(t,x)$ is measurable on the domain D.
(2) There exists a summable function $m(t)$ such that $|f(t,x)| \leq m(t)$ a.e.

Under the Modified Caratheodory Conditions, the definition of the multi-valued function $F(t,x)$ can be refined. As $m(t)$ is summable, there exists an $M < \infty$ such that $m(t) < M$, a.e. Let E be the set of points, t, for which this is true. Now, since the function $f(t,x)$ is measurable in the domain G, it is measurable also for almost all t on the cross-section G_t, defined by the intersection of the domain G with the plane $t =$ constant $\in E$. For each $t \in E$, the function $f(t,x)$, which is now considered a function of x only, is approximately continuous almost everywhere. The set, $N_0(t)$, for which this is not true, is a set of measure zero. For $t \in E$, we can write:

$$F(t,x) = \bigcap_{\delta > 0} \overline{\text{co}}\, f\Big(t, x^{\delta} \backslash N_0(t)\Big) \tag{4}$$

Defined in this way, the set $F(t,x)$ is non empty, bounded, closed, and convex. Note that for $t \in E$, the function defined in (4) is the same as the function defined in (3). For the remaining t (a set of measure 0), the set $F(t,x)$ will be defined as $\{M\}$. Note that if a function f satisfies the Caratheodory criteria, specifically, if f is continuous in x, then

$$F(t,x) = \{f(t,x)\}$$

and so under Caratheodory conditions, the Filippov and classical solutions are the same.

When considering differential equations whose right hand sides are piece-wise continuous, the problem in finding solutions only occurs at the points of discontinuity. Note that if the discontinuity is a jump discontinuity and the function value at the point of discontinuity (t_0, x_0) is such that:

$$\lim_{(t,x)\to(t_0,x_0)^+} f(t,x) \leq f(t_0,x_0) \leq \lim_{(t,x)\to(t_0,x_0)^-} f(t,x) \tag{5}$$

then a solution x_1 of the differential equation is such that

$$x_1'(t) = f(t, x_1) \in F(t, x_1)$$

for all points in the domain. Therefore, under these circumstances, a classical solution to the differential equation is also an element of the Filippov solution. The problems considered in this paper involve the signum function, and as signum is traditionally defined, discontinuities occur when the argument of the signum function equals 0. At these discontinuities, the function satisfies (5). Therefore for these differential equations, existence of a classical solution implies existence of a Filippov solution, and a classical solution is also a Filippov solution.

4 Existence of Filippov Solutions

In order to establish existence theorems for the Filippov sense of solution, some preliminary groundwork must be established.

Lemma 1. *[1] If a set M is bounded and closed, $v(t) \in M$ for $a \leq t \leq b$, then*

$$v_{\text{mean}} = \frac{1}{b-a}\int_a^b v(t)dt \in \text{co}M.$$

The same holds for the mean value of the vector-valued function $v(x)$ on any measurable set of finite measure.

PROOF: Consider the Riemann or the Lebesgue partition of the domain of integration. From the hypothesis,

$$v_{\text{mean}} = \lim S, \quad S = \sum \frac{\Delta_i}{b-a}\, v(t_i), \quad \frac{\Delta_i}{b-a} = \alpha_i \geq 0, \quad \sum \alpha_i = 1.$$

Thus, the integral sum S is a convex combination of values $v(t_i) \in M$, and therefore $S \in \text{co}M$,$\lim S \in \overline{\text{co}}M = \text{co}\, M$. □

Lemma 2. *[1] Let c be a vector, A a set and let the inequality $c \cdot x \leq \gamma$ be valid for all $x \in A$. Then this is also valid for all $x \in \overline{\text{co}}A$.*

PROOF: The inequality $c \cdot x \leq \gamma$ defines a half space Q which contains the set A. Since $\overline{\text{co}}A$ is the intersection of all half spaces containing A, then $\overline{\text{co}}A \subset Q$. Therefore, the inequality $c \cdot x \leq \gamma$ holds true for all $x \in \overline{\text{co}}A$. □

Definition 1. *A* closed ϵ-neighborhood A^ϵ *of the set A is a set of points x such that $\rho(x, A) \leq \epsilon$.*

Note that A^ϵ is a closed set and $(\overline{A})^\epsilon = A^\epsilon$. Also, for any point $a \notin A^\epsilon$, it is true that $\rho(a, A^\epsilon) = \rho(a, A) - \epsilon$.

Lemma 3. *[1] If a set A is closed and bounded, then $(\operatorname{co} A)^\epsilon = \operatorname{co}(A^\epsilon)$.*

PROOF: Choose ϵ and let $b \notin (\operatorname{co} A)^\epsilon$, thus $\rho(b, \operatorname{co} A) = \alpha > \epsilon$. There exists a point $a \in \overline{\operatorname{co}} A$ such that $\rho(b, a) = \alpha$. Place the origin at the point b and direct the x_1 axis from b to a. Fix any β such that $\epsilon < \beta < \alpha$. Note that the plane $x_1 = \beta$ separates the point b from the sets A and $\overline{\operatorname{co}} A$. Also, the plane $x_1 = \beta - \epsilon$ separates b from the set A^ϵ. Since $x \geq \beta - \epsilon$ for all $x \in A^\epsilon$,

$$(-1)x \leq \epsilon - \beta \quad \text{for all } x \in A^\epsilon$$

and by Lemma 2, it is also true for all $x \in \overline{\operatorname{co}}(A^\epsilon)$; or

$$x \geq \beta - \epsilon \quad \text{for all } x \in \overline{\operatorname{co}}(A^\epsilon),$$

Thus the point b is also separated from $\operatorname{co}(A^\epsilon)$. Therefore, $b \notin \operatorname{co}(A^\epsilon)$.

Now let $b \notin \operatorname{co}(A^\epsilon)$. Let the x_1 axis go from the point b to the nearest point c of the set $\operatorname{co}(A^\epsilon)$. Then $\rho(b, c) = \gamma > 0$; $\operatorname{co}(A^\epsilon)$ lies in the region $x_1 > \delta$, for $0 < \delta < \gamma$ as does A^ϵ. Thus, A lies in the half-space $x_1 \geq \delta + \epsilon$, as does $\operatorname{co} A$ and so $(\operatorname{co} A)^\epsilon$ lies in the half space $x_1 \geq \delta$. Thus $b \notin (\operatorname{co} A)^\epsilon$.

Therefore, $b \notin (\operatorname{co} A)^\epsilon \Leftrightarrow b \notin (\operatorname{co} A^\epsilon)$, and the equality follows. □

Definition 2. *A set-valued map is called upper semi continuous at the point p if for $p' \to p$,*

$$\sup_{f(p') \in F(p')} \rho\Big(f(p'), F(p)\Big) \to 0$$

Note this is similar to the characterization of a continuous map mapping sequences which converge to x to sequences which converge to $F(x)$. In a more intuitive way, upper semi contintuity can also be defined as follows:

Definition 3. *A set-valued map is called upper semi continuous at the point p if for any $\epsilon > 0$, there exists a $\delta > 0$ such that for each $p' \in (p)^\delta$,*

$$F(p') \subset (F(p))^\epsilon.$$

Lemma 4. *[1] Let a set D be closed, and a set-valued function $F(p)$ be bounded in a neighborhood of each point $p \in D$. The graph of $F(p)$ on D is a closed set if and only if the function $F(p)$ is upper semi continuous on D.*

PROOF: Suppose the graph of $F(p)$, on D is closed and the function is not upper semi continuous on D. Then there exist points p and p_i in D such that $p_i \to p$ and

$$\sup_{f(p_i) \in F(p_i)} \rho\Big(f(p_i), F(p)\Big) \geq \epsilon > 0, \text{ for } i = 1, 2, \ldots.$$

Thus, there exist points $q_i \in F(p_i)$ such that $\rho(q_i, F(p)) \geq \epsilon$. Since F is bounded in a neighborhood of each p, the sequence $\{q_i\}$ is bounded. Thus, there is a convergent subsequence, $q_{i_k} \to q$, for some q, and $\rho\Big(q, F(p)\Big) \geq \epsilon$. Thus points (p_{i_k}, q_{i_k}) are in the Graph of F, but their limit point (p, q) is not in the graph of F. This contradicts the hypothesis that the graph of F is closed, so the conclusion is that the function F must be upper semi continuous on D.

Now, if $F(p)$ is upper semi continuous, then let (p, q) be a limit point of its graph. If this is the case, then there exist sequences

$$p_i \to p \in D, \quad \text{and} \quad q_i \to q, \qquad \text{where } q_i \in F(p_i), \quad i = 1, 2, \dots.$$

Since $F(p)$ is upper semi continuous,

$$\rho(q_i, F(p)) \leq sup_{f(p_i) \in F(p_i)} \rho\Big(f(p_i), F(p)\Big) \to 0.$$

Thus, $\rho(q, F(p)) = 0$. Since the set $F(p)$ is closed, then $q \in F(p)$ and so the point (p, q) is in the graph of F. Therefore, the graph of F is closed. ◻

Lemma 5. *[1] Given $H(p)$, an upper semi continuous set-valued function, on the closed domain D. If for each $p \in D$, $H(p)$ is non empty, closed, and bounded, then the function $F(p) = \operatorname{co} H(p)$ is also upper semi continuous.*

PROOF: Choose $p_0 \in D$ and $\epsilon > 0$, there exists a $\delta > 0$ such that for all $p \in (p_0)^\delta$, as H is upper semi continuous, it follows that $H(p) \subset (H(p_0))^\epsilon$. By Lemma 3,

$$\operatorname{co} H(p) \subset \operatorname{co}[(H(p_0))^\epsilon] = [\operatorname{co} H(p_0)]^\epsilon.$$

Thus, $F(p) \subset (F(p_0))^\epsilon$ and F is upper semi continuous. ◻

Lemma 6. *[1] Let $f(p)$ be a bounded single-valued function, $p \in D \subset R^m$, $f(p) \in R^n$. Let for each $p_0 \in \overline{D}$, the set $H(p_0)$ be the set of all limit values of the function $f(p)$ for $p \to p_0$, supplemented by the value $f(p_0)$ in the case of $p_0 \in D$. Then the function $H(p)$ and $F(p) = \operatorname{co} H(p)$ are upper semi continuous.*

PROOF: For each $p \in \overline{D}$, the set $H(p)$ is closed, $H(\overline{D})$ is bounded. The graph of the function H is the closure of the graph of the function f and is therefore closed. Thus, by Lemma and 4 and 5, the functions H and F are upper semi continuous. ◻

Lemma 7. *[1] The set-valued function $F(t, x) = \bigcap_{\delta > 0} \overline{\operatorname{co}} f(t, x^\delta \backslash N_0(t))$ is upper semi continuous in x.*

PROOF: The function $F(t, x) = \bigcap_{\delta > 0} \overline{\operatorname{co}} f(t, x^\delta \backslash N_0(t))$ is the smallest convex set containing all limit values of the vector function $f(t, x')$ where x' spans almost the whole neighborhood of the point x. (Recall that $N_0(t)$ is the set of measure 0 over which f is not necessarily approximately continuous, so when $N_0(t)$ is excluded, f is bounded.) In Lemma 6, let $p = x$, and it follows that $F(t, x)$ is upper semi continuous in x. ◻

Definition 4. *A* support plane *P of a convex set $A \in R^n$ is an $(n-1)$-dimensional plane such that on one side of P there are no points of the set A, but they exist either on P or on the other side of P arbitrarily close to P.*

Definition 5. *A* support function *of a convex set $A \subset R^n$ is a function of a vector $v \in R^n$ defined by the equality:*

$$\psi(A, v) = \sup_{x \in A} v \cdot x.$$

Lemma 8. *[1] A closed convex set A is fully defined by specifying its support function $\psi(A,v)$. The point $a \in A$ if and only if $v \cdot a \leq \psi(A,v)$ for all v.*

PROOF: For any $v \neq 0$, a plane $v \cdot x = \gamma$, where $\gamma = \psi(A,v)$, is a support plane for the set A and a half-space $v \cdot x \leq \gamma$ contains the set A if and only if $\gamma \geq \psi(A,v)$. As the convex set A is the intersection of closed half spaces containing A, then A can be defined as the set of points a for which $v \cdot a \leq \psi(A,v)$, for all v. □

Definition 6. *A vector function $y(t)$ is called a δ-solution of an inclusion $x' \in F(t,x)$ with a function F, upper semi continuous in t,x if on a given interval, the function $y(t)$ is absolutely continuous and almost everywhere,*

$$y'(t) \in F_\delta(t,y(t)), \quad F_\delta(t,y) = [\operatorname{co} F(t^\delta, y^\delta)]^\delta$$

where $F(t^\delta, y^\delta)$ implies a union of sets $F(t_1,y_1)$ for $|t_1 - t| \leq \delta$ and $|y_1 - y| \leq \delta$.

Lemma 9. *[1] Let vector functions $x_k(t)$, $(k = 1,2,\ldots)$ be absolutely continuous for $a \leq t \leq b$ and let their graphs be contained in a bounded closed domain D. Let the set $F(t,x)$ be non empty, bounded, closed, and convex in the domain D for almost all t; let the function F be upper semi continuous in x; $|F(t,x)| \leq m(t)$ and the function $m(t)$ be summable:*

$$x_k'(t) \in [\operatorname{co} F(t,(x_k(t))^{\eta_k(t)})]^{\eta_k(t)}, \tag{6}$$

$$\eta_k(t) \geq 0 \int_a^b \eta_k(t)dt \to 0 \qquad (k \to \infty). \tag{7}$$

Then: the functions $x_k(t)$ are equi continuous on $[a,b]$ and the limit of any convergent subsequence of the function $x_k(t)$ is a solution of the inclusion $x' \in F(t,x)$.

PROOF: It follows from the hypothesis and (6), that

$$|x_k'(t)| \leq m(t) + \eta_k(t) \ a.e.$$

For any $\epsilon > 0$, there exists a $\delta > 0$ and k_0 such that for any disjoint intervals $(\alpha_i, \beta_i) \subset [a,b]$ with sum lengths less than δ, and for any $k > k_0$,

$$\sum_i \int_{\alpha_i}^{\beta_i} m(t)dt < \frac{\epsilon}{2} \quad \int_a^b \eta_k(t)dt < \frac{\epsilon}{2}.$$

Then for $k > k_0$, it follows that

$$\sum_i |x_k(\beta_i) - x_k(\alpha_i)| = \sum_i \left| \int_{\alpha_i}^{\beta_i} x_k'(t)dt \right| \leq \sum_i \left| \int_{\alpha_i}^{\beta_i} (m(t) + \eta_k(t))dt \right| < \epsilon.$$

Therefore, the functions $x_k(t)$ are equi continuous on $[a,b]$. As each $x_k(t)$ is absolutely continuous, passing to the limit, it follows that $x(t)$, the limit of $x_k(t)$ is also absolutely continuous.

Now, to show that $x'(t) \in F(t,x(t))$ almost everywhere. It follows from (7), that for any $\epsilon > 0$, the measure of the set where $|\eta_k(t)| \geq \epsilon$, tends to zero as $k \to \infty$, so that the sequence

$\eta_k(t)$ converges to zero in measure. Therefore, from the $\eta_k(t)$, a new subsequence can be chosen which converges to zero almost everywhere on $[a, b]$. Denote this new subsequence $\{\eta_i(t)\}$ and the corresponding subsequence as $\{x_i(t)\}$. As $F(t, x)$ is upper semi continuous in x, it follows from $x_i(t) \to x(t)$, $\eta_i(t) \to 0$, that

$$F(t, (x_k(t)))^{\eta_i(t)} \subset [F(t, x(t))]^{\nu_i(t)}, \quad \text{where } \nu_i(t) \to 0, \text{ for almost all } t.$$

The right hand side represents a convex set, so it also follows that

$$\operatorname{co} F\Big(t, (x_i(t))^{\eta_i(t)}\Big) \subset [F(t, x(t))]^{\nu_i(t)}$$

and

$$\Big(\operatorname{co} F(t, (x_i(t))^{\eta_i(t)}\Big)^{\eta_i(t)} \subset \Big([F(t, x(t))]^{\nu_i(t)}\Big)^{\eta_i(t)}.$$

Thus, from (6)

$$x_i'(t) \in [F(t, x(t))]^{\nu_i(t)+\eta_i(t)}.$$

Hence, for any $v \in R^n$, for almost all t

$$\limsup_{i\to\infty} x_i'(t) \cdot v = \phi(t) \leq \psi(F(t, x(t)), v), \tag{8}$$

where ψ is a support function for the function $F(t, x(t))$.

For any α, β where $a \leq \alpha < \beta \leq b$, it follows that

$$v \cdot (x_i(\beta) - x_i(\alpha)) = \int_\alpha^\beta v \cdot x_i'(t)dt \leq \int_\alpha^\beta \sup_{j\geq i}[v \cdot x_j'(t)]dt.$$

Since $\sup[v \cdot x_j'(t)]$ does not increase with increasing i and tends to the left hand side of (8) as $i \to \infty$,

$$\int_\alpha^\beta v \cdot x'(t)dt = v \cdot (x(\beta) - x(\alpha)) \leq \int_\alpha^\beta \phi(t)dt.$$

Therefore, again from (8) $v \cdot x'(t) \leq \psi(F(t, x(t)), v) a.e.$

This is also true for a countable, everywhere dense set of vectors v. Thus, from Lemma 8, $x'(t) \in F(t, x(t))$ almost everywhere; $x(t)$ is a solution to the inclusion. □

Theorem 1. *[1] Let, for almost all $t \in [t_0, t_0 + a]$ and for $|x - x_0| \leq b$,*
(1) the set $F(t, x)$ be non empty, closed, convex;
(2) the function F be upper semi continuous in x;
(3) there exists a single-valued vector function $f(t, x) \subset F(t, x)$ which is measurable in t for all x;
(4) there exist a summable function $m(t)$, such that $|f(t, x)| \leq m(t)$.

Then, on the interval $t_0 \leq t \leq t_0 + d$, where d satisfies:

$$0 < d \leq a, \qquad \phi(t_0 + d) \leq b, \qquad \phi(t) = \int_{t_0}^t m(s)ds$$

there exists a solution of the problem

$$x' \in F(t, x), \qquad x(t_0) = x_0.$$

PROOF: Define:

$$h_k = \frac{d}{k}, \quad t_{ki} = t_0 + ih_k, \qquad i = 0, 1, 2, ..., k$$

Fix k and construct $x_k(t)$ as follows: Define $x_k(t_{k0}) = x_0$. Then, assume that for some $i \geq 0$, $x_k(t_{ki}) = x_{ki}$ has been defined and

$$|x_{ki} - x_0| \leq \phi(t_{ki}) = \int_{t_0}^{t_{ki}} m(s)ds \tag{9}$$

then for $t_{ki} < t \leq t_{k,i+1}$, let

$$x_k(t) = x_{ki} + \int_{t_{ki}}^{t} f(s, x_{ki})ds \tag{10}$$

Since $|f(t,x)| \leq m(t)$ and from (9) and (10),

$$\begin{aligned} |x_k(t) - x_0| &= \left| x_{ki} + \int_{t_{ki}}^{t} (f(s, x_{ki})ds - x_0 \right| \\ &\leq \int_{t_0}^{t_{ki}} m(s)ds + \left| \int_{t_{ki}}^{t} f(s, x_{ki})ds \right| \\ &\leq \int_{t_0}^{t_{ki}} m(s)ds + \int_{t_{ki}}^{t} m(s)ds \\ &= \int_{t_0}^{t} m(s)ds = \phi(t) \leq b \end{aligned} \tag{11}$$

In this way, $x_k(t)$ is constructed inductively on the intervals $[t_{ki}, t_{k,i+1}]$, $i = 0, 1, \ldots, k-1$. On $[t_0, t_0 + d]$, the inequality (11) is valid, $x_k(t)$ are absolutely continuous, and almost everywhere on each interval,

$$x_k'(t) = u_k(t) = f(t, x_{ki}) \in F_0(t, x_{ki})$$

where $F_0(t,x)$ is a part of the set $F(t,x)$ contained in a ball of radius $m(t)$ with center at the origin. In this ball, F_0 satisfies condition (1)–(4). Now, for $t_{ki} \leq t \leq t_{k,i+1}$ and $i = 0, 1, \ldots, k-1$

$$|x_k(t) - x_k(t_{ki})| \leq \max(\phi(t_{k,i+1}) - \phi(t_{ki}))$$

Note that

$$\left| \int_{t_0}^{t_{k,i+1}} m(s)ds - \int_{t_0}^{t_{ki}} m(s)ds \right| = \left| \int_{t_{ki}}^{t_{k,i+1}} m(s)ds \right| \to 0 \quad \text{as} \quad k \to \infty\,.$$

Now, from Lemma 9, there exists a convergent subsequence and its limit is a solution of the inclusion $x' \in F_0(t,x)$, $x(t_0) = x_0$ and therefore, there exists a solution x, $x' \in F(t,x)$ on the required interval. □

It is known that if a function $f(t,x)$ satisfies the Caratheodory conditions and the function $x(t)$ is measurable, then $f(t, x(t))$ is summable. This will be stated without proof.

Theorem 2. *[1] For $t_0 \leq t \leq t_0 + a$, $|x - x_0| \leq b$, let the function $f(t,x)$ satisfy the Caratheodory conditions. Then on a closed interval $[t_0, t_0 + d]$, where $d > 0$, there exists a solution of the problem $x' = f(t,x)$, $x(t_0) = x_0$.*

In this case, one can take an arbitrary number d which satisfies the inequalities $0 < d \leq a$, $\phi(t_0 + d) \leq b$, $\phi(t) = \int_{t_0}^{t} m(s)ds$.

Now, the main theorem will be proven.

Theorem 3. *[1] Let, in an open domain G, a vector-valued function $f(t,x)$ be measurable and almost everywhere satisfy the inequality:*

$$|f(t,x)| \leq m(t)$$

with $m(t)$ summable. Then for any point $(t_0, x_0) \in G$ there exists a Filippov solution of

$$x' = f(t,x), \quad x(t_0) = x_0.$$

The solution is defined at least on the interval $[t_0 - d, t_0 + d]$, where d is such that the whole of a cylinder Z

$$|t - t_0| \leq d, \qquad |x - x_0| \leq r$$

is contained within the domain G where

$$r = \max\left\{\int_{t_0-d}^{t_0} m(t)dt, \int_{t_0}^{t_0+d} m(t)dt\right\}.$$

Proof: Let $\rho_0 = \rho(Z, \partial G)$, that is, the distance between the cylinder Z and the domain G of f. Define $\rho_k = 2^{-k}\rho_0$. Let ω_k be the volume of the ball $|y| < \rho_k$. Now define

$$f_k(t,x) = \frac{1}{\omega_k}\int_{|y|<\rho_k} f(t, x+y)dy.$$

This averaging function f_k is defined for $|x - x_0| \leq r$. For almost all $t \in [t_0 - d, t_0 + d]$, f_k is continuous in x. The function is also measurable in t and x and, therefore, measurable in t for all x. As $|f(t,x)| \leq m(t)$, it follows that $|f_k| \leq m(t)$. Thus, the function f_k in Z satisfies the Caratheodory conditions and so, for $t_0 - d \leq t \leq t_0 + d$, there exists a solution $x_k(t)$ of the differential equation, which lies in the cylinder Z. It follows from the definition of $f_k(t,x)$ and from Lemma 4, that almost everywhere,

$$f_k(t,x) \in \overline{\text{co}} f(t, x^{\rho_k} \setminus N_0(t)),$$

where $N_0(t)$ is the set of measure zero over which x is not approximately continuous. For $y \in x^{\rho_k} \setminus N_0(t)$, the function $f(t,y)$ is approximately continuous in y, hence $f(t,y) \in F(t,y)$ as defined in (4). Therefore,

$$x_k'(t) = f_k(t, x_k(t)) \in \text{co}\, F(t, (x_k(t))^{\rho_k}) \text{ a.e.}$$

As F is upper semi continuous in x, it follows from Lemma 4, that the set $F(t, (x_k(t))^{\rho_k})$ on the right-hand side above is closed. Thus, from the sequence $\{x_k(t)\}$, one can choose a uniformly convergent subsequence, and its limit is a solution of the inclusion $x' \in F(t,x)$ and therefore a solution of the differential equation. □

5 Some Examples

Return now to the example introduced at the beginning of this chapter.

$$\begin{gathered} x'(t) = g(y) - \operatorname{sgn} x(t) = f(t,x,y) \quad x(0) = 0 \qquad t \in [0, t_0] \\ |g(y)| \leq 1 \quad \text{for all } y, g \text{ is non-constant and continuous} \end{gathered}$$

It can now be shown that this equation satisfies the modified Caratheodory criteria. That is, a choice of the summable function $m(t)$ as defined below, would satisfy the inequality.

$$\begin{aligned} |f(t,x,y)| &= |g(y) - \operatorname{sgn} x(t)| \\ &\leq |g(y)| + |\operatorname{sgn} x(t)| \\ &\leq 1 + \left[\int (\operatorname{sgn} x(t))^2 dt\right]^{\frac{1}{2}} = 1 + (t - t_0)^{\frac{1}{2}} \\ &= m(t) \end{aligned}$$

Further, the continuity of $x(t)$ and $g(y)$ assures the measurability of f. Therefore, there exists a Filippov solution to this differential equation, defined on the interval specified above.

Now consider the following second order differential equation, which can be written as a vector valued function.

$$\begin{gathered} f''(t,x,x') = |x(t)| \operatorname{sgn} x'(t) + \rho(t) \\ f(t,x) = \Big(f_1(t,x,x'), f_2(t,x,x')\Big) = (x'(t), |x(t)| \operatorname{sgn} x'(t) + \rho(t)) \\ \text{for } x \in C^1, \rho \in C \ \Rightarrow x, \rho \in L^2. \end{gathered}$$

Define

$$F(t,x) = \bigcap_{\delta > 0} \ \bigcap_{\mu \mathrm{P} = 0} \overline{\mathrm{co}} f(t, x^{\delta} \backslash P)$$

It can shown that there exists a solution to this equation in the Filippov sense. That is, there is a function, $x \in$AC such that $x'(t) \in F\ (t, x(t))$.

First observe that the function f is measurable. The function f_1 is measurable because x' is continuous. Also because x' is continuous, $\operatorname{sgn} x'$ is also measurable and also $|x(t)| \operatorname{sgn} x'$ is measurable. Therefore f_2 is also measurable. Now, show that f is bounded by a summable function. Since $x \in C^1$ on $[t_0, t_1]$, there exists an M such that $|x(t)| \leq M$ for each $t \in [t_0, t_1]$ and an N such that $|x'(t)| \leq N$ for each $t \in [t_0, t_1]$. Since the function ρ is continuous on $[t_0, t_1]$, there is a K such that $|\rho(t)| \leq K$ for each $t \in [t_0, t_1]$. Now,

$$\begin{aligned} |f(t,x)| &= (|f_1|^2 + |f_2|^2)^{\frac{1}{2}} \leq \\ &\leq |x'| + ||x| \operatorname{sgn} x' + \rho(t)| \leq |x'| + ||x| \operatorname{sgn} x'| + |\rho(t)| \end{aligned}$$

$$\begin{aligned} (L^2\text{norm}) &= \left\{\int_{t_0}^{t} (x')^2 dt\right\}^{\frac{1}{2}} + \left\{\int_{t_0}^{t} [|x| \operatorname{sgn} x']^2 dt\right\}^{\frac{1}{2}} + \left\{\int_{t_0}^{t} \rho(t)^2 dt\right\}^{\frac{1}{2}} \\ &\leq \left(\int_{t_0}^{t} N^2 dt\right)^{\frac{1}{2}} + \left(\int_{t_0}^{t} M^2 dt\right)^{\frac{1}{2}} + \left(\int_{t_0}^{t} K^2 dt\right)^{\frac{1}{2}} \\ &= N(t - t_0)^{\frac{1}{2}} + M(t - t_0)^{\frac{1}{2}} + K(t - t_0)^{\frac{1}{2}} \\ &= (N + M + K)(t - t_0)^{\frac{1}{2}} \\ &\equiv m(t). \end{aligned}$$

Note: $m(t)$ is summable. Thus, by Theorem 3, there exists a solution to the differential equation in the Filippov sense.

6 Properties of Filippov Solutions

When considering a differential equation with a piece wise continuous right hand side, it is convenient to establish some important properties of the Filippov solution. These are outlined below and it will be shown that under certain conditions, Filippov solutions possess these properties.

A. Through any interior point (t_0, x_0), of the domain, there passes a solution. (true by Theorem 3)

B. Each solution lying within a given closed bounded domain is continued on both sides to reach the boundary of the domain.

C. All the solutions lying in a closed bounded domain are equicontinuous.

D. A limit of uniformly convergent sequence of solutions is a solution. (true by Lemma 9).

E. If all the solutions with given initial data $x(t_0) = x_0$ exist for $\alpha \leq t \leq \beta$, then the set of points lying on the graphs of these solutions is bounded and closed. The set of these solutions is a compact set.

Lemma 10. *Let $f(t, x)$ satisfy the modified Caratheodory conditions. Then the set of solutions of the inclusion $x' \in F(t, x)$ that lie in the domain G are equicontinuous. (Property C)*

PROOF: The set F defined as in (2) is bounded; there exists an m such that $|F(t, x)| \leq m$ in G. Thus, for all solutions in G, $|x'| \leq m$. For any $\epsilon > 0$, let $\delta = \frac{\epsilon}{m}$.

$$|x(t'') - x(t')| \leq m|t'' - t'| \leq \epsilon.$$

Therefore, the set of solutions is equicontinuous. □

Theorem 4. *Given the function $f(t, x)$ which satisfies the modified Caratheodory conditions on D, a closed, bounded domain, any solution $x(t)$ of the differential equation (1) can be extended to the boundary Γ of the domain. (Property B)*

PROOF: Define $\phi(t)=\int_{t_0}^{t} m(s)ds$; thus ϕ is absolutely continuous. Note that ϕ is uniformly continuous on any closed interval. Now, choose $p_0 = (t_0, x(t_0)) \in D$, where x is a solution to the differential equation going through p_0. Now choose $\epsilon_1 > 0$, such that $\epsilon_1 \leq \frac{1}{2}\rho(p_0, \Gamma)$. Choose c, d in the domain such that $c \leq t_0 \leq d$. Since ϕ is uniformly continuous on $[c, d]$, there exists a $\delta_1 > 0$, $\delta_1 \leq \epsilon$ such that for any $\alpha, \beta \in [c, d]$, where $|\beta - \alpha| < \delta_1$, it follows that $|\phi(\beta) - \phi(\alpha)| < \epsilon_1$. Thus,

$$|t - t_0| \leq \delta_1 \quad \text{and} \quad |x - x_0| = \left|\int_{x}^{x_0} f(t, x(t))dt\right| \leq \int_{x}^{x_0} m(t)dt = \phi(x_0) - \phi(x) < \epsilon_1.$$

Since $[c, d]$ is in the domain and $\epsilon_1 \leq \frac{1}{2}\rho(p_0, \Gamma)$, this cylinder is contained in the solution, and the solution x exists on $|t - t_0| \leq \delta_1$. Consider the point $(t_0 + \delta_1, x(t_0 + \delta_1))$. If the distance from this point to the boundary is greater than 2ϵ, follow this procedure until reaching a point $p_1 = (t_1, x(t_1))$ is obtained such that $\rho(p_1, \Gamma) < 2\epsilon_1 \leq \rho(p_0, \Gamma)$.

Now, choose $\epsilon_2 > 0$, $\epsilon_2 \leq \frac{1}{2}\rho(p_1, \Gamma) < \epsilon_1$. Choose c, d such that $c \leq t_1 \leq d$ in the domain and $\delta_2 \leq \epsilon_2$ as above. Repeating the method above, it is possible to find a sequence of cylinders within which the solution x is defined. Note that ϵ_i decreases and is bounded below by 0, so $\epsilon_i \to 0$. The sequence t_i is increasing, but bounded in D, so there exists a t^*, such that $t_i \to t^*$, and, as x is continuous, $x(t_i) \to x(t^*)$. Now $\rho(p_i, \Gamma) \to 0$ and as D is closed, $(t^*, x^*) \in \Gamma$ and in this way, $x(t)$ is extended to the boundary of the domain. ◻

Theorem 5. *Given the function $f(t, x)$ which satisfies the modified Caratheodory conditions and the differential equation* (2). *If all the solutions with given initial data $x(t_0) = \alpha$ exist for $a \leq t \leq b$, then the set of these solutions is a compact set. (Property E)*

Using the Arzela-Ascoli theorem, easily follows that the set under discussion is compact in $C(a, b)$.

7 Conclusions

The use of the Filippov definition of a solution to a differential equation allows a larger set of differential equations to be solved. The redefinition of the function allows properties of set-valued functions and differential inclusions to apply. The examples discussed in this paper have applications in vibration control and Coulomb damping, and thus further work seems to be warranted- particularly in the area of actually finding these solutions.

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