LIBERTAS MATHEMATICA, vol XXIX (2009)

GTZ Principles with Cone-Valued Metrics[1]

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Abstract. Further extensions are given for the variational principle obtained by Goepfert, Tammer and Zălinescu [Nonl. Anal., 39 (2000), 909-922] involving cone-valued metrics. These also include the related statement in Chen, Huang and Hou [J. Optimiz. Th. Appl., 106 (2000), 151-164].

Keywords: Locally convex space, (convex) cone, quasi-order, maximal element, net, separable structure, Zorn-Bourbaki principle, regularity, boundedness.

Mathematics Subject Classification (2000): Primary 03E25, 06A06, 54H25. Secondary 46A40, 54C10.

1 Introduction

Let $(Y, \mathcal{T})$ be a separated locally convex space; hence, its linear topology $\mathcal{T}$ admits a fundamental system $\mathcal{B}$ of convex neighborhoods of $0 \in Y$. Further, let Q be some convex cone of Y ($\alpha Q + \beta Q \subseteq Q$, for all $\alpha, \beta \geq 0$). The relation $(\leq_Q)$ on Y defined as

(a1) $(y_1, y_2 \in Y)$: $y_1 \leq_Q y_2$ if and only if $y_2 - y_1 \in Q$

is reflexive transitive; hence a *quasi-order*; in addition, it is *compatible* with the linear structure of Y. For each $V \subseteq Y$, let $[V] = (V + Q) \cap (V - Q)$ stand for the Q-cover of V; if $V = [V]$, then V is called *Q-full*. Note that

$$[V] \text{ is convex whenever } V \text{ is convex;} \tag{1.1}$$

cf. Peressini [21, Ch 2, Sect 1]. Finally, let X be some nonempty set. By a *Q-metric* on it we shall mean any map $d : X \times X \to Q$ with

(b1) $d(x, x) = 0$, for all $x \in X$ (reflexivity)

(c1) $d(x_1, x_3) \leq d(x_1, x_2) + d(x_2, x_3)$, $\forall x_1, x_2, x_3 \in X$ (triangular property)

(d1) $x_1, x_2 \in X$, $d(x_1, x_2) = 0 \Longrightarrow x_1 = x_2$ (sufficiency)

(e1) $d(x_1, x_2) = d(x_2, x_1)$, $\forall x_1, x_2 \in X$ (symmetry).

[1]This research was supported by Grant PN II PCE ID_387, from the National Authority for Scientific Research, Romania.

Given such an object, we may introduce the concepts of *convergence* and *Cauchy* structure as follows. Let $(I, \leq)$ be a quasi-ordered set; which, in addition, is directed (for each $i_1, i_2 \in I$ there exists $i_3 \in I$ with $i_1 \leq i_3$, $i_2 \leq i_3$). Any map $i \mapsto x(i)(= x_i)$ from I to X will be referred to as a *net* subordinated to $(I, \leq)$; and written as $(x_i)_{i\in I}$ (when $(\leq)$ is understood) or simply (x_i) (when I is understood too); cf. Kelley [15, Ch 2, Sect 2]. We say that the net $(x_i)_{i\in I}$, *d-converges* to x (and write $x_i \to x$) when: for each $B \in \mathcal{B}$ there exists $i(B) \in I$ such that $i \geq i(B) \Longrightarrow d(x_i, x) \in B$. If x is generic in this convention we say that this net is d-convergent. Further, call $(x_i)_{i\in I}$, *d-Cauchy* provided: for each $B \in \mathcal{B}$ there exists $i(B) \in I$ such that $i, j \geq i(B)$ implies $d(x_i, x_j) \in B$. Note that no relationship between these concepts is valid, in general. However, each d-convergent net is d-Cauchy provided Q is *normal*:

(1a) there exists a fundamental system of Q-full neighborhoods;

or, equivalently (under (1.1)): each $B \in \mathcal{B}$ is, in addition, Q-full. Then, we say that (X, d) is complete when each d-Cauchy net is d-convergent.

Now, let K be some convex cone of Y; supposed to be *pointed* ($K \cap (-K) = \{0\}$). By a *subcone* of K we shall mean any part H of K which is itself a convex cone in Y. Fix such an object; and consider a H-metric $d : X \times X \to H$. The relation $(\preceq)$ over $X \times Y$ introduced as

(f1) $(x_1, y_1) \preceq (x_2, y_2)$ iff $d(x_1, x_2) \leq_K y_1 - y_2$

is reflexive, transitive and antisymmetric; hence, a (partial) order. Take also some part A of $X \times Y$. For a number of both practical and theoretical reasons, it would be useful to determine sufficient conditions under which the quasi-ordered structure $(A, \preceq)$ has maximal points (in the usual sense). The natural way to be followed is that of using the Zorn-Bourbaki maximality principle. However, by the topological setting of the problem, it would be useful to express the chain inductive condition appearing there in terms of convergent/Cauchy nets. The basic result in the area was obtained by Goepfert, Tammer and Zălinescu [9, Theorem 10]. To state it, we need some conventions. Let P be some convex cone of Y. Call the net $(y_i)_{i\in I}$ in Y, $(\leq_P)$-*ascending* (or simply: P-*ascending*) if $y_i \leq_P y_j$ whenever $i \leq j$. Further, let us say that $y \in Y$ is a $(\leq_P)$-*bound* (or simply: P-*bound*) of $(y_i)_{i\in I}$ when $y_i \leq_P y$, for all $i \in I$. When y is generic in this convention, we term our net, $(\leq_P)$-*bounded (above)* (or simply: P-*bounded (above)*). Now, the general conditions to be imposed upon our data read

(1b) H is normal and (X, d) is complete

(1c) H is sequentially K-bound regular (modulo d): each H-ascending K-bounded sequence in H is d-Cauchy.

The specific hypotheses (involving A) may be written as

(1d) $P_Y(A)$ is K-bounded below [$\exists \widetilde{y} \in Y$ with $P_Y(A) \subseteq \widetilde{y} + K$]

(1e) for each $(\preceq)$-ascending net $((u_i, v_i))_{i\in I}$ in A with $u_i \to u \in X$ we have $u \in P_X(A)$ and $\exists\, v \in A(u)$ such that $(u_i, v_i) \preceq (u, v)$, $\forall i \in I$.

[Here, for each $(x, y) \in A$, $A(x)$ (respectively, $A(y)$) stands for the *x-section* (respectively, *y-section*) of (the relation) A; and P_X, P_Y are the *projection* operators from $X \times Y$ to X and Y respectively].

The announced result may then be stated as follows:

Theorem 1 *Suppose that the precise conditions are in force. Then, for each starting* $(x_0, y_0) \in A$ *there exists* $(\bar{x}, \bar{y}) \in A$ *with* **i)** $(x_0, y_0) \preceq (\bar{x}, \bar{y})$ *[hence* $y_0 \geq \bar{y}$*] and* **ii)** *if* $(x, y) \in A$ *fulfills* $(\bar{x}, \bar{y}) \preceq (x, y)$ *then* $(\bar{x}, \bar{y}) = (x, y)$.

In particular, condition (1e) holds under (1b) (the first half) and

(1f) the H-lower sections of K are closed: $K \cap (y - H)$ is closed, $\forall y \in K$

(1g) A is submonotone (modulo K): for each net $((x_i, y_i))_{i \in I}$ in A with $x_i \to x \in X$ and $(y_i)_{i \in I}$ being K-descending we have $x \in P_X(A)$ and there exists $y \in A(x)$ with $y \leq_K y_i$, for all $i \in I$.

This shows that Theorem 1 extends the related statement in Chen, Huang and Hou [4], [5] (cf. Section 6); see also Isac [12]. It is our aim in the following to show that further extensions of this result are possible when d is just a pseudometric; we refer to Section 5 for details. The basic tool for this is a Zorn-Bourbaki maximality principle (stated in Section 3) over general separable sets (cf. Section 2) and some technical facts about regular cones in topological vector spaces (delineated in Section 4). Finally, in Section 6, some particular aspects of these facts are being considered.

2 General separable structures

In the following, the concept of (general) separable structure is introduced; and some basic properties of these objects are discussed.

(A) Call the partially ordered structure $(P, \leq)$, *well ordered* if each part of P has a first element. Given a couple $(P, \leq)$, $(Q, \leq)$ of such objects, put

(a2) $(P, \leq) \sim (Q, \leq)$ iff there exists a strictly increasing bijection:$P \to Q$

(b2) $(P, \leq) \preceq (Q, \leq)$ iff $(P, \leq) \sim (Q_0, \leq)$, for some initial interval Q_0 of Q.

The former of these is an equivalence (denoted: $\text{ord}(P, \leq) = \text{ord}(Q, \leq)$); while the latter is, for the moment, a quasi-order (denoted: $\text{ord}(P, \leq) \leq \text{ord}(Q, \leq)$). The order type of $(P, \leq)$ (denoted $\text{ord}(P, \leq)$) is just its equivalence class; also referred to as an *ordinal*.

Note that the class W of all ordinals is not a set, as results from the Burali-Forti paradox; cf. Sierpinski [22, Ch 14, Sect 2]. However, when one restricts to a *Grothendieck universe* $\mathcal{G}$ (taken as in Hasse and Michler [10, Ch 1, Sect 2]) this contradictory character is removed for the class $W(\mathcal{G})$ of all *admissible* (modulo $\mathcal{G}$) ordinals (generated by (non-contradictory) well ordered parts of $\mathcal{G}$). In the following, we drop any reference to $\mathcal{G}$, for simplicity. So, by an *ordinal* in W one actually means a $\mathcal{G}$-admissible ordinal with respect to a "sufficiently large" Grothendieck universe $\mathcal{G}$. The relation on W introduced via (b2) is a good order on this class; i.e.: $(W, \leq)$ is well ordered. Note that, if ξ is an admissible ordinal and $\eta \leq \xi$

then η is an admissible ordinal too. Hence, in the formulae $W(\alpha) = \{\xi \in W; \xi < \alpha\}$, $W[\alpha, \beta] = \{\xi \in W; \alpha \leq \xi \leq \beta\}$, the symbol W in the brackets is the "absolute" class of all ordinals.

Now, an enumeration of W is realized via the *immediate successor* map $\text{suc}(M) = \min\{\xi \in W; M < \xi\}$, $M \subseteq W$. It begins with the natural numbers $N = \{0, 1, ...\}$. Their immediate successor is $\omega = \text{suc}(N)$ (the first transfinite ordinal); the next in this enumeration is $\omega + 1$, and so on. Put $W_0 = W \setminus \{0\}(= \{\xi \in W; \xi > 0\})$. This set is composed of two disjoint classes of ordinals. The former of these, W_0^1, collects all *first kind* ordinals $\xi > 0$ [in the sense: $W(\xi)$ admits a last element $\max[W(\xi)] = \xi - 1$; called the *predecessor* of ξ]. And the latter of these, W_0^2, collects all *second kind* ordinals $\xi > 0$ [in the sense: $W(\xi)$ does not admit a last element; or, equivalently: $\lambda < \xi \Longrightarrow \lambda + 1 < \xi$]; this is also referred to as $\xi > 0$ being a *limit* ordinal.

The basic operations with ordinals may be introduced in a synthetic way as follows. Let α, β be two ordinals; and $(A, \leq)$, $(B, \leq)$, well ordered structures with $\text{ord}(A, \leq) = \alpha, \text{ord}(B, \leq) = \beta$, $A \cap B = \emptyset$. Then **a)** $\alpha + \beta = \text{ord}(A \cup B, \leq)$, where the associated order is given by the *concatenation* procedure: $x < y$ iff either $[x \in A, y \in B]$ or $[(x, y \in A, x < y), (x, y \in B, x < y)]$; **b)** $\alpha \cdot \beta = \text{ord}(A \times B, \leq)$, where the associated order is defined by the lexicographic procedure: $(x, y) < (u, v)$ iff either $[y < v]$ or $[x < u, y = v]$. Further details may be found in Kuratowski and Mostowski [17, Ch 7, Sect 5].

(B) In parallel to this, we may (construct and) enumerate the class of all admissible cardinals. Let P and Q be nonempty sets; we put

(c2) $P \sim Q$ iff there exists a bijection:$P \to Q$

(d2) $P \preceq Q$ iff $P \sim Q_0$, for some part $Q_0 \subseteq Q$.

The former of these (denoted $\text{card}(P) = \text{card}(Q)$) is an equivalence; while the latter (denoted as: $\text{card}(P) \leq \text{card}(Q)$) is a quasi-order. Denote also

(e2) $P \prec Q$ if and only if $P \preceq Q$ and $\neg(P \sim Q)$.

This relation is *irreflexive* ($\neg(P \prec P)$, for each P) and *transitive*; hence a *strict order* (indicated as: $\text{card}(P) < \text{card}(Q)$). For each P, the *power* of it (denoted $\text{card}(P)$) is just its equivalence class; also referred to as a *cardinal*; and the class of all these is denoted by Z.

Note that, by the Zermelo well ordering principle [26] each cardinal is attached to well ordered structures; i.e., it is an *aleph*. As a consequence, the relation on Z introduced via (d2) is a good order on this class; i.e.: $(Z, \leq)$ is well ordered. In addition, its associated non-reflexive relation is obtainable via (e2) above. In fact, Z itself may be viewed as a family of initial ordinals, as follows. Let $\alpha > 0$ be an (admissible) ordinal; we say that it is an *initial* one if $\text{card}(W(\xi)) < \text{card}(W(\alpha))$, for each $\xi < \alpha$. The class of all these, completed with $\{0\}$ is nothing else than Z above. Now, the enumeration we are looking for is realized via the immediate successor (in Z) map $\text{SUC}(M) = \min\{\eta \in Z; M < \eta\}$, $M \subseteq Z$. Precisely, it begins with the natural numbers $N = \{0, 1, ...\}$. The immediate successor (in Z) of all these is again $\omega = \text{SUC}(N)$ (the first transfinite cardinal). To describe the remaining ones, we may introduce via transfinite recursion the function $\lambda \vdash \aleph_\lambda$ from W to Z as

$$\aleph_0 = \omega; \ \ \aleph_\lambda = \text{SUC}\{\aleph_\xi; \xi < \lambda\}, \ \lambda > 0.$$

Note that, in such a case, the order structure of $Z(\omega, \leq)$ is identical with the one of W; cf. Just and Weese [14, Ch 11, Sect 11.2].

Any nonempty part P with $\text{card}(P) < \omega$ ($\text{card}(P) = \omega$) is termed *finite* (*effectively countable*); the union of these ($\text{card}(P) \leq \omega$) is referred to as P is *countable*. When $P = W(\xi)$, all such properties will be transferred to ξ. Generally, take some ordinal γ; and put $\Gamma = \aleph_\gamma$. Any nonempty part P with $\text{card}(P) < \Gamma$ ($\text{card}(P) = \Gamma$) is termed Γ-*finite* (*effectively* Γ-*countable*); the union of these ($\text{card}(P) \leq \Gamma$) is referred to as P is Γ-*countable*. As before, when $P = W(\xi)$, all such properties will be transferred to ξ.

Let J be a subset of W (endowed with the induced order) and M be some nonempty set. Any map $j \mapsto u_j$ from J to M will be termed a J-*net*; when $J = W(\lambda)$ (for some $\lambda \in W$) we shall talk about a λ-*net*. Let also μ be some ordinal with $\lambda \leq \mu$. Any λ-net $(a_\xi; \xi < \lambda)$ in M is referred to as μ-*subordinated* (in short: μ-*sub*); and if $\lambda < \mu$, the net in question will be referred to as μ-*strongly subordinated* (in short: μ-*strongsub*).

Denote by Δ the immediate successor (in Z) of Γ [$\Delta = \text{SUC}(\Gamma)$; hence $\Delta = \aleph_{\gamma+1}$, if $\Gamma = \aleph_\gamma$]. By the very definition above, one has

$$\xi \text{ is } \Gamma\text{-countable, for each } \xi < \Delta; \quad \text{but } \Delta \text{ is not } \Gamma\text{-countable.} \tag{2.1}$$

A basic consequence of this is precise in the statement below (to be found, e.g., in Kuratowski and Mostowski [17, Ch 3, Sect 4]).

Proposition 1 *The following are valid:*

i) *The ordinal* Δ *cannot be attained via* Γ*-sub net limits of* Γ*-countable ordinals. In other words: if* $(\alpha_\xi; \xi < \theta)$ *is an ascending* θ*-net (where* $\theta \leq \Gamma$*) of* Γ*-countable ordinals then*

$$\alpha = \sup_\xi(\alpha_\xi)(= \lim_\xi(\alpha_\xi)) \tag{2.2}$$

is Γ*-countable too.*

ii) *Each second kind* Γ*-countable ordinal is attainable via such nets. In other words: if (the* Γ*-countable)* $\alpha < \Delta$ *is of second kind, there must be a strictly ascending* Γ*-sub net* $(\alpha_\xi; \xi < \theta)$ *(where* $\theta \leq \Gamma$*) of* Γ*-countable ordinals with the property (2.2).*

(C) Let M be a nonempty set; and $(\leq)$, some *order* (=antisymmetric quasi-order) on it. By a $(\leq)$-*chain* of M we shall mean any (nonempty) part A of M with $(A, \leq)$ well ordered (see above). Note that any such object may be written as $A = \{a_\xi; \xi < \lambda\}$, where the net $\xi \vdash a_\xi$ is strictly ascending ($\xi < \eta \Longrightarrow a_\xi < a_\eta$); the uniquely determined ordinal λ is just $\text{ord}(A, \leq)$. Let μ be another ordinal. If $\lambda \leq \mu$, we say that the $(\leq)$-chain $(A, \leq)$ is μ-*subordinated* (in short: μ-*sub*); and if $\lambda < \mu$, the same chain $(A, \leq)$ will be referred to as μ-*strongly-subordinated* (in short: μ-*strongsub*). A basic particular case of these conventions is the following. Let $\gamma \geq 0$ be arbitrary fixed; and put $\Gamma = \aleph_\gamma$, $\Delta = \text{SUC}(\Gamma)$ (hence $\Delta = \aleph_{\gamma+1}$). Note that, by the very definition above (and (2.1)) A is Γ-countable iff $(A, \leq)$ is Δ-strongsub. This in particular happens when $(A, \leq)$ is Γ-sub. The following characterization of this last concept is useful for us.

Proposition 2 *The* $(\leq)$*-chain* A *is* Γ*-sub if and only if*

$$\begin{aligned} &A = \{b(\xi); \xi < \theta\}, \textit{ where } \theta \in W[\omega, \Gamma] \textit{ and} \\ &\xi \vdash b(\xi) \textit{ is ascending } (\xi < \eta \Longrightarrow b(\xi) \leq b(\eta)). \end{aligned} \tag{2.3}$$

Proof *i)* Assume that A is Γ-sub. By definition, $A = \{a_\xi; \xi < \lambda\}$, where $\lambda \le \Gamma$ and $\xi \vdash a_\xi$ is strictly ascending. If λ is finite, then (2.3) is accessible via $\theta = \omega$ and $[b(\xi) = a_\xi, \xi < \lambda; b(\xi) = a_{\lambda-1}, \xi \ge \lambda]$. And, if λ is infinite, we just take $\theta = \lambda$ and $[b(\xi) = a_\xi, \xi < \theta]$, so as to get (2.3). *ii)* Assume A is as in (2.3); and put $E = \{\min(b^{-1}(x)); x \in A\}$. The map $\xi \vdash b(\xi)$ is a strictly increasing bijection from $(E, \le)$ to $(A, \le)$; wherefrom $\mathrm{ord}(E, \le) = \mathrm{ord}(A, \le)$. On the other hand, $\mathrm{ord}(E, \le) \le \mathrm{ord}(W(\theta), \le) = \theta \le \Gamma$; because $E \subseteq W(\theta)$ (see, e.g., Sierpinski [22, Ch 13, Sect 5]). And this, in conjunction with a preceding relation, shows that A is Γ-sub. □

Let P, Q be nonempty parts with $P \supseteq Q$. We say that P is *majorized* by Q (and write $P \propto Q$) provided Q is cofinal in P (for each $x \in P$, there exists $y \in Q$ with $x \le y$). The $(\le)$-chain $S \subseteq M$ is called *upper* Γ*-countable* in case:

(f2) $S \propto T$, for some Γ-sub $(\le)$-chain $T \subseteq S$.

Clearly, this happens if S is Γ-sub. As a completion, we have

Proposition 3 *The generic relation holds*

$$(\forall(\le)\text{-chain})\ \Gamma\text{-countable} \Longrightarrow \text{upper } \Gamma\text{-countable}. \tag{2.4}$$

Hence, the $(\le)$*-chain* $S \subseteq M$ *is upper* Γ*-countable if and only if*

$$S \propto T, \text{ for some } \Gamma\text{-countable } (\le)\text{-chain } T \subseteq S. \tag{2.5}$$

Proof Let $S = \{s(\xi); \xi < \lambda\}$ be the representation of this $(\le)$-chain where $\lambda := \mathrm{ord}(S, \le) < \Delta$. If λ is a first kind ordinal, we are done; because $T = \{s(\lambda - 1)\}$ is then cofinal in S. Assume now λ is a second kind ordinal. By Proposition 1 there exists a strictly ascending Γ-sub net $(\lambda_\xi; \xi < \theta)$ (where $\theta \le \Gamma$) with $\lambda = \sup_\xi(\lambda_\xi)$. But then, $T = \{s(\lambda_\xi); \xi < \theta\}$ is a Γ-sub $(\le)$-chain (of S) cofinal in S; i.e., we are again done. □

Remark 1 The reciprocal of (2.4) is not in general true; just take any $(\le)$-chain S of M with $\Delta \le \mathrm{ord}(S, \le)=$ first kind ordinal.

(D) Let us now return to our initial setting. We say that the order structure $(M, \le)$ is Γ*-separable* if

(g2) any $(\le)$-chain of M is upper Γ-countable.

For example, this holds (under (2.4)) whenever

(h2) $(M, \le)$ is *strongly* Γ*-separable*: any $(\le)$-chain of M is Γ-countable.

In fact, the reciprocal holds too; so that, we may formulate

Proposition 4 *Under these conventions,*

$$(\forall(M, \le)=\text{ordered structure})\ \Gamma\text{-separable} \Longleftrightarrow \text{strongly } \Gamma\text{-separable}. \tag{2.6}$$

Proof Assume that $(M, \leq)$ is Γ-separable; and let $S = \{s(\xi); \xi < \lambda\}$ be some $(\leq)$-chain of M; where $\lambda := \mathrm{ord}(S, \leq)$. If, by absurd, S is not Γ-countable, we must have $\lambda \geq \Delta$. The initial segment (of S) $U = \{s(\xi); \xi < \Delta\}$ is not Γ-countable too; cf. (2.1). On the other hand, by hypothesis, U is upper Γ-countable; so, there exists a strictly ascending Γ-sub net $(\xi_\nu; \nu < \theta)$ (where $\theta \leq \Gamma$) of ranks in $W(\Delta)$ with $U \propto \{s(\xi_\nu); \nu < \theta\}$; hence $\Delta = \lim_\nu(\xi_\nu)$. This, however, cannot be accepted, in view of Proposition 1. Hence, S is Γ-countable; and the proof is complete. □

Remark 2 From this result it follows that the notion of Γ-separable structure is a transfinite extension of the concept of ω-separable structure; which has been introduced by Zhu, Fan and Zhang [27].

Now, call the cardinal Γ, *separable-admissible* (in short: sep-admissible) for $(M, \leq)$ whenever $(M, \leq)$ is Γ-separable [or, equivalently (see above): strongly Γ-separable]. The class of all these, $\mathrm{Sep}(M, \leq)$, is nonempty; because $\mathrm{card}(M)$ is an element of it. In addition, it is hereditary after the cardinal magnitude; i.e.: $\Gamma \in \mathrm{Sep}(M, \leq)$ and $\Gamma \leq \Delta$ imply $\Delta \in \mathrm{Sep}(M, \leq)$. The minimal element of this set, $\mathrm{sep}(M, \leq) = \min \mathrm{Sep}(M, \leq)$ is therefore well defined as a cardinal number; it will be referred to as the *separability cardinal* of $(M, \leq)$. By the remark above, we have $\mathrm{sep}(M, \leq) \leq \mathrm{card}(M)$; but, the converse relation may be false, in general. However, in many practical situations, $\mathrm{card}(M)$ is a good "approximation" for $\mathrm{sep}(M, \leq)$.

(E) In the following, we shall give a useful example of such structures. Let $\mathcal{I}(M) := \{(x, x); x \in M\}$ stand for the identical relation over M. By an *almost uniformity* (on M) we shall mean any family $\mathcal{U}$ of parts in $M \times M$ with $\mathcal{I}(M) \subseteq \cap\mathcal{U}$. Suppose that we fixed such an object. The basic conditions to be needed further may be written as

(2a) $\mathcal{U}$ is Γ-*pseudometrizable*: there exists a Γ-countable subfamily $\mathcal{V} \subseteq \mathcal{U}$, cofinal in $(\mathcal{U}, \supseteq)$ $[\forall U \in \mathcal{U}, \exists V \in \mathcal{V}\colon U \supseteq V]$;

(2b) $\mathcal{U}$ is sufficient: $\cap\mathcal{U} = \mathcal{I}(M)$.

For the next one, we need some preliminaries. Call the (ascending) net $(a_\xi; \xi < \lambda)$, $\mathcal{U}$-*Cauchy*, when: $\forall U \in \mathcal{U}, \exists \mu = \mu(U)$, such that $\mu \leq \xi \leq \eta \Longrightarrow (a_\xi, a_\eta) \in U$. Likewise, call the (ascending) sequence $(b_n; n < \omega)$, $\mathcal{U}$-*asymptotic*, in case: $\forall U \in \mathcal{U}, \exists k = k(U)$, such that $n \geq k \Longrightarrow (b_n, b_{n+1}) \in U$. The following auxiliary fact is useful for us.

Lemma 1 *The global conditions below are equivalent each other*

(2c) each ascending net is $\mathcal{U}$*-Cauchy*

(2d) each ascending sequence is $\mathcal{U}$*-asymptotic.*

The verification is directly obtainable from these definitions; so, we do not give details. By definition, either of the underlying properties will be referred to as: $\mathcal{U}$ is (strongly) *regular*.

We are now in position to give the promised example.

Proposition 5 *Assume that there exists an almost uniformity* $\mathcal{U}$ *over* M *which is* Γ*-pseudometrizable, sufficient and (strongly) regular. Then,* $(M, \leq)$ *is (strongly)* Γ*-separable.*

Proof Without loss, one may assume that $\mathcal{U}$ itself is Γ-countable; i.e., written as a θ-net $(U_\xi; \xi < \theta)$, where $\theta < \Delta$. (Otherwise, we simply replace $\mathcal{U}$ by $\mathcal{V}$). The case $\theta < \omega$ is clear; so, it remains to discuss the alternative $\theta \geq \omega$. Let S be some $(\leq)$-chain in M. If there exists a last element $s = \max(S)$, we are done; so, without restriction, one may assume that

(2e) for each $x \in S$ there exists $y \in S$ with $x < y$.

This, and the (strong) regularity of $\mathcal{U}$, yields (cf. Turinici [23])

$$\begin{array}{l} \forall x \in S, \forall U \in \mathcal{U}, \text{ there exists } y = y(x, U) \in S(x, <) \\ \text{such that: } p, q \in S,\ y \leq p \leq q \Longrightarrow (p, q) \in U. \end{array} \tag{2.7}$$

Let $a \in S$ be arbitrary fixed. By (2e) and (2.7), there exists (in S) $a_0 > a$ with $[p, q \in S, a_0 \leq p \leq q \Longrightarrow (p, q) \in U_0]$. Further, by the same relations, there exists (in S) $a_1 > a_0$ with $[p, q \in S, a_1 \leq p \leq q \Longrightarrow (p, q) \in U_1]$; and so on. Generally, assume that for the ordinal $\mu < \theta$, we constructed a net $(a_\xi; \xi < \mu)$ in S with: for each $\lambda < \mu$,

(2f) $\xi < \lambda$ implies $a_\xi < a_\lambda$

(2g) $p, q \in S,\ a_\lambda \leq p \leq q \Longrightarrow (p, q) \in U_\lambda$.

Two possibilities may occur.

j) μ is a first kind ordinal: $\lambda = \mu - 1$ exists. Again by (2e) and (2.7), there exists (in S) $t > a_\lambda$ with $[p, q \in S, t \leq p \leq q \Longrightarrow (p, q) \in U_\mu]$. Taking $a_\mu = t$, (2f)+(2g) are fulfilled with (with μ in place of λ).

jj) μ is a second kind ordinal: $\mu - 1$ does not exist. By the choice of θ, the $(\leq)$-chain (in S) $T = \{a_\xi; \xi < \mu\}$ is Γ-countable. If T is cofinal in S, we are done; because (cf. Proposition 3) S is upper Γ-countable. Otherwise,

(2h) $a_\xi < s$, for all $\xi < \mu$ and some $s \in S$.

Again by (2e)+(2.7), there exists (in S) $t > s$ with $[p, q \in S, t \leq p \leq q \Longrightarrow (p, q) \in U_\mu]$. Putting $a_\mu = t$, (2f)+(2g) are fulfilled (with μ in place of λ).
As a consequence, the process above either stops at a certain stage $\mu < \theta$ (and then, we are done); or else (in the opposite situation) it is continuable over all of $W(\theta)$; i.e., (2f)+(2g) hold, for each $\lambda < \theta$. We claim that $T = \{a_\xi; \xi < \theta\}$ is cofinal in S; and this, combined with the Γ-countable property of the same, completes the argument. Assume not; i.e., (2h) is true (with θ in place of μ). By (2e), there exists $t \in S$ with $t > s$; hence $t \neq s$. On the other hand, by the choice of $(a_\xi; \xi < \theta)$ one has $(s, t) \in U_\xi$, for all $\xi < \theta$; hence $s = t$ (by the sufficiency condition). The obtained facts involving (s, t) are contradictory. Hence, (2h) cannot hold (with θ in place of μ); and our claim follows. □

A basic particular construction of this type may be described along the following lines. By a *pseudometric* over M we mean any map $d : M \times M \to R_+$. Call this object *reflexive* provided $d(x, x) = 0$, $\forall x \in M$. Let $D = (d_\lambda; \lambda < \alpha)$ be a family of reflexive pseudometrics; where $\alpha \geq \omega$. Then $\mathcal{U}(D) = \{U(\lambda, r); \lambda < \alpha, r > 0\}$, where $U(\lambda, r) = \{(x, y) \in M \times$

$M; d_\lambda(x,y) < r\}$, $\lambda < \alpha$, $r > 0$, is an almost uniformity over M. The sufficiency condition for this object is characterized as: D is *sufficient* $[d_\lambda(x,y) = 0, \forall \lambda < \alpha] \Longrightarrow x = y]$. On the other hand, the subfamily $\mathcal{V} = \{U(\lambda, 2^{-n}); \lambda < \alpha, n < \omega\}$ is cofinal in $(\mathcal{U}, \supseteq)$; and this, in conjunction with $\text{card}(W(\alpha) \times W(\omega)) = \text{card}(W(\alpha))$ (cf. Alexandrov [1, Ch 3, Sect 6]) shows that $\mathcal{U}$ is Γ-pseudometrizable, where $\Gamma = \text{card}(W(\alpha))$. A translation of Proposition 5 in terms of $D = (d_\lambda; \lambda < \alpha)$ is immediate; we do not give details.

In particular, when $\Gamma = \omega$, these developments reduce to the ones in Turinici [25]; see also Zhu and Li [28].

3 Zorn-Bourbaki principles

Let M be a nonempty set; and $(\leq)$, some *order* (antisymmetric quasi-order) on it. Call the point $z \in M$, $(\leq)$-*maximal* when

(a3) $z \leq w \in M \Longrightarrow z = w$; or, equivalently: $z < x$ is false, $\forall x \in M$.

[Here, $(<)$ is the *strict order* attached to $(\leq)$]. Sufficient conditions for the existence of such elements may be obtained as follows. Call the (nonempty) part A of M, a *linear* $(\leq)$-*chain* provided $(A, \leq)$ is linearly ordered [$\forall x, y \in A$: either $x \leq y$ or $y \leq x$]; and a *(natural)* $(\leq)$-*chain*, when $(A, \leq)$ is well ordered.

Theorem 2 *Suppose that one of the conditions below holds*

(3a) each linear $(\leq)$*-chain (of* M*) is bounded above*

(3b) each $(\leq)$*-chain (of* M*) is bounded above.*

Then, $(\leq)$ *is a normal order, in the sense: for each* $u \in M$ *there exists a* $(\leq)$*-maximal* $v \in M$ *with* $u \leq v$.

(A) Some remarks are in order. The first explicit formulation of Theorem 2 in terms of (3a) was given in 1914 by Hausdorff [11, Ch 6, Sect 1]; a slight different version of it was obtained in 1922 by Kuratowski [16]. Note that the quoted authors regarded Theorem 2 only as a handy tool in solving various existence problems in the setting of (AC)(= the Axiom of Choice). Finally, again under the lines of (3a), we must mention the 1935 contribution due to Zorn [29]; who regarded Theorem 2 as an axiom. The version of this result involving (3b) was stated in Bourbaki [2]; who also established its equivalence with the Well Ordering Principle in Zermelo [26] (equivalent with (AC)). For this reason, it is natural that Theorem 2 be referred to as the Zorn-Bourbaki (maximal) principle. Note that, in the context of (AC), we have: (3b) $\Longrightarrow$ (3a) (hence (3b) $\Longleftrightarrow$ (3a)); see e.g. Felgner [8]. Further historical aspects may be found in Moore [18, Ch 4, Sect 4] and the references therein.

(B) Now, as results from the developments in Section 2, the verification of (3b) for (cardinal-) countable chains only will suffice (for its validity) in many concrete cases. This suggests us considering maximality principles over ordered structures with such regularity properties. So, let $(M, \leq)$ be a (partially) ordered structure; and fix some sep-admissible cardinal $\Gamma = \aleph_\gamma$ (for some $\gamma \geq 0$). [The best choice of this object is $\Gamma = \text{sep}(M, \leq)$; but, it is not in general accessible. So, we may choose for the moment any cardinal $\Gamma \geq \mathrm{card}(M)$; which, as precise, is sep-admissible]. By definition,

$$\begin{array}{l}(M, \leq) \text{ is (strongly) } \Gamma\text{-separable: each } (\leq)\text{-chain } S \subseteq M \\ \text{is majorized by some } \Gamma\text{-sub } (\leq)\text{-chain } T \subseteq S.\end{array} \tag{3.1}$$

Remember that, by Proposition 4, this also reads

$$\text{each } (\leq)\text{-chain of } M \text{ is } \Gamma\text{-countable.} \tag{3.2}$$

Further, assume that

(3c) $(M, \leq)$ is sequentially Γ-inductive: each Γ-sub $(\leq)$-chain of M is bounded from above (modulo $(\leq)$).

Note that, by Proposition 2 this writes

(3d) each ascending θ-net (where $\omega \leq \theta \leq \Gamma$) is bounded above.

So, in the particular case of $\Gamma = \omega$, its associated notion is identical with the concept of sequentially inductive (ordered) structure in Turinici [23]. Moreover, by Proposition 3, it may be also written as

(3e) each Γ-countable $(\leq)$-chain of M is bounded above (modulo $(\leq)$).

Putting these together gives (3b); so that, we get:

Theorem 3 *Suppose that the sep-admissible cardinal* Γ *is such that (3d) holds. Then, conclusion of Theorem 2 is retainable.*

For the moment, Theorem 3 is deductible from Theorem 2. The reverse inclusion is also true; just take $\Gamma = \mathrm{card}(M)$ in the above statement. Hence Theorem 3 and Theorem 2 are logically equivalent. In other words: the enlargement of Theorem 2 assured by Theorem 3 is technical in nature.

(C) An interesting completion of these facts may be given along the following lines. Let M be a nonempty set; and $(\leq)$, a quasi-order (reflexive and transitive relation) over it. The associated relation

(b3) $(x, y \in M)$: $x \prec y$ iff $x \leq y$ and $\neg(y \leq x)$

is *irreflexive* ($\neg(x \prec x)$, $\forall x \in M$) and transitive; hence a *strict order*. As a consequence, its completion

(c3) $(x, y \in M)$: $x \preceq y$ iff either $x \prec y$ or $x = y$

is an order on M. For an element $z \in M$, its $(\preceq)$-maximal property is the one in (a3) (with $(\preceq)$ in place of $(\leq)$); which, in terms of $(\leq)$ means:

(d3) $z \leq w \Longrightarrow w \leq z$ (or, equivalently: $M(z, \leq) \subseteq M(z, \geq)$);

referred to as: z is $(\leq)$-maximal. [This must be not confused with the strong $(\leq)$-maximality of z, which may is introduced as in (a3) (the first part): $z \leq x \Longrightarrow z = x$]. Concrete circumstances for the existence of such points are obtainable from Theorem 3 above. For practical reasons, it would be useful having one of the conditions (3c)-(3e) expressed in terms of our quasi-order. For each θ-net $(u_\xi; \xi < \theta)$ in M, define its ascending/boundedness (modulo $(\leq)$) properties in the usual way.

Theorem 4 *Let the sep-admissible cardinal* Γ *for* $(M, \preceq)$ *be such that (3d) holds with respect to the ambient quasi-order* $(\leq)$. *Then, for each* $u \in M$ *there exists* $v = v(u) \in M$ *with* **(i)** $u \leq v$ *and* **(ii)** $M(v, \leq) \subseteq M(v, \geq)$ *(or, equivalently:* $v \leq x$ *is false, whenever so is* $x \leq v$*).*

Proof Let P be a Γ-sub $(\preceq)$-chain in M; hence, it may be represented as a net $(u_\xi; \xi < \mu)$ where $\mu \leq \Gamma$ and the map $\xi \mapsto u_\xi$ is strictly ascending:

$$\xi < \eta \text{ implies } u_\xi \prec u_\eta \text{ [hence } u_\xi \leq u_\eta,\ \neg(u_\eta \leq u_\xi)].$$

When μ is a first kind ordinal, we are done; because $P \preceq u_\lambda$, where $\lambda = \mu - 1$. Assume now that μ is a second kind ordinal. By the strict ascending property above, the net $(u_\xi; \xi < \mu)$ is ascending (modulo $(\leq)$); wherefrom, by hypothesis, $u_\xi \leq v$, for all $\xi < \mu$ and some $v \in M$. On the other hand, again by the property in question, $v \leq u_\xi$ is impossible for all $\xi < \mu$; since for each η with $\xi < \eta < \mu$ we should have $u_\eta \leq v \leq u_\xi$ (hence $u_\eta \leq u_\xi$), in contradiction with a previous relation. Hence, v is an upper bound of P (modulo $(\preceq)$; and as such, (3c) follows (with respect to the order $(\preceq)$). But then, all is clear from Theorem 3, applied to the structure $(M, \preceq)$. □

Note finally that further variants of these results are possible for transitive relations; as well as for amorphous ones. But, for our purposes, the above discussed facts will suffice.

4 Regular cones in tvs

Let Y be a (real) linear space; and $\mathcal{T}$ be some linear topology over it [i.e.: the linear space operations are continuous]. Usually, $\mathcal{T}$ may be generated by a certain family $\mathcal{B}$ of (nonempty) parts of Y with

(a4) $(\mathcal{B}, \supseteq)$ is directed (for each $B_1, B_2 \in \mathcal{B}$ there exists $B_3 \in \mathcal{B}$ with $B_1 \cap B_2 \supseteq B_3$); [wherefrom, $\mathcal{B}$ is a filterbase]

(b4) each $B \in \mathcal{B}$ is balanced ($[-1, 1]B \subseteq B$); hence, in particular, B is symmetric ($B = -B$) and contains the origin ($0 \in B$)

(c4) each $B \in \mathcal{B}$ is absorbing [$\forall y \in Y$, $\exists \varepsilon = \varepsilon(y) > 0$: $[-\varepsilon, \varepsilon]y \subseteq B$]

(d4) for each $B \in \mathcal{B}$ there exists $D \in \mathcal{B}$ with $D + D \subseteq B$.

In fact, under these requirements, there may be constructed a (uniquely determined) linear topology on Y such that $\mathcal{B}$ should represent a fundamental system of balanced zero neighborhoods; cf. Cristescu [6, Ch 1, Sect 2]. Conversely, given a linear topology $\mathcal{T}$ on Y, the family

$\mathcal{B}$ of all balanced zero neighborhoods fulfills (a4)-(d4) above. This allows us expressing $\mathcal{T}$-concepts in terms of $\mathcal{B}$, and vice versa. Now, starting from this, let us construct a family $U(\mathcal{B}) = \{U(B); B \in \mathcal{B}\}$ of relations on Y according to: $U(B) = \{(a, b) \in Y^2; b - a \in B\}$, $B \in \mathcal{B}$. The following properties are immediate (if we take into account the preceding ones):

$$U(B_3) \supseteq U(B_1) \cap U(B_2), \text{ where } B_1, B_2, B_3 \text{ are as in (a4)} \tag{4.1}$$

$$U(B) \supseteq \mathcal{I}(Y) := \{(y, y); y \in Y\}, \text{ for all } B \in \mathcal{B} \tag{4.2}$$

$$U(B) = U(B)^{-1}, \text{ for each } B \in \mathcal{B} \tag{4.3}$$

$$U(D) \circ U(D) \subseteq U(B), \text{ where } D, B \text{ are as in (d4).} \tag{4.4}$$

As a consequence, $U(\mathcal{B})$ is a fundamental system of entourages for a uniformity $\mathcal{U}(\mathcal{T})$ on Y introduced as in Bourbaki [3, Ch 2, Sect 1] [obtainable by taking super-sets of the parts in $U(\mathcal{B})$]. However, all basic concepts related to it may be phrased in terms of $U(\mathcal{B})$; hence, ultimately, in terms of $\mathcal{B}$.

(A) Further aspects in this direction are to be carried out in a conical setting; and involve the Cauchy (asymptotic) property of a net (sequence). So, let K be some *convex cone* of Y; and $(\leq_K)$ stand for the induced quasi-order. Given the net $(a_\xi; \xi < \lambda)$ in Y, call it $(\leq_K)$-ascending (descending) [or, simply: K-ascending (descending)] when $\xi < \eta \Longrightarrow a_\xi \leq_K a_\eta$ ($a_\xi \geq_K a_\eta$); the union of these will be referred to as $(a_\xi; \xi < \lambda)$ is $(\leq_K)$-monotone [or simply: K-monotone]. Also, call this net $(\leq_K)$-bounded above (below) [or, simply: K-bounded above (below)] if $a_\xi \leq_K v$ ($a_\xi \geq_K u$) for all $\xi < \lambda$ and some $v \in Y$ ($u \in Y$); the intersection of these will be referred to as $(a_\xi; \xi < \lambda)$ is $(\leq_K)$-bounded [or simply: K-bounded].

Now, given the K-monotone net $(a_\xi; \xi < \lambda)$ in Y, the Cauchy property of it with respect to the above uniformity means: $\forall B \in \mathcal{B}$, $\exists \mu = \mu(B)$ such that $\mu \leq \xi \leq \eta \Longrightarrow a_\eta - a_\xi \in B$; it is referred to as $\mathcal{B}$-Cauchy (because only the elements of $\mathcal{B}$ are entering here). Note that, by (4.3)+(4.4), this property may be written in the simplified form:

(e4) $\forall B \in \mathcal{B}$, $\exists \nu = \nu(B)$ such that $\xi \geq \nu \Longrightarrow a_\xi - a_\nu \in B$.

Likewise, for the K-monotone sequence $(b_n; n < \omega)$, the asymptotic property with respect to the same uniformity means:

(f4) $\forall B \in \mathcal{B}$, $\exists m = m(B)$ such that $n \geq m \Longrightarrow b_{n+1} - b_n \in B$.

As before, it will be referred to as a $\mathcal{B}$-asymptotic one (for the sequence in question).

(B) Let H be a subcone of K. We claim that the global conditions below are equivalent:

(4a) each K-bounded H-ascending net (in Y) is $\mathcal{B}$-Cauchy

(4b) each K-bounded H-ascending sequence (in Y) is $\mathcal{B}$-asymptotic.

In fact, (4a) $\Longrightarrow$ (4b) is clear. For the converse relation, assume that the K-bounded H-ascending net $(a_\xi; \xi < \lambda)$ (in Y) is not $\mathcal{B}$-Cauchy. By the remark involving (e4), it follows that there must be some $B \in \mathcal{B}$ with

$$\text{for each } \xi < \lambda, \text{ there exists } \eta > \xi \text{ such that } a_\eta - a_\xi \in B^c. \tag{4.5}$$

(Here, by definition, $B^c := Y \setminus B$). Consequently, a sequence $(b_n; n < \omega)$ may be found (with the aid of these terms) so as:

$$b_{n+1} - b_n \in B^c, \text{ for all } n < \omega. \tag{4.6}$$

In addition, $(b_n; n < \omega)$ is by construction K-bounded and H-ascending. But, in this case, (4b) is contradicted; hence the claim. On the other hand, (b4) tells us that the above conditions are equivalent with, respectively,

(4c) each K-bounded H-descending net (in Y) is $\mathcal{B}$-Cauchy

(4d) each K-bounded H-descending sequence (in Y) is $\mathcal{B}$-asymptotic.

By convention, (4a)/(4c) will be referred to as H is K-bound regular. We therefore obtained that another description of this property is (4b)/(4d); note that it extends the one due to Nemeth [19] in the locally convex case.

Another useful characterization of the concept in question is to be given under the lines below. For each sequence $(h_n; n < \omega)$ in Y, let $(k_n := \sum_{i \leq n} h_i; n < \omega)$ stand for the partial sum sequence attached to it.

Lemma 2 *The subcone H of K is K-bound regular if and only if*

(4e) H is asymptotically K-regular: whenever $(h_n; n < \omega)$ in H and $B \in \mathcal{B}$ fulfill $h_n \in B^c$ for infinitely many n, the associated partial sum sequence cannot be K-bounded.

Proof Assume that H is K-bound regular; and let $(h_n; n < \omega) \subseteq H$, $B \in \mathcal{B}$ be as in the premise of (4e). The associated partial sum sequence $(k_n; n < \omega)$ in H is H-ascending. Suppose by contradiction that it is K-bounded: $k_n \leq_K b$, for all n and some $b \in Y$. By the K-bound regularity of H (expressed in terms of (4b)) it follows that there exists $m = m(B)$ so that $h_{n+1} = k_{n+1} - k_n \in B$, for all $n \geq m$; in contradiction with the choice of $(h_n; n < \omega)$; hence the necessity. Conversely, assume that H is asymptotically K-regular; and let $(a(\xi); \xi < \lambda)$ be a K-bounded H-monotone net in Y which is not $\mathcal{B}$-Cauchy. Again by the remark involving (e4), there must be some $B \in \mathcal{B}$ with the property (4.5). Now, there is no loss in assuming that $(a_\xi; \xi < \lambda)$ is H-ascending. Then (see above) we arrive at a K-bounded H-ascending sequence $(b_n; n < \omega)$ [if we single out the terms appearing in (4.5)] so that (4.6) be valid. Denote $h_n = b_{n+1} - b_n, n < \omega$. The sequence $(h_n; n < \omega)$ is in H and satisfies $h_n \in B^c$, for all n; so, by (4e), the associated partial sum sequence $(k_n = b_{n+1} - b_0; n < \omega)$ is not K-bounded. On the other hand, this sequence is K-bounded, as a translate of $(b_n; n < \omega)$. The contradiction at which we arrived yields the sufficiency. □

5 Main result

With these information at hand, we may now return to the questions formulated in Section 1.

(A) Let Y be a (real) vector space; and $\mathcal{T}$ be some linear topology over it; remember that (cf. Section 4) it is obtainable from a family of (nonempty) sets $\mathcal{B}$ taken as in (a4)-(d4).

Further, let K be a pointed convex cone of Y; the associated relation $(\leq_K)$ is an order; also denoted as $(\leq)$, when K is understood. Take also some subcone H of K; note that (by these hypotheses) H is pointed too; hence $(\leq_H)$ is again order. The working condition to be accepted here is

(5a) H is K-bound regular (cf. Section 4).

Remember that there is another way of expressing it, subsumed to Lemma 2 above.

(B) Let X be a nonempty set. By a H-*almost metric* (in short: H-*ametric*) over X we shall mean any map $d : X \times X \to H$ with the properties (b1)-(d1) (relative to H). In other words, a H-ametric is a H-metric without symmetry. Suppose that we introduced such an object. Define

(a5) $U(d; B) = \{(x, y) \in X \times X; d(x, y) \in B\},\ B \in \mathcal{B}$.

The family $U(d; \mathcal{B}) = \{U(d; B); B \in \mathcal{B}\}$ fulfills conditions (4.1)+(4.2) (under these notations); but not (4.3)+(4.4), in general. So, it is not a fundamental system of entourages for a uniformity over X; we shall term $U(d; \mathcal{B})$, the *pseudo-uniformity* on X (generated by d and $\mathcal{B}$). Note that, even in this case, the concept of convergent and Cauchy net is meaningful. Precisely, we say that the net $(u_\xi; \xi < \lambda)$ (in X) d-*converges* to $u \in X$ (and we write this as $u_\xi \to u$) if: for each $B \in \mathcal{B}$, there exists $\mu = \mu(B)$ such that $\xi \geq \mu$ implies $d(u_\xi, u) \in B$. When u is generic in this convention, the net in question is called d-*convergent*. Further, call the net $(u_\xi; \xi < \lambda)$ (in X) d-*Cauchy* provided: for each $B \in \mathcal{B}$, there exists $\mu = \mu(B)$ such that $\mu \leq \xi \leq \eta$ implies $d(u_\xi, u_\eta) \in B$. We stress that, by the choice of our data, the d-convergence property cannot imply the d-Cauchy one in general. However, one says that (X, d) is complete when each d-Cauchy net is d-convergent.

(C) Let $(\preceq)$ be the (partial) order on $X \times Y$ introduced as in (f1); and A be some (nonempty) part of $X \times Y$. As in Section 1, we are interested to determine sufficient conditions under which the ordered structure $(A, \preceq)$ has maximal points (in the precise sense). The basic tool for this is (again) the Zorn-Bourbaki principle; but under the "local" form described in Section 3. Technically speaking, the specific conditions to be used are (1d) and an "ordinal" variant of (1e). For an exact formulation, we need some preliminaries. Remember that by $\mathrm{Sep}(A, \preceq)$ we denoted the class of all sep-admissible ordinals for $(A, \preceq)$; and by $\mathrm{sep}(A, \preceq)$, the first element of it.

Lemma 3 *Assume that the general conditions above are valid, as well as (1d). Then*

$$\mathrm{sep}(A, \preceq) \leq \mathrm{sep}(P_Y(A), \geq) \leq \mathrm{sep}(K, \geq). \tag{5.1}$$

Proof By definition, we have $(x_1, y_1) \prec (x_2, y_2) \Longrightarrow y_1 > y_2 \Longleftrightarrow y_1 - \widetilde{y} > y_2 - \widetilde{y}$. Here, $(\prec)$ and $(>)$ are the strict orders attached to $(\preceq)$ and $(\geq)$ respectively. As a consequence of this, the image (via P_Y) of a $(\preceq)$-chain in A is a $(\geq)$-chain in $P_Y(A)$; and the image (via the translation $y \mapsto y - \widetilde{y}$) of a $(\geq)$-chain in $P_Y(A)$ is a $(\geq)$-chain in K. This, and the definition of the separability index ends the argument. □

The following fact has a practical meaning for us. Bet $B \in \mathcal{B}$ be arbitrary fixed. We say that the point $(x_0, y_0) \in A$ is B-*almost minimum* (in short: B-*aminimum*) when (b5) $(x', y'), (x", y") \in A$, $(x_0, y_0) \preceq (x', y') \preceq (x", y") \Longrightarrow d(x', x") \in B$.

Lemma 4 *Let the same (general) conditions be in use; as well as (1d)+(5a). Then, for each* $(x_0, y_0) \in A$ *and each* $B \in \mathcal{B}$ *there exists a* B*-aminimum point* $(x_B, y_B) \in A$ *with* $(x_0, y_0) \preceq (x_B, y_B)$.

Proof Suppose that this is not true:

(5b) each $(x, y) \in A$ with $(x_0, y_0) \preceq (x, y)$ is not B-aminimum.

Applying this condition to $(x, y) = (x_0, y_0)$, there must be (x_1, y_1) and (x_2, y_2) in A with $(x_0, y_0) \preceq (x_1, y_1) \preceq (x_2, y_2)$, $d(x_1, x_2) \in B^c$. Again by this condition (applied to $(x, y) = (x_2, y_2)$) there exist (x_3, y_3) and (x_4, y_4) in A with $(x_2, y_2) \preceq (x_3, y_3) \preceq (x_4, y_4)$, $d(x_3, x_4) \in B^c$. The procedure may continue indefinitely. It gives us a $(\preceq)$-ascending sequence $((x_n, y_n); n < \omega) \subseteq A$ with:

$$h_{2n+1} \in B^c, \forall n; \text{ where } h_n := d(x_n, x_{n+1}), n < \omega. \tag{5.2}$$

On the other hand, the choice of our sequence gives $h_n \le y_n - y_{n+1}$, for each $n < \omega$; wherefrom (by (1d))

$$k_n := \sum_{i \le n} h_i \le y_0 - y_{n+1} \le y_0 - \widetilde{y}, \ \forall n < \omega. \tag{5.3}$$

The obtained facts are contradictory in the light of Lemma 2. Hence, (5b) must be rejected; and conclusion follows. □

Fix in the following some sep-admissible cardinal Γ for $(K, \ge)$. We assume

(5c) for each $(\preceq)$-ascending θ-net $((u_\xi, v_\xi); \xi < \theta)$ in A (where $\theta \le \Gamma$) with $(u_\xi; \xi < \theta)$ d-Cauchy, $\exists\ (u, v) \in A$: $(u_\xi, v_\xi) \preceq (u, v)$, $\forall \xi < \theta$.

The main result of this exposition is (under the same general conditions):

Theorem 5 *Assume that (1d)+(5a)+(5c) hold. Then*

(I) *for each* $(x_0, y_0) \in A$ *there exists* $(\bar{x}, \bar{y}) \in A$ *with* **i)** $(x_0, y_0) \preceq (\bar{x}, \bar{y})$, **ii)** $(\bar{x}, \bar{y}) \preceq (x, y) \in A \Longrightarrow (\bar{x}, \bar{y}) = (x, y)$, **iii)** $\bar{y}$ *is maximal in* $(A(\bar{x}), \ge)$.

(II) *for each* $(x_*, y_*) \in A$ *and* $B \in \mathcal{B}$ *there exists a* B*-aminimum point* $(x_0, y_0) \in A$ *with* $(x_*, y_*) \preceq (x_0, y_0)$. *Moreover, if* $(\bar{x}, \bar{y}) \in A$ *is the point given by* **(I)**, *we have in addition* **iv)** $d(x_0, \bar{x}) \in B$.

Proof It will suffice verifying the first part; because the second one follows at once by Lemma 4. Note that, in view of Lemma 3, the cardinal Γ appearing in (5c) is sep-admissible for $(A, \preceq)$. Let $((u(\xi), v(\xi)); \xi < \theta)$ be a $(\preceq)$-ascending θ-net in A (where $\theta \le \Gamma$); i.e.,

(5d) $d(u(\xi), u(\eta)) \le v(\xi) - v(\eta)$, whenever $\xi \le \eta (< \theta)$.

We claim that the projection net $(u(\xi); \xi < \theta)$ is d-Cauchy. Suppose not: there must be some $B \in \mathcal{B}$ such that

(5e) $\forall \mu$, $\exists (\xi, \eta)$ with $\mu \leq \xi \leq \eta$, $d(u(\xi), u(\eta)) \in B^c$.

By the same way as in Lemma 4, there exists an ascending sequence of ranks $(\nu_n; n < \omega)$ in $W(\theta)$ so that the sequence $((x_n, y_n); n < \omega)$ in A, where $(x_n = u(\nu_n), y_n = v(\nu_n); n < \omega)$ is $(\preceq)$-ascending and fulfills (5.2)+(5.3). But, as precise, these relations are in contradiction with (5a) (via Lemma 2); hence (5e) cannot hold and our claim follows. By (5c), our θ-net is bounded from above in A (with respect to $(\preceq)$). Summing up, Theorem 3 applies to $(A, \preceq)$ and Γ. It gives us, for the starting point $(x_0, y_0) \in A$, some other element $(\bar{x}, \bar{y}) \in A$ with the properties **i)** and **ii)**. Let $y \in A(\bar{x})$ be such that $y \leq \bar{y}$. By definition, this gives $(\bar{x}, \bar{y}) \preceq (\bar{x}, y)$; wherefrom (by **ii)**) $(\bar{x}, \bar{y}) = (\bar{x}, y)$; that is, $\bar{y} = y$; hence, **iii)** holds as well. □

It is worth stressing that, in the reasoning above, the multiplication properties of Y were not used. So, Theorem 5 is valid when Y is just a topological Abelian group. Note that the corresponding version of this result extends a related variational principle in Nemeth [20].

6 Particular aspects

Roughly speaking, the obtained result is an "abstract" one. So, for technical reasons, it would be useful having "concrete" realizations of it; which should also give its technical connections with some of the existing contributions in the area.

Let $(Y, \mathcal{T})$ be a (real) topological vector space; where the linear topology $\mathcal{T}$ is obtainable from a family $\mathcal{B}$ of (nonempty) sets taken as in (a4)-(d4). Further, let K be a pointed cone in Y; and $(\leq)$, its associated order. Take also a subcone H of K; it is pointed too (hence the associated relation $(\leq_H)$ is again an order). Further, let X be a nonempty set; and $d : X \times X \to H$ be some H-ametric over it (cf. (b1)-(d1)). Define an order $(\preceq)$ over $X \times Y$ under (f1); and let A be some (nonempty) part of $X \times Y$. The specific conditions to be accepted for the moment are (1d)+(5a).

(A) Concerning the specific condition (5c), it would be natural that the point $u \in X$ appearing there to be obtained as limit of $(u_\xi; \xi < \theta)$. So, let $\Gamma \in \mathrm{Sep}(K, \geq)$ be fixed in the sequel. Assume that

(6a) (X, d) is Γ-complete:
each d-Cauchy θ-net in X (where $\theta \leq \Gamma$) is d-convergent.

Then, the appropriate version of (5c) to be considered is

(6b) for each $(\preceq)$-ascending θ-net $((u_\xi, v_\xi); \xi < \theta)$ in A (where $\theta \leq \Gamma$) with $u_\xi \to u$ we have $u \in P_X(A)$ and $\exists v \in A(u)$: $(u_\xi, v_\xi) \preceq (u, v)$, $\forall \xi < \theta$.

Putting these together, we have the following "convergence type" variant of the statement above:

Theorem 6 *Let the precise conditions be in use; as well as (6a)+(6b). Then, conclusions of Theorem 5 are retainable.*

Proof (Sketch) Let $((u_\xi, v_\xi); \xi < \theta)$ be a $(\preceq)$-ascending net in A (where $\theta \leq \Gamma$) with $(u_\xi; \xi < \theta)$ d-Cauchy. By (6a), $u_\xi \to u$ for some $u \in X$; and this, along with (6b), gives

$u \in P_X(A)$ and $\exists v \in A(u)$ with $(u_\xi, v_\xi) \preceq (u, v)$, $\forall \xi < \theta$. In other words, (5c) holds for our data; and, from this, all is clear via Theorem 5 above. □

In particular, when $(Y, \mathcal{T})$ is a locally convex space and the H-ametric d is symmetric (hence a H-metric) Theorem 6 reduces to Theorem 1. However, we must note that the argument used here is rather different from the one appearing there; in addition, the normality of H may be dropped.

(B) Let us now return to the specific condition (6b). Note that, as a direct consequence of its last relation, $v_\xi \geq v$, $\forall \xi < \theta$. So, the condition below is deductible from (6b) above:

(6c) for each $(\preceq)$-ascending θ-net $((u_\xi, v_\xi); \xi < \theta)$ in A (where $\theta \leq \Gamma$) with $u_\xi \to u$ we have $u \in P_X(A)$ and $\exists v \in A(u)$: $v_\xi \geq v$, $\forall \xi < \theta$.

It is then legitimate to ask under which supplementary conditions is (6b) deductible from this one. An appropriate answer is to be obtained under

(6d) K is closed (hence, so is $(-K)$).

Remember that, the closure of any subset V of Y may be expressed as

$$\mathrm{cl}(V) = \cap\{V + B; B \in \mathcal{B}\} = \cap\{V - B; B \in \mathcal{B}\}; \tag{6.1}$$

see for instance Cristescu [6, Ch 1, Sect 2].

Theorem 7 *Let the same specific conditions hold; as well as (6a), (6c), (6d). Then, conclusions of Theorem 5 are retainable.*

Proof As precise, it will suffice verifying that (6b) is valid under (6c)+(6d). So, let $((u_\xi, v_\xi); \xi < \theta)$ be some $(\preceq)$-ascending net in A (where $\theta \leq \Gamma$) with $u_\xi \to u$. By (6c), $u \in P_X(A)$ and there exists $v \in A(u)$ with $v_\xi \geq v$, $\forall \xi < \theta$. We claim that (u, v) is our desired point. In fact, let $\lambda < \theta$ be arbitrary fixed. From the choice of our data, we have $d(u_\lambda, u_\eta) \leq v_\lambda - v_\eta \leq v_\lambda - v$, if $\lambda \leq \eta < \theta$. This, along with the triangular property of d, yields

$$d(u_\lambda, u) \leq d(u_\lambda, u_\eta) + d(u_\eta, u) \leq v_\lambda - v + d(u_\eta, u), \text{ whenever } \lambda \leq \eta < \theta.$$

Let $B \in \mathcal{B}$ be arbitrary fixed. As $u_\xi \to u$, there exists $\mu = \mu(B) \geq \lambda$ in such a way that $\eta \geq \mu \Longrightarrow d(u_\eta, u) \in B$. Taking $\eta \geq \mu$ in the relation above gives

$$d(u_\lambda, u) - [v_\lambda - v] \in -K + d(u_\eta, u) \subseteq -K + B.$$

As B was arbitrarily chosen in $\mathcal{B}$, one gets

$$d(u_\lambda, u) - [v_\lambda - v] \in \mathrm{cl}(-K) = -K; \text{ i.e., } (u_\lambda, v_\lambda) \preceq (u, v).$$

Hence, (6b) holds; and the conclusion follows. □

In particular, (6c) holds under

(6e) A is (Γ, K)-submonotone: for each θ-net $((u_\xi, v_\xi); \xi < \theta)$ in A (where $\theta \le \Gamma$) with $u_\xi \to u$ and $(v_\xi; \xi < \theta)$ K-descending, we have $u \in P_X(A)$ and there exists $v \in A(u)$ with $v_\xi \ge v$, $\forall \xi < \theta$.

Note that, in such a case, Theorem 7 includes the related statement in Chen, Huang and Hou [4, Theorem 4.1] (the alternative **(I)**) provided (in addition), $(Y, \mathcal{T})$ is locally convex and d is symmetric (hence a H-metric).

(C) An interesting variant of (6e) is to be reached under compactness assumptions about the sections of A. Precisely, assume that

(6f) $(Y, \mathcal{T})$ is Hausdorff separated ($\cap \mathcal{B} = \{0\}$)

(6g) $A(x)$ is compact, for each $x \in \mathrm{Dom}(A)$.

Call A, (Γ, K)-*almost submonotone* (in short: (Γ, K)-*asubmonotone*) if

(6h) for each θ-net $((u_\xi, v_\xi); \xi < \theta)$ in A (where $\theta \le \Gamma$) with $u_\xi \to u$ and $(v_\xi; \xi < \theta)$ K-descending we have $u \in P_X(A)$ and $A(u)(v_\xi, \ge) \ne \emptyset$, for each $\xi < \theta$.

The following auxiliary fact is available.

Lemma 5 *Assume that (6d)+(6f)+(6g) hold. Then, (6h) implies (6e).*

Proof Let the θ-net $((u_\xi, v_\xi); \xi < \theta)$ in A be as in the premise of this implication. By (6f)+(6g), $A(u)$ is closed in Y; see, for instance, Bourbaki [3, Ch 1, Sect 9.3]. This, along with (6d), tells us that $G_\xi := A(u)(v_\xi, \ge)$ is nonempty closed; hence, a fortiori, closed in $A(u)$ (endowed with the relative topology); in addition, the family $(G_\xi; \xi < \theta)$ has the finite intersection property. By the compactness of $A(u)$ we therefore derive $\cap\{G_\xi; \xi < \theta\} \ne \emptyset$; see, e.g., Kelley [15, Ch 5, Sect 1]. Clearly, any point v in this intersection fulfills $(u, v) \in A$ and $v_\xi \ge v$, $\forall \xi < \theta$; so that, we are done. □

Now, by simply combining this with the preceding statement, one gets (under the general/specific conditions of Theorem 7):

Theorem 8 *Assume that (in addition) (6a), (6d) and (6f)-(6h) hold. Then, conclusions of Theorem 5 are retainable.*

In particular, when $(Y, \mathcal{T})$ is a locally convex space, this result includes the related one in Chen, Huang and Hou [4, Theorem 4.1] (the alternative **(II)**) provided (in addition), d is symmetric (hence a H-metric).

(D) Finally, some remarks must be made about our data.

i) Let $k^0 \in H \setminus \{0\}$ be some point; and $e : X \times X \to R_+$ be an *almost metric* (in short: *ametric*) over X; i.e., it has all the properties of a (standard) metric, except symmetry. Then, the application $d(x, y) = k^0 e(x, y)$, $(x, y \in X)$, is a H-ametric on X. The corresponding version of Theorem 5 under this choice is the main result in Goepfert, Tammer and Zălinescu [9, Theorem 1].

ii) Let $F : X \to Y$ be some function and $A = \mathrm{gr}(F)(:= \{(x, F(x)); x \in X\})$. The corresponding version of Theorem 7 under this choice is just the statement in Nemeth [19], Isac [13] and Turinici [24]; which as shown, extend Ekeland's variational principle [7].

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