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Brezis-Browder Principles and Equilibrium Points[1]

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Abstract. Some pseudometric versions of the Brezis-Browder ordering principle [Adv. Math., 21 (1976), 355-364] are discussed. An application of these facts to equilibrium points is also included.

Keywords: Quasi-order, maximal element, sequential inductivity, pseudometric, Cauchy/asymptotic sequence, regularity, convergence structure, equilibrium point.

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1 Introduction

Let M be a nonempty set. Take a *quasi-order* $(\leq)$ (i.e.: reflexive and transitive relation) over it, as well as a function $x \mapsto \psi(x)$ from M to $R_+ := [0, \infty[$. Call the point $z \in M$, $(\leq, \psi)$-*maximal* when: $w \in M$ and $z \leq w$ imply $\psi(z) = \psi(w)$. A basic result about the existence of such points is the 1976 Brezis-Browder ordering principle [5]:

Theorem 1 *Suppose that*

(1a) $(M, \leq)$ *is sequentially inductive:*
each ascending sequence has an upper bound (modulo $(\leq)$*)*

(1b) ψ *is* $(\leq)$*-decreasing* $(x \leq y \Longrightarrow \psi(x) \geq \psi(y))$.

Then, for each $u \in M$ *there exists a* $(\leq, \psi)$*-maximal* $v \in M$ *with* $u \leq v$.

This statement, including the well known Ekeland's variational principle [9], found some useful applications to convex and nonconvex analysis (cf. the above references). So, it cannot be surprising that many extensions of Theorem 1 were proposed; cf. Bae, Cho and Yeom [3], Altman [1] and Anisiu [2]. These are interesting from a technical perspective. However, we must emphasize that, in all concrete situations when a maximality principle of this type is to be applied, a substitution of it by the Brezis-Browder's is always possible. This raises the question of to what extent are such enlargements of Theorem 1 effective. As we shall see

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below (in Section 2) the answer is negative for most of these. On the other hand, there do exist metrical maximality principles which are not comparable with Theorem 1; see Kang and Park [13]. It is our second aim in this exposition to show (cf. Section 3) that all such statements may be viewed as particular cases of an "asymptotic" type version of Theorem 1. Finally, in Section 4, an application of these facts is given to equilibrium points in the Oettli-Thera sense [17]. Further aspects will be delineated elsewhere.

2 Equivalents of Theorem 1

Let $(M, \leq)$ be a quasi-ordered structure; and $x \mapsto \varphi(x)$ stand for a function between M and $R_+ \cup \{\infty\} = [0, \infty]$.

Proposition 1 *Assume (1a) and (1b) are true, as well as*

(2a) $(M, \leq)$ *is almost regular (modulo* φ*):*
$\forall x \in M$, $\forall \varepsilon > 0$, $\exists y = y(x, \varepsilon) \geq x$ *with* $\varphi(y) \leq \varepsilon$.

Then, for each $u \in M$ *there exists* $v \in M$ *with* $u \leq v$ *and* $\varphi(v) = 0$ *(hence* v *is* $(\leq, \varphi)$*-maximal).*

Proof By (2a), there must be some $z \geq u$ with $\varphi(z) < \infty$. Clearly, (1a)+(1b) apply to $M(z, \leq) := \{x \in M; z \leq x\}$ and $(\leq, \varphi)$. So, for $z \in M(z, \leq)$ there exists $v \in M(z, \leq)$ with **i)** $z \leq v$ (hence $u \leq v$) and **ii)** v is $(\leq, \varphi)$-maximal in $M(z, \leq)$. Suppose by contradiction that $\gamma := \varphi(v) > 0$; and fix some β in $]0, \gamma[$. By (2a) again, there must be $y = y(v, \beta) \geq v$ (hence $y \in M(z, \leq)$) with $\varphi(y) \leq \beta < \gamma (= \varphi(v))$. This cannot be in agreement with the second conclusion above. Hence, $\varphi(v) = 0$; and we are done. □

As a consequence, Proposition 1 is deductible from Theorem 1. But, the converse inclusion is also true; to verify it, we need some conventions. By a (generalized) *pseudometric* over M we shall mean any map $d : M \times M \to R_+ \cup \{\infty\}$. Fix such an object; supposed to be *reflexive* $[d(x, x) = 0, \forall x \in M]$. Call $z \in M$, $(\leq, d)$*-maximal*, if: $u, v \in M$ and $z \leq u \leq v$ imply $d(u, v) = 0$. Note that, if d is (in addition) *sufficient* $[d(x, y) = 0 \Longrightarrow x = y]$, the $(\leq, d)$-maximal property becomes: $w \in M, z \leq w \Longrightarrow z = w$ (and reads: z is *strongly* $(\leq)$-*maximal*). So, existence results involving such points may be viewed as "metrical" versions of the Zorn-Bourbaki maximality principle (cf. Moore [16, Ch 4, Sect 4]). To get sufficient conditions for these, one may proceed as below. Let (x_n) be an ascending sequence in M. The d-Cauchy property for it is introduced in the usual way $[\forall \varepsilon > 0, \exists n(\varepsilon)$ such that $n(\varepsilon) \leq p \leq q \Longrightarrow d(x_p, x_q) \leq \varepsilon]$. Also, call (x_n), d*-asymptotic* when $d(x_n, x_{n+1}) \to 0$, as $n \to \infty$. Clearly, each (ascending) d-Cauchy sequence is d-asymptotic too. The reverse implication is also true when all such sequences are involved; i.e., the global conditions below are equivalent

(2b) each ascending sequence is d-Cauchy

(2c) each ascending sequence is d-asymptotic.

By definition, either of these will be referred to as $(M, \leq)$ is *regular* (modulo d). Note that this property implies its relaxed version

(2d) $(M, \leq)$ is weakly regular (modulo d): $\forall x \in M, \forall \varepsilon > 0$,
$\exists y = y(x, \varepsilon) \geq x$ such that $y \leq u \leq v \Longrightarrow d(u, v) \leq \varepsilon$.

The following ordering principle is then available (cf. Kang and Park [13]):

Proposition 2 *Assume that $(M, \leq)$ is sequentially inductive and weakly regular (modulo d). Then, for each $u \in M$ there exists a $(\leq, d)$-maximal $v \in M$ with $u \leq v$.*

Proof Let us introduce the function (from M to $R_+ \cup \{\infty\}$)

(a2) $\varphi_d(x) = \sup\{d(u, v); x \leq u \leq v\},\ x \in M$.

Clearly, (1b) holds for this object, as well as (2a) (if one takes (2d) into account). Hence, Proposition 1 is applicable to M and $(\leq, \varphi_d)$. This, added to [$\varphi_d(z) = 0$ iff z is $(\leq, d)$-maximal] gives the desired conclusion. □

As a direct consequence of this, we get the maximality principle in Turinici [20] (see also Conserva and Rizzo [6]):

Proposition 3 *Assume that $(M, \leq)$ is sequentially inductive and regular (modulo d). Then, conclusion of Proposition 2 is holding.*

So far, Proposition 3 is a logical consequence of Theorem 1. The reciprocal of this is also true, by simply taking $d(x, y) = |\psi(x) - \psi(y)|, x, y \in M$ (where ψ is the above one). We therefore established the inclusional chain: Theorem 1 $\Longrightarrow$ Prop 1 $\Longrightarrow$ Prop 2 $\Longrightarrow$ Prop 3 $\Longrightarrow$ Theorem 1. Hence, all these ordering principles are nothing but logical equivalents of the Brezis-Browder's [5]. (This also includes the related statements in Szaz [18] and Tataru [19]; which extend the one in Dancs, Hegedus and Medvegyev [7]). Further aspects may be found in Hamel [11, Ch 4]; see also Hyers, Isac and Rassias [12, Ch 5].

3 Asymptotic extensions

The developments above raise the question of whether or not extensions of Theorem 1 (or its variants) exist without being reducible to it.

(A) Any attempt of solving it must begin from the sequential inductivity condition (1a). Precisely, an examination of the argument in Proposition 2 shows that one may impose it *asymptotically* (i.e., to sequences (x_n) with $\varphi_d(x_n) \to 0$) for the written conclusion be retainable. So, it is natural to ask whether this has a general character. A positive answer may be given under the lines below. Let again M be some nonempty set. Take a quasi-order $(\leq)$ over it, as well as a function $\varphi : M \to R_+ \cup \{\infty\}$. The following counterpart of Proposition 1 is now available.

Theorem 2 *Assume that (1b) and (2a) are true, as well as*

(3a) $(M, \leq)$ is sequentially inductive (modulo φ): each ascending sequence (x_n) with $\varphi(x_n) \to 0$ has an upper bound (modulo $(\leq)$).

Then, for each $u \in M$ there exists $v \in M$ with $u \leq v$ and $\varphi(v) = 0$ (hence v is $(\leq, \varphi)$-maximal).

Proof By (2a), it is not hard to construct an ascending (modulo $(\leq)$) sequence (u_n) with ($u \leq u_0$ and) $\varphi(u_n) \leq 2^{-n}, \forall n$ (hence $\varphi(u_n) \to 0$). Let v stand for an upper bound (modulo $(\leq)$) of this sequence (assured by (3a)). This element has all properties we need. □

Now, (1a) is a particular case of (3a). This tells us that Proposition 1 (hence Theorem 1 as well) is a particular case of Theorem 2. The reciprocal question (Th 1 $\Longrightarrow$ Th 2) remains open; we conjecture that the answer is negative. To explain our position, it will be useful to consider

Example 1 Let $R^2 = R \times R$ stand for the Cartesian plane; and $(\leq)$ denote the partial order induced by the convex cone R^2_+. Further, put $M = A \cup B$, where $A = \{u_n := (n, 0); n \geq 0\}$, $B = \{v_n := (n, 2^{-n}); n \geq 0\}$; and take the function (from M to $R_+ \cup \{\infty\}$): $\varphi(z) = \infty$, if $z \in A$ and $\varphi(z) = 0$, if $z \in B$. Clearly, (1b) is retainable; as well as (2a) (in view of $u_n \leq v_n$, for all $n \geq 0$). Further, (1a) cannot hold; for, e.g., the ascending sequence (u_n) is not bounded above; so that, Proposition 1 is not applicable to $(M, \leq)$ and φ. However, $(M, \leq)$ fulfills (3a); hence, Theorem 2 applies to the same data.

(B) The following version of this result is to be noted. Let M be a nonempty set; and $\mathcal{P}(M)$ stand for the class of its subsets. According to Du [8] any function $\mu : \mathcal{P}(M) \to R_+ \cup \{\infty\}$ with

(a3) $\mu(\emptyset) = 0$; $\mu(A) \leq \mu(B)$ if $A \subseteq B$

will be called *sizing-up*. Assume that we fixed such an object; and let $(\leq)$ be a quasi-order on M. Then, the function φ_μ from M to $R_+ \cup \{\infty\}$ given as

(b3) $\varphi_\mu(x) = \mu(M(x, \leq))$, $x \in M$,

fulfills (1b). An application of Theorem 2 yields the following practical maximality principle:

Proposition 4 *Suppose that*

(3b) $(M, \leq)$ is almost regular (modulo φ_μ): $\forall x \in M$, $\forall \varepsilon > 0$, $\exists y = y(x, \varepsilon) \geq x$ such that $\mu(M(y, \leq)) \leq \varepsilon$

(3c) $(M, \leq)$ is sequentially inductive (modulo φ_μ): each ascending sequence (x_n) with $\mu(M(x_n, \leq)) \to 0$ has an upper bound (modulo $(\leq)$).

Then, for each $u \in M$ there exists $v \in M$ with $u \leq v$ and $\mu(M(v, \leq)) = 0$ (hence v is $(\leq, \varphi_\mu)$-maximal).

In particular, (3c) holds under

(3d) each ascending sequence (x_n) with $\mu(\{x_n, x_{n+1}, ...\}) \to 0$ as $n \to \infty$ has an upper bound.

Then, Proposition 4 is just the basic maximal principle in Du [8]; which, among others, includes Brezis-Browder's [5] (Theorem 1). An interesting aspect refers to Theorem 2 being deductible from Proposition 4; we conjecture that the answer is positive.

(C) A basic particular case of these facts corresponds to the construction in Section 2. Precisely, let $d : M \times M \to R_+ \cup \{\infty\}$ be a reflexive (generalized) pseudometric (over M); and $\varphi_d : M \to R_+ \cup \{\infty\}$, its associated by (a2) function. Clearly, (1b) holds in this context; and the almost regularity (modulo φ_d) condition (2a) is just the one in (2d). Putting these together, it results the following maximality statement involving these data.

Theorem 3 *Assume that $(M, \leq)$ is sequentially inductive (modulo φ_d) and weakly regular (modulo d). Then, for each $u \in M$ there exists a $(\leq, d)$-maximal $v \in M$ with $u \leq v$.*

As before, the sequential inductivity (modulo φ_d) holds under (1a); wherefrom, Theorem 3 includes Proposition 2. An interesting question is that of the reciprocal inclusion being also retainable; further aspects will be treated elsewhere.

(B) Now, the pseudometric setting above is also appropriate for discussing the sequential inductivity (modulo φ_d) condition. This will necessitate some conventions. Denote by $\mathcal{S}(M)$, the class of all sequences (x_n) in M. By a (sequential) *convergence structure* on M we mean, as in Kasahara [14], any part $\mathcal{C}$ of $\mathcal{S}(M) \times M$ with the properties

(c3) $x_n = x, \forall n \in N \Longrightarrow ((x_n); x) \in \mathcal{C}$

(d3) $((x_n); x) \in \mathcal{C} \Longrightarrow ((y_n); x) \in \mathcal{C}$, for each subsequence (y_n) of (x_n).

In this case, $((x_n); x) \in \mathcal{C}$ will be denoted $x_n \xrightarrow{\mathcal{C}} x$; and referred to as: x is the $\mathcal{C}$*-limit* of (x_n). When x is generic in this convention, we say that (x_n) is $\mathcal{C}$*-convergent*. Assume that we fixed such an object and let $(\leq, d)$ be taken as before. Call the subset Z of M, $(\leq)$-*closed* (modulo $\mathcal{C}$) when the $\mathcal{C}$-limit of each ascending sequence in Z is an element of it. Further, let us say that $(\leq)$ is *self-closed* (modulo $\mathcal{C}$) when $M(x, \leq)$ is $(\leq)$-closed (modulo $\mathcal{C}$), for each $x \in M$; or, equivalently: the $\mathcal{C}$-limit of each ascending sequence is an upper bound of it. Finally, term the (reflexive) pseudometric d, $(\leq)$-*complete* (modulo $\mathcal{C}$) when each ascending d-Cauchy sequence is $\mathcal{C}$-convergent.

We may now give an appropriate answer to the posed question.

Theorem 4 *Suppose that $(\leq)$ is self-closed (modulo $\mathcal{C}$), d is $(\leq)$-complete (modulo $\mathcal{C}$) and $(M, \leq)$ is weakly regular (modulo d). Then, conclusions of Theorem 3 are retainable.*

Proof We claim that, under the accepted conditions, Theorem 3 is applicable to $(M, \leq; d)$; precisely, that $(M, \leq)$ is sequentially inductive (modulo φ_d). Let (x_n) be an ascending sequence in M with $\varphi_d(x_n) \to 0$. In particular, it is an ascending d-Cauchy sequence (in M); so that (by the $(\leq)$-completeness (modulo $\mathcal{C}$) of d) $x_n \xrightarrow{\mathcal{C}} y$, for some $y \in M$. Combining with the self-closeness (modulo $\mathcal{C}$) of $(\leq)$ yields $x_n \leq y$, for all n; and this proves the claim. $\square$

Now, a good choice for our convergence structure is $\mathcal{C} = (\xrightarrow{d})$ [introduced as: $x_n \xrightarrow{d} x$ whenever $d(x_n, x) \to 0$ as $n \to \infty$; and called the *convergence* structure attached to d]. Any (modulo $(\xrightarrow{d})$) property will be referred to as a (modulo d) one. In addition, d is $(\leq)$-complete (modulo d) will be simply referred to as: d is $(\leq)$-complete.

Proposition 5 *Suppose that* $(\leq)$ *is self-closed (modulo* d*),* d *is* $(\leq)$*-complete and* $(M, \leq)$ *is weakly regular (modulo* d*). Then, for each* $u \in M$ *there exists a* $(\leq, d)$*-maximal* $v \in M$ *with* $u \leq v$*.*

In particular, when d is *triangular* $[d(x, z) \leq d(x, y) + d(y, z), \forall x, y, z \in M]$, and *symmetric* $[d(x, y) = d(y, x), \forall x, y \in M]$, Proposition 5 is just the statement in Kang and Park [13]; which, in turn, includes the maximality principle by Granas and Horvath [10]. Further aspects of structural nature may be found in Zhu and Li [23]; see also Turinici [21].

4 Equilibrium points

In the following, an application of these facts to equilibrium points is given.

Let I be some nonempty index set; and $\{(X_i, d_i); i \in I\}$ be a family of metric spaces. Denote $X = \prod\{X_i; i \in I\}$ (the Cartesian product); each element of it will be written as $x = (x_i; i \in I)$. Define a map $d : X \times X \to R_+ \cup \{\infty\}$ as

(a4) $d(x, y) = \max\{d_i(x_i, y_i); i \in I\}$, $x = (x_i; i \in I)$, $y = (y_i; i \in I) \in X$.

This is a generalized metric on X, in the Luxemburg sense [15]. For each $i \in I$, take an extended real valued function $G_i : X \times X_i \to R \cup \{-\infty, \infty\}$, fulfilling (for each $i \in I$)

(4a) $G_i(x, x_i) \leq 0$, for all $x = (x_i; i \in I) \in X$

(4b) $G_i(x, z_i) \leq G_i(x, y_i) + G_i(y, z_i)$, for each $x = (x_i; i \in I)$, $y = (y_i; i \in I)$, $z = (z_i; i \in I)$ in X (when the right member exists).

The relation $(\preceq)$ on X introduced as

(b4) $x \preceq y$ iff $G_i(x, y_i) + d_i(x_i, y_i) \leq 0$, $\forall i \in I$

is reflexive (by (4a)) and transitive (by (4b)); hence a quasi-order. (Unfortunately, it is not anti-symmetric (hence, not an order), in general). Now, it would be useful to establish under which supplementary conditions is the quasi-ordered (generalized) metric structure $(X, \preceq; d)$ endowed with the maximality properties in Section 3. This will necessitate some new conventions and auxiliary facts. For each $i \in I$, let $\mathcal{S}_i$ stand for the relation (over $X \times X_i$)

(c4) $(x, y_i) \in \mathcal{S}_i$ iff $G_i(x, y_i) + d_i(x_i, y_i) \leq 0$.

Note that, in such a case, the quasi-order $(\preceq)$ introduced by (b4) is just the product of these relations:

$$x \preceq y \text{ iff } (x, y_i) \in \mathcal{S}_i, \forall i \in I. \tag{4.1}$$

For each $i \in I$, $x \in X$, let $\mathcal{S}_i(x)$ denote the section of $\mathcal{S}_i$ at x. Clearly,

$$x \preceq y \text{ iff } \mathcal{S}_i(x) \supseteq \mathcal{S}_i(y), \text{ for all } i \in I. \tag{4.2}$$

Further, let us denote for $i \in I$,

(d4) $\sigma_i(x) = \sup\{-G_i(x, y_i); y_i \in \mathcal{S}_i(x)\}$, $x \in X$.

We have $\sigma_i(x) \geq 0$, $x \in X$, in view of (4a); hence, this map is with values in $R_+ \cup \{\infty\}$. The alternative $\sigma_i(X) = \{\infty\}$ (for some $i \in I$) cannot be excluded; so, to avoid it, we must impose a condition like

(4c) $\mathrm{Dom}(\sigma_i) := \{x \in X; \sigma_i(x) < \infty\} \neq \emptyset$, $\forall i \in I$.

In addition, we accept that such points are "uniformly" available:

(4d) $U := \cap\{\mathrm{Dom}(\sigma_i); i \in I\}$ is nonempty (in X).

The following evaluation will be in effect for us.

Lemma 1 *Let $x \in U$, $y \in X$ be two points with $x \preceq y$. Then*

$$d_i(x_i, y_i) \leq -G_i(x, y_i) \leq \sigma_i(x) - \sigma_i(y),\ \forall i \in I. \tag{4.3}$$

Hence, in particular, $y \in U$; wherefrom: $X(x, \preceq) = U(x, \preceq)$.

Proof Let $i \in I$ and $z_i \in \mathcal{S}_i(y)$ be arbitrary fixed; note that, by (4.2), we also have $z_i \in \mathcal{S}_i(x)$. The triangular inequality (4b) gives $G_i(x, z_i) \leq G_i(x, y_i) + G_i(y, z_i)$. Passing to infimum over $z_i \in \mathcal{S}_i(y)$ gives $-\sigma_i(x) \leq G_i(x, y_i) - \sigma_i(y)$; and from this, all is clear. □

We are now in position to answer the above posed problem.

Theorem 5 *Assume that (in addition)*

(4e) $y_i \mapsto G_i(x, y_i)$ is d_i-lsc on X_i, $\forall x \in X$, $\forall i \in I$

(4f) d is $(\preceq)$-complete (cf. Section 3).

Then, for each $u \in U$, there exists $v \in U$ in such a way that

$$u \preceq v;\ \text{i.e.: } G_i(u, v_i) + d_i(u_i, v_i) \leq 0,\ \text{for all } i \in I \tag{4.4}$$

$$\begin{aligned} &v \preceq w \in X \text{ implies } v = w;\\ &\text{hence: } \forall w \in X \setminus \{v\},\ \exists i \in I \text{ with } G_i(v, w_i) + d_i(v_i, w_i) > 0. \end{aligned} \tag{4.5}$$

Proof We show that conditions of Proposition 5 are fulfilled by the structure $(U, \preceq; d)$; and, from this, all is done.

Step 1. Let us first establish that $(\preceq)$ is self-closed on U. Let $(x^n = (x_i^n; i \in I); n \in N)$ be an ascending (modulo $(\preceq)$) sequence in U, converging (modulo d) to some point $x = (x_i; i \in I)$ of X:

(4g) $x^n \preceq x^m$ [i.e.: $G_i(x^n, x_i^m) + d_i(x_i^n, x_i^m) \leq 0$, $\forall i \in I$], whenever $n \leq m$

(4h) $x^n \to x$ as $n \to \infty$ [hence: $d_i(x_i^n, x_i) \to 0$ as $n \to \infty$, for each $i \in I$].

Passing to liminf as $m \to \infty$ in the former of these gives (by (4e)) $x^n \preceq x$, for all n; wherefrom $x \in U$, by Lemma 1. Hence the conclusion.

Step 2. Further, we establish that d is $(\preceq)$-complete on U. In fact, let $(x^n = (x_i^n; i \in I); n \in N)$ be an ascending (modulo $(\preceq)$) d-Cauchy sequence in U. By (4f), there exists $x = (x_i; i \in I)$ in X with the property (4h). This, along with Step 1, gives $x^n \preceq x$, $\forall n \in N$; hence $x \in U$; and we are done.

Step 3. Finally, one shows hat $(U, \preceq)$ is weakly regular (modulo d); cf. Section 2. Let $x \in U$ be arbitrary fixed; as well as some $\varepsilon > 0$. By definition, for each $i \in I$ there must be some $y_i \in \mathcal{S}_i(x)$ with

$$-G_i(x, y_i) > \sigma_i(x) - \varepsilon \text{ (so that: } G_i(x, y_i) + \sigma_i(x) < \varepsilon). \tag{4.6}$$

We claim that the point $y = (y_i; i \in I)$ in X has all needed properties. For the moment, $x \preceq y$ (cf. (4.1)); wherefrom (4.3) holds for the couple (x, y). This gives on the one hand $\sigma_i(y) \leq G_i(x, y_i) + \sigma_i(x) < \varepsilon$, $\forall i \in I$; whence, in particular, $y \in U$. On the other hand, let $w = (w_i; i \in I)$ be some point in U with $y \preceq w$. By (4.3) again

$$d_i(y_i, w_i) \leq \sigma_i(y) - \sigma_i(w) \leq \sigma_i(y) < \varepsilon, \forall i \in I;$$

wherefrom $d(y, w) \leq \varepsilon$. Combining with the properties of the metric d gives the desired conclusion.

Summing up, Proposition 5 is indeed applicable to the precise data. This, along with Lemma 1 and the definition of a $(\preceq, d)$-maximal point, ends the argument. □

The obtained result may be viewed as a refinement of the one in Du [8] based on Proposition 4. A basic particular case of it corresponds to the index set I being a singleton. So, let (X, d) be a metric space. Take an extended real valued function $G : X \times X \to R \cup \{-\infty, \infty\}$ fulfilling

(4i) $G(x, x) \leq 0$, for all $x \in X$

(4j) $G(x, z) \leq G(x, y) + G(y, z)$, for all $x, y, z \in X$.

The relation $(\preceq)$ on X introduced as

(e4) $x \preceq y$ iff $G(x, y) + d(x, y) \leq 0$,

is reflexive and transitive; hence a quasi-order. Denote further

(f4) $\sigma(x) = \sup\{-G(x, y); y \in X(x, \preceq)\}$, $x \in X$;

and assume that

(4k) $\mathrm{Dom}(\sigma) := \{x \in X; \sigma(x) < \infty\} \neq \emptyset$.

We now have (cf. Oettli and Thera [17]):

Theorem 6 *Assume that (in addition) (4f) holds, as well as*

(4m) $y \mapsto G(x, y)$ is d-lsc on X, for each $x \in X$.

Then, for each $u \in \mathrm{Dom}(\sigma)$ *there exists* $v \in \mathrm{Dom}(\sigma)$ *in such a way that*

$$u \preceq v;\ i.e.:\ G(u,v) + d(u,v) \le 0 \tag{4.7}$$

$$\text{for each } w \in X \setminus \{v\} \text{ one has } G(v,w) + d(v,w) > 0. \tag{4.8}$$

In particular, let $\varphi : X \to R \cup \{\infty\}$ be some inf-proper function

(4n) $\mathrm{Dom}(\varphi) \neq \emptyset$ and $\varphi_* := \inf[\varphi(X)] > -\infty$.

Denote by G the map (from $X \times X$ to $R \cup \{-\infty, \infty\}$)

(g4) $G(x,y) = \varphi(y) - \varphi(x), \quad x, y \in M \quad$ (where $\infty - \infty = 0$).

The associated by (d4) function is just

(h4) $\sigma(x) = \varphi(x) - \varphi_*, x \in X$ (hence $\mathrm{Dom}(\sigma) = \mathrm{Dom}(\varphi)$).

In addition, (4m) holds whenever φ is d-lsc on X. Note that this yields the variational principle in Ekeland [9]. But, the reciprocal implication is also true; cf. Bao and Khanh [4]. Further aspects may be found in Turinici [22].

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