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# Almost Periodic Functions Defined on $\mathbb{R}^n$ with Values in $p$-Fréchet Spaces, $0 < p < 1$

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**Abstract.** In this paper theory of almost periodic functions defined on $\mathbb{R}^n$ with values in $p$-Fréchet spaces, $0 < p < 1$ is developed.

**Keywords:** $p$-Fréchet spaces, relatively dense set, relatively compact set, $p$-norm etc.

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## 1 Introduction

Harald Bohr's interest in which functions could be represented by a Dirichlet series, i.e. of the form $\sum_{n=1}^{\infty} a_n e^{-\lambda_n z}$, where $a_n, z \in \mathbb{C}$ and $(\lambda_n)_{n\in\mathbb{N}}$ is a monotone increasing sequence of real numbers (series which play an important role in complex analysis and analytic number theory), led him to develop a theory of almost periodic real (and complex) functions, between the years 1923 and 1926. The theory of almost periodic functions was strongly extended to abstract spaces, see for example the monographs [9], [17], [18] for Banach space valued functions and [7], [17], [19] for complete locally convex (Fréchet) space valued functions. Also, in the paper [5] (see also [18, Chapter 3]), the theory has been extended to the case of fuzzy-number-valued functions. The purpose of this paper is to extend the main properties of almost periodic functions that are defined on $\mathbb{R}^n$ with values in Banach spaces ( see e.g. [21, Chapter 9] ), to the class of almost periodic functions that are defined on $\mathbb{R}^n$ with values in other important abstract spaces in Functional Analysis, namely the $p$-Fréchet spaces, $0 < p < 1$, which are non-locally convex spaces.

## 2 Preliminaries

It is well known that an $F$-space $(X, +, ., \|.\|)$ is a linear space (over the field $K = \mathbb{R}$ or $K = \mathbb{C}$) such that $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in X$, $\|x\| = 0$ if and only if $x = 0$, $\|\lambda x\| \leq |\lambda|\|x\|$, for all scalars $\lambda$ with $|\lambda| \leq 1, x \in X$, and with respect to the metric $D(x, y) = \|x - y\|$, $X$ is a complete metric space (see e.g. [10, p.52] or [13]). Obviously $D$ is invariant under translations. In addition, if there exists $0 < p < 1$ with $\|\lambda x\| = |\lambda|^p\|x\|$, for all $\lambda \in K$ and $x \in X$, then $\|.\|$ will be called a $p$-norm and $X$ will be called $p$-Fréchet space.

(This is only a slight abuse of terminology. Note that in e.g. [3] these spaces are called $p$-Banach spaces). In this case, it is immediate that $D(\lambda x, \lambda y) = |\lambda|^p D(x, y)$, for all $x, y \in X$, $\lambda \in K$. It is known that $F$-spaces are not necessarily locally convex spaces. Three classical examples of $p$-Fréchet spaces, non-locally convex, are the Hardy space $H_p$ with $0 < p < 1$ that consists in the class of all analytic functions $f : \mathbb{D} \longrightarrow \mathbb{C}$, $\mathbb{D} = \{z \in \mathbb{C}; |z| \le 1\}$ with the property

$$\|f\| = \frac{1}{2\pi} \sup\{\int_0^{2\pi} |f(re^{it})|^p dt, r \in [0, 1)\} < +\infty$$

the sequences space $l^p$

$$l^p = \{x = (x_n)_n; \|x\| = \sum_{n=1}^{\infty} |x_n| < \infty\}$$

for $0 < p < 1$, and the $L^p[0, 1]$, $0 < p < 1$, given by

$$L^p[0, 1] = \{f : [0, 1] \longrightarrow \mathbb{R}; \|f\| = \int_0^1 |f(t)|^p dt < \infty\}$$

More generally, we may consider $L^p(\Omega, \Sigma, \mu)$, $0 < p < 1$, based on a general measure space $(\Omega, \Sigma, \mu)$, with the $p$-norm given by $\|f\| = \int_\Omega |f|^p d\mu$. Some important characteristics of the $F$-spaces are given by the following remark.

**Remark 1.** *Three fundamental results in Functional Analysis hold for $F$-spaces too : the Principle of Uniform Boundedness (see e.g. [10, p.52]), the Open Mapping Theorem and the Closed Graph Theorem (see e.g. [13, p.9-10]). But on the other hand, the Hahn-Banach Theorem fails in non-locally convex $F$-spaces. More exactly, if in an $F$-space the Hahn-Banach theorem holds, then that space is necessarily locally convex space (see e.g. [13, Chapter 4]).*

# 3 Basic Definition and Properties

In this section we develop the theory of almost periodic functions that are defined on $\mathbb{R}^n$ and taking values in a $p$-Fréchet space, $0 < p < 1$ and also we use the same notation for a $p$-norm in a $p$-Fréchet space with $0 < p < 1$ and a norm in $\mathbb{R}^n$. Everywhere in this section, $(X, +, ., \|.\|)$ will be a $p$-Fréchet space with $0 < p < 1$ (over the field $K = \mathbb{R}$ or $K = \mathbb{C}$). Also, denote $D(x, y) = \|x - y\|$. Although most of the results in this paper are similar to the results for the functions defined on $\mathbb{R}^n$ with values in Banach space ( see e.g. [21, Chapter 9] ) because triangle inequality of the norm is used to prove them but yet we have given the detailed proofs of these results. In the previous sections it was mentioned that the metric $D(x, y) = \|x - y\|$ is invariant under translations and satisfies $D(\lambda x, \lambda y) = |\lambda|^p D(x, y)$, for all $x, y \in X$, $\lambda \in K$. Moreover $D$ has some additional properties given in the following:

**Theorem 1.** *(i) $D(cx, cy) \le D(x, y)$ for $|c| \le 1$,*

*(ii) $D(x + u, y + v) \le D(x, y) + D(u, v)$,*

*(iii)* $D(kx, ky) \leq D(rx, ry)$ *for* $k, r \in \mathbb{R}$, $0 < k \leq r$,

*(iv)* $D(kx, ky) \leq kD(x, y)$, $\forall k \in \mathbb{N}$, $k \geq 1$,

*(v)* $D(cx, cy) \leq (|c| + 1)D(x, y)$, $\forall\ c \in \mathbb{R}$.

*Proof.* Properties (i) and (iii) are obvious, we shall give proofs of (ii), (iv) and (v) for the proof of (ii) we have

$$\begin{aligned} D(x+u, y+v) &= D(x + (u-v) + v, y+v) \\ &= D(x+u-v, y) = D(y, x+u-v) \\ &\leq D(y, x) + D(x, x+u-v) \\ &= D(x, y) + D(x+v, x+u) = D(x, y) + D(v, u) \end{aligned}$$

(iv) Since $0 < p < 1$, we have $D(kx, ky) = |k|^p D(x, y) \leq kD(x, y)$, for all $k \geq 1$.

(v) If $|c| < 1$, then $D(cx, cy) = |c|^p D(x, y) \leq |c|D(x, y) \leq (|c|+1)D(x, y)$. If $|c| \geq 1$ then we get

$$D(cx, cy) = |c|^p D(x, y)|c|D(x, y) \leq\ (|c| + 1)D(x, y)$$

which proves the theorem. □

Now we recall some definitions and theorems about the Euclidean $n$-dimensional space before giving the definition and properties of almost periodic functions that are defined on $\mathbb{R}^n$ and taking values in the a $p$-Fréchet space, $0 < p < 1$. Let $\mathbb{R}^n$ the usual Euclidean $n$-dimensional space. The elements of $\mathbb{R}^n$ are the $n$-tuples $x = (x_1, x_2, ..., x_n)$ and a norm of $x \in \mathbb{R}^n$ is given by

$$\|x\| = (x_1^2 + x_2^2 + ... + x_n^2)^{\frac{1}{2}}$$

A closed ball $\overline{B}(x_0; r)$ in $\mathbb{R}^n$ with center $x$ and radius $r > 0$ is defined by the set

$$\overline{B}(x_0; r) = \{x \in \mathbb{R}^n : \|x - x_0\| \leq r\}$$

**Definition 1.** *A set $P$ is said to be relatively dense in $\mathbb{R}^n$ if there exists a number $r > 0$ such that $P \cap \overline{B}(x; r) \neq \emptyset$, for all $x \in \mathbb{R}^n$.*

**Remark 2.** *Super sets of a relatively dense set are relatively dense.*

We also have the following two important theorems for the sequel. For the detailed proofs of these theorems see [21, Chapter 9].

**Theorem 2.** *A subset $P$ of $\mathbb{R}^n$ is relatively dense in $\mathbb{R}^n$ if and only if, for some $r > 0$, we have the relation $\mathbb{R}^n = \underset{p \in P}{\cup} \overline{B}(p; r)$.*

**Theorem 3.** *A subset $P$ of $\mathbb{R}^n$ is relatively dense if and only if there exists a compact set $S \subset \mathbb{R}^n$ such that $S + P = \mathbb{R}^n$ ( vector sum of $S$ and $P$ )*

**Definition 2.** *A function $f : \mathbb{R}^n \longrightarrow X$ is said to be continuous at $x_0 \in \mathbb{R}$ such that $D(f(x), f(x_0)) < \epsilon$, whenever $\|x - x_0\| < \delta$.*

**Remark 3.** *From the triangle inequality satisfied by the p-norm* $\|.\|$ *, it easily follows that* $|\|x\| - \|y\|| \leq \|x - y\|$*, which immediately implies that if* $f$ *is continuous at* $x_0$*, then the real valued function* $x \longmapsto \|f(x)\|$ *is also continuous at* $x_0$*.*

**Definition 3.** *A continuous function* $f : \mathbb{R}^n \longrightarrow X$*, is called B-almost periodic function, if for every* $\epsilon > 0$*, there exists a real number* $r = r(\epsilon) > 0$*, such that in any ball* $\overline{B}(x; r)$ *of radius* $r = r(\epsilon)$ *contains at least one point* $y$ *with*

$$D(f(x + y), f(x)) < \epsilon, \forall\ x \in \mathbb{R}^n$$

**Remark 4.** *By using the concept of relatively dense set the above definition can also be rewritten as: A continuous function* $f : \mathbb{R}^n \longrightarrow X$*, is called B-almost function if for every* $\epsilon > 0$*, there exist a relatively dense set, which we denote by* $T(f; \epsilon)$*, such that*

$$\sup_{x \in \mathbb{R}^n} D(f(x + y), f(x)) < \epsilon, \forall y \in T(f; \epsilon)$$

$$or D(f(x + y), f(x)) < \epsilon, \forall y \in T(f; \epsilon), \forall\ x \in \mathbb{R}^n$$

*The elements of set* $T(f; \epsilon)$ *are called* $\epsilon$*-translation vectors.*

The following is a direct consequence of the above remark

**Remark 5.** *If* $f : \mathbb{R}^n \longrightarrow X$ *is B-almost periodic and if* $y_1 \in T(f; \epsilon_1)$*,* $y_2 \in T(f; \epsilon_2)$*, then* $y_1 + y_2 \in T(f; \epsilon_1 + \epsilon_2)$*.*
*This is because of the fact that*

$$\begin{aligned} D(f(x + y_1 + y_2), f(x)) &= \|f(x + y_1 + y_2) - f(x)\| \\ &\leq \|f(x + y_1 + y_2) - f(x + y_1)\| + \|f(x + y_1) - f(x)\| \\ &= D(f(x + y_1 + y_2), f(x + y_1)) + D(f(x + y_1), f(x)) \\ &< \epsilon_1 + \epsilon_2, \forall x \in \mathbb{R}^n \end{aligned}$$

**Theorem 4.** *If* $f$ *is B-almost periodic, then the functions* $\lambda f$*, (* $\lambda$ *is any scalar ),* $\overset{\vee}{f} : \mathbb{R}^n \longrightarrow X$ *defined by* $\overset{\vee}{f}(x) = f(-x)$*,* $F_h(x) = f(x + h)$ *and* $G(x) = \|f(x)\|$*,* $x \in \mathbb{R}^n$ *are also B-almost periodic.*

*Proof.* (i) Since $f : \mathbb{R}^n \longrightarrow X$ is B-almost periodic function so for every $\epsilon > 0$ we can find a relatively dense set $T(f; \epsilon)$ such that

$$D(f(x + y), f(x)) < \epsilon, \forall\ y \in T(f; \epsilon), \forall x \in \mathbb{R}^n$$

If $\lambda = 0$ then there is nothing to prove, so we suppose that $\lambda \neq 0$. Now $\forall\ y \in T(f; \epsilon), \forall\ x \in \mathbb{R}^n$, we have

$$D(\lambda f(x + y), \lambda f(x)) = \|\lambda f(x + y) - \lambda f(x)\| < |\lambda|^p \epsilon,\ 0 < p < 1$$

This shows that $T(f; \epsilon) \subset T(f; |\lambda|^p \epsilon)$ i.e. for every $\epsilon > 0$ we can find a relatively dense set $T(f; |\lambda|^p \epsilon)$ such that

$$D(\lambda f(x + y), \lambda f(x)) < |\lambda|^p \epsilon, \forall\ y \in T(f; |\lambda|^p \epsilon),\ \forall\ x \in \mathbb{R}^n, 0 < p < 1$$

Hence it is proved that $\lambda f$ is B-almost periodic.

(ii) Since $f$ B-almost periodic function, it follows that for every $\epsilon > 0$ we may find a relatively dense set $T(f;\epsilon)$ such that

$$D(f(x+y), f(x)) < \epsilon, \forall\ y \in T(f;\epsilon), \forall\ x \in \mathbb{R}^n$$

Let $-x = x^{'}$, therefore we have

$$D(f(-x^{'}+y), f(-x^{'})) < \epsilon, \forall\ y \in T(f;\epsilon), \forall\ x^{'} \in \mathbb{R}^n$$

or

$$D(f(-(x^{'}-y), f(-x^{'})) < \epsilon, \forall\ y \in T(f;\epsilon), \forall\ x^{'} \in \mathbb{R}^n$$

Replacing $x^{'}$ by $x$ we get

$$D(\overset{\vee}{f}(x-y), \overset{\vee}{f}(x)) < \epsilon, \forall\ y \in T(f;\epsilon), \forall\ x \in \mathbb{R}^n$$

This implies that $T(f;\epsilon) \subset T(\overset{\vee}{f};\epsilon)$. That is for every $\epsilon > 0$ we can find a relatively dense set $T(\overset{\vee}{f};\epsilon)$ such that

$$D(\overset{\vee}{f}(x-y), \overset{\vee}{f}(x)) < \epsilon, \forall\ y \in T(\overset{\vee}{f};\epsilon), \forall\ x \in \mathbb{R}^n$$

and hence it is proved that $\overset{\vee}{f}$ is B-almost periodic with $-y$ as $\epsilon$-translation vector.

(iii) Since $f$ is B-almost periodic therefore for any $\epsilon > 0$, we can find a relatively dense set $T(f;\epsilon)$ such that

$$D(f(x+y), f(x)) < \epsilon, \forall\ y \in T(f;\epsilon), \forall\ x \in \mathbb{R}^n$$

Replacing $x$ by $x+h$ we get

$$D(f(x+h+y), f(x+h)) < \epsilon, \forall\ y \in T(f;\epsilon), \forall\ x \in \mathbb{R}^n$$

$$D(F_h(x+y), F_h(x)) < \epsilon, \forall\ y \in T(f;\epsilon), \forall\ x \in \mathbb{R}^n$$

This implies that $T(f;\epsilon) \subset T(F_h;\epsilon)$ Therefore for every $\epsilon > 0$ we can find a relatively dense set $T(F_h;\epsilon)$ such that

$$D(F_h(x+y), F_h(x)) < \epsilon, \forall\ y \in T(F_h;\epsilon), \forall\ x \in \mathbb{R}^n$$

and thus $F_h$ is proved to be B-almost.

(iv) Since $f$ is B-almost periodic therefore for any $\epsilon > 0$, we can find a relatively dense set $T(f;\epsilon)$ such that

$$D(f(x+y), f(x)) < \epsilon, \forall\ y \in T(f;\epsilon), \forall\ x \in \mathbb{R}^n$$

Now

$$\begin{aligned}|G(x+y) - G(x)| &= |\|f(x+y)\| - \|f(x)\|| \\ &\leq \|f(x+y) - f(x)\| \\ &= D(f(x+y), f(x)) < \epsilon, \forall\ y \in T(f;\epsilon), \forall\ x \in \mathbb{R}^n\end{aligned}$$

This shows that $T(f;\epsilon) \subset T(G;\epsilon)$. Therefore for every $\epsilon > 0$ we can find a relatively dense set $T(G;\epsilon)$ such that

$$|G(x+y) - G(x)| < \epsilon, \forall\ y \in T(G;\epsilon), \forall\ x \in \mathbb{R}^n$$

this proves the almost periodicity of the function $G(x) = \|f(x)\|$, $x \in \mathbb{R}^n$. This completes the proof of the theorem. □

**Theorem 5.** *Let $f : \mathbb{R}^n \longrightarrow X$ be B-almost periodic, then the range of $f$ is bounded in $X$.*

*Proof.* Since $D(f(x), f(y)) \leq D(f(x), 0_X) + D(0_X, f(y)) = \|f(x)\| + \|f(y)\|$, therefore it is sufficient to prove that $\|f(x)\| \leq M_1$, $\forall\ x \in \mathbb{R}^n$ where $M_1$ is some positive real number . Let, for any given $\epsilon > 0$, the associated relatively dense set be $T(f;\epsilon)$. By Theorem 2 we have $\mathbb{R}^n = \bigcup_{y \in T(f;\epsilon)} \overline{B}(y;r)$ for some $r = r(\epsilon) > 0$. Therefore for any $x \in \mathbb{R}^n$, $\exists\ y \in T(f;\epsilon)$ such that $\|x - y\| \leq r$. Then, if $x^{'} = x - y$, we have $\|x^{'}\| \leq r$, $y \in T(f;\epsilon)$. Now

$$f(x) = f(x^{'} + y) = f(x^{'} + y) - f(x^{'}) + f(x^{'})$$

Therefore we have

$$\begin{aligned} \|f(x)\| &= \|f(x^{'} + y) - f(x^{'}) + f(x^{'})\| \\ &\leq \|f(x^{'} + y) - f(x^{'})\| + \|f(x^{'})\| \\ &= D(f(x^{'} + y), f(x^{'})) + \|f(x^{'})\| \end{aligned} \tag{1}$$

Since $x^{'} \longmapsto \|f(x^{'})\|$ is continuous function on the compact set $\overline{B}(0;r)$, hence it is bounded there in. Take now $y \in T(f;\epsilon)$ and $x^{'} \in \overline{B}(0;r) \subset \mathbb{R}^n$, then by B-almost periodicity of $f$ we have

$$D(f(x^{'} + y), f(x^{'})) < \epsilon$$

Thus from (1) we have

$$D(f(x), 0_X) = \|f(x)\| \leq \varepsilon + \sup_{\|x^{'}\| \leq r} \|f(x^{'})\| = M_1(\text{say}), \forall\ x \in \mathbb{R}^n$$

This completes the proof of the theorem. □

**Theorem 6.** *(i) If $f : \mathbb{R}^n \longrightarrow X$ is B-almost periodic, then $f$ is uniformly continuous over $\mathbb{R}^n$.*

*(ii) If $(f_k)_k$ is sequence of B-almost periodic functions, $f_k : \mathbb{R}^n \longrightarrow X$, $1 \leq k < \infty$. If $f_k \longrightarrow f$ uniformly to $f$ on $\mathbb{R}^n$ then $f$ is B- almost periodic.*

*Proof.* (i) Let $f : \mathbb{R}^n \longrightarrow X$ be B-almost periodic function and let for any $\epsilon > 0$, its associated relatively dense set be $T(f;\frac{\epsilon}{3})$. Therefore by Theorem 3, any $x^{'} \in \mathbb{R}^n$ can be written as $x^{'} = x + y$, where $y \in T(f;\frac{\epsilon}{3})$, $\|x\| \leq r$, for some real number $r = r(\frac{\epsilon}{3}) > 0$. Now by the uniform continuity of $f$ on the closed ball $\overline{B}(0;r) = \{x \in \mathbb{R}^n : \|x\| \leq r\}$ we can find $\delta = \delta(\frac{\epsilon}{3}) > 0$ such that

$$D(f(u_1)\ , f(u_2)) < \frac{\epsilon}{3}, \text{ whenever } \|u_1 - u_2\|\ < \delta \text{ and } \|u_1\| \leq\ r, \|u_1\| \leq\ r$$

Take now any pair $x_1^{'}, x_2^{'} \in \mathbb{R}^n$ such that $\|x_1^{'} - x_2^{'}\| < \delta$. Then for any given $y \in T(f; \frac{\epsilon}{3})$, we obtain the decomposition $x_1^{'} = x_1 + y$, $x_2^{'} = x_2 + y$,where $\|x_1\| \leq r, \|x_2\| \leq r$ in view of Theorem 3. Therefore it follows that

$$\|x_1 - x_2\| = \|(x_1^{'} - y) - (x_2^{'} - y)\|\ = \|x_1^{'} - x_2^{'}\| < \delta.$$

and

$$D(f(x_1)\ , f(x_2)) < \frac{\epsilon}{3}, \forall\ x_1, x_2 \in \mathbb{R}^n, \|x_1\| \leq r, \|x_2\| \leq r$$

Now for any $y \in T(f; \frac{\epsilon}{3})$ and $\forall\ x_1^{'}, x_2^{'} \in \mathbb{R}^n$ such that $\|x_1^{'} - x_2^{'}\| < \delta$, we get

$$\begin{aligned} D(f(x_1^{'}), f(x_2^{'})) &= D(f(x_1 + y), f(x_2 + y)) \\ &= \|f(x_1 + y) - f(x_2 + y)\| \\ &\leq \|f(x_1 + y) - f(x_1)\| + \|f(x_1) - f(x_2)\| + \|f(x_2) - f(x_2 + y)\| \\ &= D(f(x_1 + y), f(x_1)) + D(f(x_1), f(x_2)) + D(f(x_2 + y), f(x_2)) \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon \end{aligned}$$

This proves that $f$ is uniformly continuous over $\mathbb{R}^n$.

(ii) Since $f_k(x) \longrightarrow f(x)$ uniformly over $\mathbb{R}^n$ as $k \longrightarrow \infty$ so for any $\epsilon > 0$ we can find a natural number $k_0$ such that

$$\forall\ k \geq k_0 \Rightarrow D(f_k(x), f(x)) < \frac{\epsilon}{3}$$

Since $f_k : \mathbb{R}^n \longrightarrow X$ is almost periodic for $k = 1, 2, 3, ...$, so for already chosen $\epsilon > 0$ we can find a relatively dense set $T(f_k; \frac{\epsilon}{3})$ such that

$$D(f_k(x + y), f_k(x)) < \frac{\epsilon}{3}, \forall\ y \in T(f; \frac{\epsilon}{3}), \forall\ x \in \mathbb{R}^n, k = 1, 2, 3....$$

Now $\forall\ y \in T(f_k; \frac{\epsilon}{3})$ and $\forall x \in \mathbb{R}^n$, we have

$$\begin{aligned} D((x + y), f(x)) &= \|f(x + y) - f(x)\| \\ &\leq \|f(x + y) - f_k(x + y)\| + \|f_k(x + y) - f_k(x)\| + \|f_k(x) - f(x)\| \\ &= D(f(x + y), f_k(x + y)) + D(f_k(x + y), f_k(x)) + D(f_k(x), f(x)) \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} \\ &= \epsilon \end{aligned}$$

Hence for the relatively dense $T(f; \epsilon)$ we have

$$D(f(x + y), f(x)) < \epsilon, \forall\ y \in T(f; \epsilon), \forall\ x \in \mathbb{R}^n$$

Thus $f$ is proved to be almost periodic. □

**Theorem 7.** *If $f : \mathbb{R}^n \longrightarrow X$ is B-almost periodic function then given any $\epsilon > 0$ we can find two positive numbers $r = r(\epsilon)$ and $\delta = \delta(\epsilon)$ such that any ball $\overline{B}(a; r)$ in $\mathbb{R}^n$ contains a ball of radius $\delta$ which is contained in $T(f; \epsilon)$.*

*Proof.* Since $f$ is almost periodic function therefore for any $\epsilon > 0$ there exists relatively dense set $T(f; \frac{\epsilon}{2})$ in $\mathbb{R}^n$ and the associated number $R(\frac{\epsilon}{2}) = R > 0$ such that

$$\overline{B}(x; R) \cap T(f; \frac{\epsilon}{2}) \neq \emptyset, \ \forall x \in \ \mathbb{R}^n$$

By the uniform continuity of $f$ over $\mathbb{R}^n$ we can find $\delta(\frac{\epsilon}{2}) = \delta$ such that if $h \in \mathbb{R}^n$ and $\|h\| < \delta$ then

$$D(f(x + h), f(x)) < \frac{\epsilon}{2}, \ \forall x \in \ \mathbb{R}^n$$

We shall now prove that $r(\epsilon) = R(\frac{\epsilon}{2}) + 2\delta(\frac{\epsilon}{2})$ and $\delta(\epsilon)$ are our desired numbers. In fact given $a \in \mathbb{R}^n$, take $z \in \mathbb{R}^n$ with $\|z\| = \delta$, then $\exists\ y \in T(f; \frac{\epsilon}{2}) \cap \overline{B}(z + a; R)$ and hence $\|y - a\| \leq R + \delta < r$, so that $y \in \overline{B}(a; r)$. Furthermore, $\forall\ h \in \mathbb{R}^n$, $\|h\| < \delta$, $\|y + h - a\| \leq R + \delta + \delta = r$, hence $y + h \in \overline{B}(a; r)$. Therefore the whole ball $\overline{B}(y; \delta)$ is contained in the ball $\overline{B}(a; r)$. Finally, any vector in this ball belongs to $T(f; \epsilon)$; this is because, if $y + h$ with $\|h\| \leq \delta$ is such a vector then $\forall\ x \in \mathbb{R}^n$, we have

$$\begin{aligned} D(f(x + y + h), f(x)) &= \|f(x + y + h) - f(x)\| \\ &\leq \|f(x + y + h) - f(x + h)\| + \|f(y + h) - f(x)\| \\ &= D(f(x + y + h), f(x + h)) + D(f(y + h), f(x)) \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \end{aligned}$$

where we have used the facts that $y \in T(f; \frac{\epsilon}{2})$, $\|h\| < \delta$ and the uniform continuity of $f$ over $\mathbb{R}^n$. This proves the result. □

**Theorem 8.** *If $f : \mathbb{R}^n \longrightarrow X$ is B- almost periodic is, then the range$\{\ f(x) : x \in \mathbb{R}^n\}$ of $f$ is relatively compact in $X$.*

*Proof.* In complete metric spaces, the relatively compact sets coincides with totally bounded sets, it is sufficient to show that the values of the functions can be embedded in a finite number of spheres of radius $2\varepsilon$. Since $f : \mathbb{R}^n \longrightarrow X$ is B-almost periodic therefore by Theorem 2 for any $\epsilon > 0$ we can find a relatively dense set $T(f; \epsilon)$ such that for some $r = r(\epsilon) > 0$ we have $\mathbb{R}^n = \underset{y \in T(f;\epsilon)}{\cup} \overline{B}(y; r)$. Therefore for any $x \in \mathbb{R}^n$, $\exists\ y \in T(f; \epsilon)$ such that we have$\|x - y\| \leq r$. Let $x^{'} = x - y$ which implies that $x = x^{'} + y$ where $\|x^{'}\| \leq r$, $y \in T(f; \epsilon)$. By the continuity of $f$, the set $\{f(x^{'}) : x^{'} \in \overline{B}(0; r)\}$ is compact in $X$. But in a $p$-Fréchet space,$0 < p < 1$, every compact set is totally bounded, therefore there exist $k$ elements $f(x_1), f(x_2), ..., f(x_k)$ in $X$ where $\|x_i\| \leq r$, $1 \leq i \leq k$ such that for every $x^{'} \in \overline{B}(0; r)$ we have

$$f(x^{'}) \in \overset{k}{\underset{i=1}{\cup}} \overline{B}(f(x_i), \epsilon)$$

Take now an arbitrary $x^{'} \in \overline{B}(0; r) \subset \mathbb{R}^n$ and consider $y \in T(f; \epsilon)$ then we have

$$D(f(x^{'} + y), f(x^{'})) < \epsilon$$

Choose $x_l$ among $x_1, x_2, ..., x_k$ such that

$$f(x^{'}) \in \overline{B}(f(x_l), \epsilon) \Longrightarrow D(f(x^{'}), f(x_l)) \leq \epsilon,\ l = 1, 2, 3, ..., k$$

Now for any arbitrary $x \in \mathbb{R}^n$ we have

$$\begin{aligned} D(f(x), f(x_l)) &= \|f(x) - f(x_l)\| \\ &= \|f(x^{'} + y) - f(x_l)\| \\ &\leq \|f(x^{'} + y) - f(x^{'})\| + \|f(x^{'}) - f(x_l)\| \\ &= D(f(x^{'} + y), f(x^{'})) + D(f(x^{'}), f(x_l)) \\ &< 2\epsilon \end{aligned}$$

Therefore we have $f(x) \in \overline{B}(f(x_l), 2\epsilon), 1 \leq l \leq k$. Since $x$ is an arbitrary vector in $\mathbb{R}^n$, we conclude that

$$\{f(x) : x \in \mathbb{R}^n\} \subset \bigcup_{i=1}^{k} \overline{B}(f(x_i), 2\epsilon)$$

This completes the proof the theorem. □

**Remark 6.** *Let $f : \mathbb{R}^n \longrightarrow X$ be B-almost periodic and let us consider the sequence of values $(f(x_k))_{k\in\mathbb{N}}$. Denote $A = \{f(x_k); k \in \mathbb{N}\}$ and take the closure $\overline{A} \subset \overline{f(\mathbb{R}^n)} \subset X$, it follows that $\overline{A}$ is compact, so $\overline{A}$ is sequentially compact too (since $(X, D)$ is a metric space), which by $A \subset \overline{A}$ implies that the sequence $(f(x_k))_{k\in\mathbb{N}}$ has convergent subsequence in $X$.*

**Theorem 9.** *If $f : \mathbb{R}^n \longrightarrow X$ is B-almost periodic and $g : X \longrightarrow Y$ is continuous on $f(\mathbb{R}^n)$, where $X$ is p-Fréchet space and $Y$ is a q-Fréchet space with metrics $D$ and $D^{'}$ respectively, $0 < p, q < 1$, (with q not necessarily equal to p), then $h : \mathbb{R}^n \longrightarrow Y$ , defined by $h(x) = g(f(x))$, $x \in \mathbb{R}^n$ is B-almost periodic.*

*Proof.* Since range $f(\mathbb{R}^n)$ of $g$ is closed and bounded so it is compact subset of $X$ and any continuous function defined on a compact set is uniformly continuous. Therefore $g : f(\mathbb{R}^n) \longrightarrow Y$ is uniformly continuous. Hence for given $\epsilon > 0$ we can find $\delta(\epsilon) > 0$ such that

$$D^{'}(g(x_1), g(x_2)) < \epsilon, \text{ whenever } \|x_1 - x_2\| < \delta,\ \forall x_1, x_2 \in f(\mathbb{R}^n)$$

Since $f : \mathbb{R}^n \longrightarrow X$ is B-almost periodic so for $\delta(\epsilon) > 0$ we can find a relatively dense set $T(f; \delta)$ such that

$$D(f(x + y), f(x)) < \delta, \forall x \in \mathbb{R}^n, y \in T(f; \delta)$$

It follows that

$$D^{'}(h(x + y), h(x)) = D^{'}(g(f(x + y)), g(f(x))) < \epsilon$$

whenever

$$D(f(x + y), f(x)) < \delta, \forall x \in \mathbb{R}^n, y \in T(f; \delta)$$

This implies that $T(f; \delta) \subset T(h; \epsilon)$. That is we can find a relatively dense set $T(h; \epsilon)$ such that

$$D^{'}(h(x + y)), h(x)) < \epsilon, \forall x \in \mathbb{R}^n, y \in T(h; \epsilon)$$

Therefore $h$ is proved to be B-almost periodic. □

**Theorem 10.** *Let $f : \mathbb{R}^n \longrightarrow X$ be a B-almost periodic function. Then for every sequence $(x_k)_k$ in $\mathbb{R}^n$, there exists a subsequence $(x_k^{'})_k$ such that $(f(x+x_k^{'}))_k$ is uniformly convergent over $\mathbb{R}^n$.*

*Proof.* Let $(x_k)_k$ be a given sequence in $\mathbb{R}^n$ and consider the sequence $(f_{x_k})_k$ functions $f_{x_k} : \mathbb{R}^n \longrightarrow X$ defined by $f_{x_k}(x) = f(x + x_k), 1 \leq k < \infty$. Let $S = \{ \eta_k; k \in \mathbb{N} \}$ be a countable dense set in $\mathbb{R}^n$. Since the range $\{f(x) : x \in \mathbb{R}^n\}$ of $f$ is relatively compact therefore we can extract from $(f(\eta_1 + x_k))_k$ a convergent subsequence. Let $(f_{x_1,k})_k$ be a subsequence of the sequence $(f_{x_k})_k$ which is convergent at $\eta_1$. We apply the same argument to the sequence $(f_{x_1,k})_k$ to choose a subsequence $(f_{x_2,k})_k$ which converges at $\eta_2$. We continue the process, and consider the diagonal sequence $(f_{x_k,k})_k$ which converges at each $\eta_k$ in $S$. Call this last sequence $(f_{x_k^{'}})_k$. We shall prove that this last sequence is uniformly convergent over $\mathbb{R}^n$, that is, we shall prove that for every $\epsilon > 0$, $\exists$ a natural number $k_0 = k_0(\epsilon)$ such that

$$D(f(x + x_k^{'}), f(x + x_l^{'})) < \epsilon, \forall\ k, l \geq k_0, \forall\ x \in \mathbb{R}^n$$

Since $f$ uniformly continuous over $\mathbb{R}^n$, therefore for already chosen $\epsilon > 0$ we can find $\delta = \delta(\frac{\epsilon}{5}) > 0$, such that

$$\|x - x^{'}\| < \delta \Longrightarrow D(f(x), f(x^{'})) < \frac{\epsilon}{5}, \forall\ x, x^{'} \in \mathbb{R}^n \tag{2}$$

Since $\overline{B}(0; r)$ is compact in $\mathbb{R}^n$ so we can suppose that $\overline{B}(0; r)$ is contained in the union of finite number of $\upsilon$ balls (say) of radii smaller than $\delta$ and choose from each ball a point of $S$, we obtain $S_0 = \{\xi_1, \xi_2, ...\xi_\upsilon\}$. Since $S_0$ is finite set, $(f_{x_k^{'}})_k$ is uniformly convergent over $S_0$ ; therefore there exists a natural number $k_0 = k_0(\frac{\epsilon}{5})$ such that

$$D(f(\xi_i + x_k^{'}), f(\xi_i + x_l^{'})) < \frac{\epsilon}{5}, \forall\ i, 1 \leq i \leq \nu \text{ and } k, l \geq k_0 \tag{3}$$

Let $x \in \mathbb{R}^n$ be arbitrary and $y \in T(f; \frac{\epsilon}{5})$ then $D(f(x + y), f(x)) < \frac{\epsilon}{5}$ this is because of B-almost periodicity of $f$. Let us choose $\eta_i$ such that $\|x + y - \xi_i\| < \delta$ then from (2) we have

$$D(f(x + y + x_k^{'}), f(\xi_i + x_k^{'})) < \frac{\epsilon}{5}\ , \forall\ k, 1 \leq i \leq \nu \tag{4}$$

Now $\forall\ k, l \geq k_0$ and $\forall\ x \in \mathbb{R}^n$ we have

$$\begin{aligned}
D(f(x + x_k^{'}), f(x + x_l^{'})) &= \|f(x + x_k^{'}) - f(x + x_l^{'})\| \\
&\leq \|f(x + x_k^{'}) - f(x + x_k^{'} + y)\| \\
&+ \|f(x + x_k^{'} + y) - f(\xi_i + x_k^{'})\| \\
&+ \|f(\xi_i + x_k^{'}) - f(\xi_i + x_l^{'})\| \\
&+ \|f(\xi_i + x_l^{'}) - f(x + y + x_l^{'})\| \\
&+ \|f(x + y + x_l^{'}) - f(x + x_l^{'})\| \\
&< \frac{\epsilon}{5} + \frac{\epsilon}{5} + \frac{\epsilon}{5} + \frac{\epsilon}{5} + \frac{\epsilon}{5} \\
&= \epsilon
\end{aligned}$$

[By (3), (3) and almost periodicity of $f$ ]. Which proves the uniform convergence of the sequence $(f(x+x_k^{'}))_k$ over $\mathbb{R}^n$. □

**Theorem 11.** *A function $f : \mathbb{R}^n \longrightarrow X$ is B-almost periodic if and only if for every sequence $(x_k)_k$ in $\mathbb{R}^n$, there exists a subsequence $(x_k^{'})_k$ such that $(f(x+x_k^{'}))_k$ is uniformly convergent over $\mathbb{R}^n$.*

*Proof.* The condition is necessary by Theorem 10. We now prove that it is sufficient. Suppose, by contradiction, that $f$ is not B-almost periodic, so there exists an $\epsilon > 0$ such that for every real number $r > 0$, there exists a closed ball $\overline{B}(a;r)$ which contains no element of $T(f;\epsilon)$. Consider now an arbitrary vector $x_1 \in \mathbb{R}^n$ and take $r_2 - \|x_1\| > 0$ so $\exists$ a ball $\overline{B}(x_2;r_2)$ which is disjoint of $T(f;\epsilon)$. Note that $x_2 - x_1 \in \overline{B}(x_2;r_2)$ hence $x_2 - x_1 \notin T(f;\epsilon)$. Next take $r_3 > \|x_1\| + \|x_2\|$ and find a ball $\overline{B}(x_3;r_3)$ which is disjoint of $T(f;\epsilon)$. Now both the $x_3 - x_1$ and $x_2 - x_3$ belong to $\overline{B}(x_3;r_3)$ but $x_3 - x_1, x_2 - x_3 \notin T(f;\epsilon)$. Continuing this procedure, we can find an infinite sequence $(x_k)_k \subset \mathbb{R}^n$ such that $\forall\ l,m \in \mathbb{N}$ , $l \neq m \Longrightarrow x_l - x_m \notin T(f;\epsilon)$. It also follows that

$$D(f(x+x_l-x_m), f(x)) \geq \epsilon, \forall\ x \in \mathbb{R}^n,\ \forall\ l,m \in \mathbb{N}, l \neq m$$

If we put $y = x - x_l$ we get

$$D(f(y+x_l), f(y+x_m)) \geq \epsilon,\ \forall\ y \in \mathbb{R}^n, \forall\ l,m \in \mathbb{N}, l \neq m \tag{5}$$

Suppose there exists a subsequence $(x_k^{'})_k$ of $(x_k)_k$ such that $(f(x+x_k^{'}))_k$ converges uniformly over $\mathbb{R}^n$. Then for already chosen $\epsilon > 0$, there exists a natural number $p_0 = p_0\ (\epsilon)$ such that

$$\forall\ l,m\ \geq p_0\ \Rightarrow D(f(x+x_l^{'}), f(x+x_m^{'})) < \epsilon, \forall\ x \in \mathbb{R}^n$$

This contradicts (5) and so the sufficiency of condition is established. Hence the theorem is proved. □

We also have the following interesting theorem

**Theorem 12.** *Let $f : \mathbb{R}^n \longrightarrow X$ be a B-almost periodic function. Then for every sequence $(x_k)_k \subset \mathbb{R}^n$ there exists a subsequence $(x_k^{'})$ such that for every $\epsilon > 0$, the inequality*

$$D(f(x+x_k^{'}), f(x+x_j^{'})) < \epsilon, \forall x \in \mathbb{R}^n, \forall k,j \in \mathbb{N} \textit{ is satisfied.}$$

*Proof.* Let $\epsilon > 0$ then by the almost periodicity of $f$ there exists a real number $r = r(\epsilon) > 0$ such that every closed ball of radius $r$ there exists $y$ such that

$$D(f(x+y), f(x)) < \frac{\epsilon}{3}, \forall\ x \in \mathbb{R}^n$$

Consider now a given sequence $(x_k)_k$ in $\mathbb{R}^n$ then by Theorem 3, $x_k \in \mathbb{R}^n$ can be written as $x_k = z_k + y_k$, $y_k \in T(f;\frac{\epsilon}{3})$ and $\|z_k\| \leq r$, for all $k$, where $T(f;\frac{\epsilon}{3})$ is relatively dense set associated with $f$. Moreover, using the uniform continuity of $f$ over $\mathbb{R}^n$, we can find $\delta > 0$ such that

$$\|x_1 - x_2\| < 2\delta \Rightarrow D(f(x_1), f(x_2)) < \frac{\epsilon}{3},\ \forall\ x_1, x_2 \in \mathbb{R}^n$$

Note that $z_k \in \overline{B}(0;r)$ for all $k$. Since $\overline{B}(0;r)$ is a closed and bounded subset of finite-dimensional normed space $\mathbb{R}^n$ so it must be compact in $\mathbb{R}^n$. Therefore the sequence $(z_k)_k$ has a convergent subsequence, say $(z_{k_i})_i$. Let $\lim\limits_{i\longrightarrow\infty} z_{k_i} = z$, which shows that $z \in \overline{B}(0;r)$. Now consider the subsequence $(z_{k_i})_i$ ( we use the same notation ) with

$$\|z_{k_i} - z\| < \delta, i = 1, 2, 3....$$

and let $(x_{k_i})_i$ be the corresponding subsequence of $(x_k)_k$ with

$$x_{k_i} = z_{k_i} + y_{k_i},\ i = 1, 2, 3, ...$$

Let us now prove that

$$D(f(x + x_{k_i}), f(x + x_{k_j})) < \epsilon$$

for all $x \in \mathbb{R}^n$ and for all $i$, $j$. Now

$$\begin{aligned} D(f(x + x_{k_i}), f(x + x_{k_j})) &= D(f(x + y_{k_i} + z_{k_i}), f(x + y_{k_j} + z_{k_j})) \\ &= \|f(x + y_{k_i} + z_{k_i}) - f(x + y_{k_j} + z_{k_j})\| \\ &\le \|f(x + y_{k_i} + z_{k_i}) - f(x + z_{k_i})\| \\ &+ \|f(x + z_{k_i}) - f(x + z_{k_j})\| \\ &+ \|f(x + z_{k_j}) - f(x + y_{k_j} + z_{k_j})\| \\ &= D(f(x + y_{k_i} + z_{k_i}), f(x + z_{k_i})) \\ &+ D(f(x + z_{k_i}), f(x + z_{k_j})) \\ &+ D(f(x + z_{k_j}), f(x + y_{k_j} + z_{k_j})) \end{aligned} \tag{6}$$

Since $f$ is almost periodic and $y_{k_i}$, $y_{k_j} \in T(f; \frac{\epsilon}{3})$, $x + z_{k_i}, x + z_{k_j} \in \mathbb{R}^n$, $\forall\ i, j \in \mathbb{N}$, therefore we have

$$D(f(x + y_{k_i} + z_{k_i}), f(x + z_{k_i})) < \frac{\epsilon}{3} \text{ and } D(f(x + y_{k_j} + z_{k_j}), f(x + z_{k_j})) < \frac{\epsilon}{3}$$

Also since $f$ is uniformly continues and $\|(x + z_{k_i}) - (x + z_{k_j})\| = \|z_{k_i} - z_{k_j}\| = \|z_{k_i} - z + z - z_{k_j}\| \le \|z_{k_i} - z\| + \|z - z_{k_j}\| < \delta + \delta = 2\delta$, therefore we have

$$D(f(x + z_{k_i}), f(x + z_{k_j})) < \frac{\epsilon}{3}, \forall x \in \mathbb{R}^n, \forall\ i, j \in \mathbb{N}$$

Hence from (6) we get

$$D(f(x + x_{k_i}), f(x + x_{k_j})) < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon\ , \forall x \in \mathbb{R}^n, \forall\ i, j \in \mathbb{N}$$

If we put $x_{k_i} = x_i^{'}$ and $x_{k_j} = x_j^{'}$, we get

$$D(f(x + x_k^{'}), f(x + x_j^{'})) < \epsilon, \forall x \in \mathbb{R}^n, \forall k, j \in \mathbb{N}$$

This completes the proof of the theorem. □

**Definition 4.** *A function $f : \mathbb{R}^n \longrightarrow X$ is said to be normal if for any sequence $(x_k)_k \subset \mathbb{R}^n$ one can extract a subsequence $(x_k^{'})$ such that the sequence $(f(x_k^{'} + x))_k$ of translated functions is convergent uniformly on $\mathbb{R}^n$.*

**Remark 7.** *From Theorem 11 it is obvious that $f$ is normal if and only if it is B-almost periodic.*

**Corollary 1.** *(i) The sum $f + g$ of two B-almost periodic functions $f$ and $g$ defined on $\mathbb{R}^n$ with values in the p-Fréchet space $X$, with $0 < p < 1$, is B-almost periodic function.*

*(ii) If $f_1$; $f_2 : \mathbb{R}^n \longrightarrow X$ are B-almost periodic, then the function $F : \mathbb{R}^n \longrightarrow X \times X$, defined by $F(x) = (f_1(x), f_2(x))$, $x \in \mathbb{R}^n$, is B-almost periodic.*

*Proof.* (i) Let $(x_k)_k$ be sequence in $\mathbb{R}^n$. By almost periodicity of $f$ and $g$ there exists a subsequence $(x_k^{'})_k \subset (x_k)_k$ such that both $(f(x + x_k^{'}))_k$ and $(g(x + x_k^{'}))_k$ are uniformly convergent in $x \in \mathbb{R}^n$ in view of Theorem 11. Consequently $((f + g)(x + x_k^{'}))_k$ is also uniformly convergent in $x \in \mathbb{R}^n$. This completes the proof again by Theorem 11.

(ii) Let $(x_k)_k$ be sequence in $\mathbb{R}^n$. In view of Theorem 11 one can find a subsequence $(x_k^{'})_k \subset (x_k)_k$ such that both $(f(x + x_k^{'}))_k$ and $(g(x + x_k^{'}))_k$ are uniformly convergent in $x \in \mathbb{R}^n$. Thus $(F(x + x_k^{'}))_{k=1}^{\infty} = ((f_1(x + x_k^{'}), f_2(x + x_k^{'})))_k$ is uniformly convergent in $x \in \mathbb{R}^n$, which proves the almost periodicity of $F$. □

**Theorem 13.** *If $f_1$; $f_2 : \mathbb{R}^n \longrightarrow X$ are B-almost periodic, then $\epsilon > 0$, there exist common $\epsilon$-translation numbers for $f_1$ and $f_2$*

*Proof.* Let $X$ be a $p$-Fréchet space with $D(x, y) = \|x - y\|$, $0 < p < 1$. Consider the Cartesian product $X \times X = X^2$ with $\|(x, y)\|_{X^2} = \|x\| + \|y\|$, $(x, y) \in X^2$ then obviously it is a $p$-norm on $X^2$. Endowed with the metric $\overline{D} : X^2 \times X^2 \longrightarrow \mathbb{R}_+ \cup \{0\}$, defined by

$$\begin{aligned}\overline{D}((x_1, y_1), (x_2, y_2)) &= \|(x_1, y_1) - (x_2, y_2)\|_{X^2}\\ &= \|(x_1 - x_1, x_2 - y_2)\|_{X^2}\\ &= \|x_1 - x_2\| + \|y_1 - y_2\|\\ &= D(x_1, x_2) + D(y_1, y_2),\ \forall\ (x_1, y_1), (x_2, y_2) \in\ X \times X\end{aligned}$$

$X \times X$ is a $p$-Fréchet space. As $f_1$; $f_2 : \mathbb{R}^n \longrightarrow X$ are B-almost periodic, therefore the function $F : \mathbb{R}^n \longrightarrow X \times X$, defined by $F(x) = (f_1(x), f_2(x))$, $x \in \mathbb{R}^n$, is B-almost periodic by Corollary (1). Let $y$ be a $\epsilon$-translation vector of $F$ then we have

$$\overline{D}(F(x + y), F(x)) < \epsilon,\ \forall\ x \in \mathbb{R}^n$$

Since

$$\begin{aligned}\overline{D}(F(x + y), F(x)) &= \overline{D}((f_1(x + y), f_2(x + y)), (f_1(x), f_2(x)))\\ &= D((f_1(x + y), f_1(x)) + D((f_2(x + y), f_2(x)) < \epsilon,\ \forall\ x \in \mathbb{R}^n\end{aligned}$$

therefore we have

$$D((f_1(x + y), f_1(x)) < \epsilon \text{ and } D((f_2(x + y), f_2(x)) < \epsilon, \forall x \in \mathbb{R}^n$$

This proves that $f_1$and $f_2$ have common $\epsilon$-translation numbers and this completes the proof of the theorem as well. □

**Remark 8.** *Corollary (1) and Theorem (13) hold true even for $k$ functions where $k > 2$.*

In what follows we denote the set of all continuous and bounded functions from $\mathbb{R}^n \longrightarrow X$ by $C_b(\mathbb{R}^n, X)$ and the set of all B-almost periodic functions from $\mathbb{R}^n \longrightarrow X$ by $AP(X)$, where $X$ is a $p$-Fréchet space, $0 < p < 1$, that is $C_b(\mathbb{R}^n, X) = \{f : \mathbb{R}^n \longrightarrow X;\ f$ is continuous and bounded $\}$ and $AP(X) = \{f : \mathbb{R}^n \longrightarrow X;\ f$ is B-almost periodic $\}$.

**Remark 9.** *If we define $\|f\|_b = \sup\{\|f(x)\|; x \in \mathbb{R}^n\}$, $f \in C_b(\mathbb{R}^n, X)$ then $\|f\|_b < +\infty$. Obviously $\|.\|_b$ is a p-norm on the space $C_b(\mathbb{R}^n, X)$. Since $(X, D)$ is a complete metric space, it follows that $C_b(\mathbb{R}^n, X)$ becomes a complete metric space with respect to the metric $D_b(f, g) = \|f - g\|$, that is $(C_b(\mathbb{R}^n, X), \|.\|_b)$ is a p-Fréchet space, $0 < p < 1$. By Theorem 5 it follows that $AP(X) \subset C_b(\mathbb{R}^n, X)$, that is $AP(X)$ is a linear subspace of $C_b(\mathbb{R}^n, X)$. By Theorem 6 (ii) it also follows that $AP(X)$ is closed linear subspace of the the space $C_b(\mathbb{R}^n, X)$. Therefore $(AP(X), D_b)$ is a complete metric space and hence $(AP(X), \|.\|_b)$ turns out to be a p-Fréchet space, $0 < p < 1$.*

In what follows we consider the notion of Bochner's transform. The Bochner's transform of a function $f \in C_b(\mathbb{R}^n, X)$ is denoted by $B(f) = \widetilde{f}$ and is defined by $\widetilde{f} : \mathbb{R}^n \longrightarrow C_b(\mathbb{R}^n, X)$, $\widetilde{f}(s) \in C_b(\mathbb{R}^n, X)$ and $\widetilde{f}(s)(x) = f(x + s)\ \forall\ x \in \mathbb{R}^n$. The properties of Bochner's transform are given in the following theorem

**Theorem 14.** *(i) $\|\widetilde{f}(s)\|_b = \|f(. + s)\|_b = \|\widetilde{f}(0)\|_b$, for all $s \in \mathbb{R}^n$*

*(ii) $\|\widetilde{f}(s+y) - \widetilde{f}(s)\|_b = \sup\{\|f(x+y) - f(x)\|; x \in \mathbb{R}^n\} = \|\widetilde{f}(s) - \widetilde{f}(0)\|_b$, for all $s, y \in \mathbb{R}^n$;*

*(iii) $f$ is B-almost periodic iff $\widetilde{f}$ is B-almost periodic, with the same set of $\epsilon$-translation numbers;*

*(iv) $\widetilde{f}$ is B-almost periodic iff there exists a relatively dense sequence $\{x_n; n \in \mathbb{N}\}$ such the sequence $\{\widetilde{f}(x_n); n \in \mathbb{N}\}$ of functions, is relatively compact in $(C_b(\mathbb{R}^n, X), \|.\|_b)$;*

*(v) $\widetilde{f}$ is B-almost periodic iff $\widetilde{f}(\mathbb{R}^n)$ is relatively compact in $(C_b(\mathbb{R}^n, X), \|.\|_b)$;*

*(vi) (Bochner's criterion) $f$ is B-almost periodic iff $\widetilde{f}(\mathbb{R}^n)$ is relatively compact in $(C_b(\mathbb{R}^n, X), \|.\|_b)$.*

*Proof.* (i) Since $\|f\|_b = \sup\{\|f(x)\|; x \in \mathbb{R}^n\}$, therefore we have

$$\begin{aligned}
\|\widetilde{f}(s)\|_b &= \sup\{\|\widetilde{f}(s)(x)\|; x \in \mathbb{R}^n\} \\
&= \sup\{\|f(x + s)\|; x \in \mathbb{R}^n\} \\
&= \sup\{\|f(y)\|; y \in \mathbb{R}^n\} \\
&= \sup\{\|f(y + 0)\|; y \in \mathbb{R}^n\} \\
&= \sup\{\|f(0)(y)\|; y \in \mathbb{R}^n\} \\
&= \|\widetilde{f}(0)\|_b,\ \forall\ s \in \mathbb{R}^n
\end{aligned}$$

Also

$$\begin{aligned}\|\widetilde{f}(.+s)\|_b &= \sup\{\|\widetilde{f}(.+s)(x)\|; x\in\mathbb{R}^n\}\\ &= \sup\{\|f(x+.+s)\|; x\in\mathbb{R}^n\}\\ &= \sup\{\|f(y)\|; y\in\mathbb{R}^n\}\\ &= \sup\{\|f(y)\|; y\in\mathbb{R}^n\}\\ &= \sup\{\|f(y+0)\|; y\in\mathbb{R}^n\}\\ &= \sup\{\|\widetilde{f}(0)(y)\|; y\in\mathbb{R}^n\}\\ &= \|\widetilde{f}(0)\|_b,\ \forall\ s\in\mathbb{R}^n\end{aligned}$$

Hence it is proved that

$$\|\widetilde{f}(s)\|_b = \|f(.+s)\|_b = \|\widetilde{f}(0)\|_b,\ \forall\ s\in\mathbb{R}^n$$

(ii) Again by using the definition of $\|.\|_b$ we have

$$\begin{aligned}\|\widetilde{f}(s+y)-\widetilde{f}(s)\|_b &= \sup\{\|\widetilde{f}(s+y)(x)-f(s)(x)\|; x\in\mathbb{R}^n\}\\ &= \sup\{\|f(x+s+y)-f(x+s)\|; x\in\mathbb{R}^n\}\\ &= \sup\{\|f(z+y)-f(z)\|; z\in\mathbb{R}^n\}\\ &= \sup\{\|f(z+y)-f(0+z)\|; z\in\mathbb{R}^n\}\\ &= \sup\{\|f(y)(z)-f(0)(z)\|; z\in\mathbb{R}^n\}\\ &= \|\widetilde{f}(y)-\widetilde{f}(0)\|_b\ \forall\ y,s\in\mathbb{R}^n\end{aligned}$$

(iii) This is an immediate consequence of (ii)

(iv) If $\widetilde{f}(s)$ is almost periodic, it follows by Remark 6 that for every sequence $(x_k)_k$ in $\mathbb{R}^n$, the set $\{\widetilde{f}(x_k) : k\in\mathbb{N}\}$ is relatively compact in $(C_b(\mathbb{R}^n,X),\|.\|_b)$. Conversely suppose that there exists a relatively dense sequence $(x_k)_k$ in $\mathbb{R}^n$ such that the set $\{\widetilde{f}(x_k) : k\in\mathbb{N}\}$ is relatively compact in the complete metric space $(C_b(\mathbb{R}^n,X),\|.\|_b)$. This is equivalent with the fact that $\{\widetilde{f}(x_k) : k\in\mathbb{N}\}$ is totally bounded in $(C_b(\mathbb{R}^n,X),\|.\|_b)$. Therefore due to total boundedness of the set $\{\widetilde{f}(x_k) : k\in\mathbb{N}\}$ in $C_b(\mathbb{R}^n,X)$, it is possible to fined $\nu$ vectors $\{x_{1,0},x_{2,0},...x_{\nu,0}\}$ in $(x_k)_k$ such that $\widetilde{f}(x_k)\in\bigcup_{j=1}^{\nu}\overline{B}(\widetilde{f}(x_{j,0}),\varepsilon)$, $\forall\ k=1,2,3,....$ We divide the whole sequence $\{\widetilde{f}(x_k) : k\in\mathbb{N}\}$ into $\nu$ subsequences $\{\widetilde{f}(x_{j,k}) : k\in\mathbb{N}\}$, $j=1,2,3,...,\nu$, where $(x_{j,k})_k\subset(x_k)_k$ is such that $\|\widetilde{f}(x_{j,k})-\widetilde{f}(x_{j,0})\|_b<\varepsilon$ which by Theorem 14(ii) implies that $\|\widetilde{f}(x_{j,k}-x_{j,0})-\widetilde{f}(0)\|_b<\varepsilon$. It follows that $y_{j,k}=x_{j,k}-x_{j,0}\in T(\widetilde{f},\varepsilon)$. Now we show that the set $\bigcup_{j=1}^{\nu}\{y_{j,k} : k\in\mathbb{N}\}$ is relatively dense in $\mathbb{R}^n$. Let $r>0$ be such that any ball $\overline{B}(x,r)$ in $\mathbb{R}^n$contains some $x_k$. We say that in any ball $\overline{B}(x,r+\overline{r})$ ,where $\overline{r}=\sup_{1\le j\le\nu}\{\|x_{j,0}\|\}$, contains some $y_{j,k}$. In fact, in $B(x,r)$ there is some $x_k$ which is an $x_{j,k}$ for some $j$ and $k$. Hence $\|x_{j,k}-x\|\le r$ and it follows that $\|x_{j,k}-x_{j,0}-x\|\le\|x_{j,k}-x\|+\|x_{j,0}\|\le r+\overline{r}$, so that $x_{j,k}-x_{j,0}\in\overline{B}(x,r+\overline{r})$. We note that the relatively dense set contained in $T(\widetilde{f},\varepsilon)$ was obtained by taking differences of elements in $(x_k)_k$. Also assume that $f\in C_b(\mathbb{R}^n,X)$ is almost periodic then $\{\widetilde{f}(x) : x \in \mathbb{R}\}$ is relatively compact and in particular $\{\widetilde{f}(w_k) : w_k \in \mathbb{Z}^n\}$ is relatively compact sequence in $C_b(\mathbb{R}^n, X)$. It follows that, $\forall\ \varepsilon > 0$, the set $T(f, \varepsilon)$ contains relatively dense set formed from the elements in $\mathbb{Z}^n$. This proves the B-almost periodicity of $\widetilde{f}$.

(v) The necessity follows from Theorem 8. The sufficiency is a direct consequence of (iv).

(vi) It is a consequence of (iii) and (v) □

The following theorem is also sufficient criterion for almost periodicity.

**Theorem 15.** *Let $f \in C_b(\mathbb{R}^n, X)$ and $\{\widetilde{f}(x_k) : k \in \mathbb{N}\}$ be a relatively compact sequence in $X$ for a relatively dense sequence $(x_k)_k$ in $\mathbb{R}^n$. Assume that*

$$cD(f(x + x_p), f(x + x_q)) \leq D(f(x_p), f(x_q)), \forall\ x \in \mathbb{R}^n,\ \forall\ p, q \in \mathbb{N}$$

*holds true for some $c > 0$. Then $f$ is B-almost periodic.*

*Proof.* From the inequality in the statement we have

$$\begin{aligned} D(f(x_p), f(x_q)) &= \|f(x_p) - f(x_q)\| \\ &\geq c\|f(x + x_p) - f(x + x_q)\| \\ &\geq c \sup\{\|f(x + x_p) - f(x + x_q)\|; x \in \mathbb{R}^n\} \\ &= c \sup\{\|\widetilde{f}(x_p)(x) - \widetilde{f}(x_q)(x)\|; x \in \mathbb{R}^n\} \\ &= cD_b(\widetilde{f}(x_p), \widetilde{f}(x_q)), \forall p, q \in \mathbb{N} \end{aligned}$$

Since the set $\{f(x_k) : n \in \mathbb{N}\}$ is relatively compact, it has a convergent subsequence $\{f(x_k^{'}) : n \in \mathbb{N}\}$ which is Cauchy sequence in the complete metric space $(X, D)$, so it must be convergent. It follows that $\{\widetilde{f}(x_k^{'}) : n \in \mathbb{N}\}$ is a Cauchy sequence in $(C_b(\mathbb{R}^n, X), \|.\|_b)$. With this together with Theorem 14, (iv), it follows that $\widetilde{f}(s)$ is B-almost periodic, which by Theorem 14, (ii), implies that $f$ is B-almost periodic.

This completes the proof of the theorem. □

**Corollary 2.** *Let the function $f : \mathbb{R}^n \longrightarrow X$ have relatively compact range and also assume that*

$$c \sup\{D(f(x + y), f(x)) : x \in \mathbb{R}^n\} \leq \inf\{D(f(x + y), f(x)) : x \in \mathbb{R}^n\}$$

*for some $c > 0$. Then $f$ is B-almost periodic.*

*Proof.* Let $(x_k)_k$ be any relatively dense sequence in $\mathbb{R}^n$ then $\{f(x_k) : n \in \mathbb{N}\}$ is relatively compact. Moreover if $y = x_p - x_q$, from the given inequality we get

$$c \sup\{D(f(x + x_p - x_q), f(x)) : x \in \mathbb{R}^n\} \leq \inf\{D(f(x + x_p - x_q), f(x)) : x \in \mathbb{R}^n\}$$

Take $u = x - x_q$ then

$$\begin{aligned} c \sup\{D(f(u + x_p), f(u + x_q)) : u \in \mathbb{R}^n\} &\leq \inf\{D(f(u + x_p), f(u + x_q)) : u \in \mathbb{R}^n\} \\ &\leq D(f(x_p), f(x_q)) \end{aligned} \tag{7}$$

but

$$\begin{aligned}\sup\{D(f(u+x_p),f(u+x_q)) : u\in\mathbb{R}^n\} &= \sup\{\|f(u+x_p)-f(u+x_q)\| : u\in\mathbb{R}^n\}\\ &= \sup\{\|\widetilde{f}(x_p)(u)-\widetilde{f}(x_q)(u)\| : u\in\mathbb{R}^n\}\\ &= \|\widetilde{f}(x_p)-\widetilde{f}(x_q)\|_b\\ &= D_b(\widetilde{f}(x_p),\widetilde{f}(x_q))\end{aligned}$$

Therefore from (7) we have

$$cD_b(\widetilde{f}(x_p),\widetilde{f}(x_q)) \leq D(f(x_p),f(x_q)),\ \forall p,q\in\mathbb{N}$$

From Theorem 15 the B-almost periodicity is proved. □

**Definition 5.** *Let $f:\mathbb{R}^n \longrightarrow X$ be an almost periodic function. A sequence $(x_k)_k$ in $\mathbb{R}^n$ is said to be regular with respect to $f$ if the sequence of translates $(f(x+x_k))_k$ is uniformly convergent on $\mathbb{R}^n$ which is equivalent to the convergence of $(\widetilde{f}(x_k))_k$ in $C_b(\mathbb{R}^n,X)$.*

**Theorem 16.** *Let $f:\mathbb{R}^n \longrightarrow X$ be an almost periodic function and assume that for a sequence $(x_k)_k$ in $\mathbb{R}^n$ there exists the limit $\lim\limits_{k\to\infty} f(\eta_p+x_k)=g_p\in X$ for all elements in the sequence $(\eta_p)_p$ which is dense in $\mathbb{R}^n$ then $(x_k)_k$ is regular to $f$.*

*Proof.* Suppose that $(\widetilde{f}(x_k))_k$ is not convergent then there exists $\varepsilon_0>0$ and one may extract sequences $(p_l)_l$ and $(q_l)_l$ from $(p)_p$, such that $\|\widetilde{f}(x_{p_k})-\widetilde{f}(x_{q_k})\|_b\geq\varepsilon_0$, $k=1,2,3....$ As the range of $\widetilde{f}$ is relatively compact, we may assume that $\lim\limits_{k\to\infty}\widetilde{f}(x_{p_k})=\widetilde{g}_1$, $\lim\limits_{k\to\infty}\widetilde{f}(x_{q_k})=\widetilde{g}_2$, where $D_b(\widetilde{g}_1,\widetilde{g}_2)\geq\varepsilon_0$. Hence $\lim\limits_{k\to\infty} f(x+x_{p_k})=g_1(x)$, $\lim\limits_{k\to\infty} f(x+x_{q_k})=g_2(x)$ uniformly on $\mathbb{R}^n$. On the other hand we get $\lim\limits_{k\to\infty} f(\eta_j+x_{p_k})=g_1(\eta_j)=g_j=\lim\limits_{k\to\infty} f(\eta_j+x_{q_k})=g_2(\eta_j)$, $\forall\ j=1,2,3,....$ From the continuity of $g_1$ and $g_2$ and the density of $(\eta_j)_j$ in $\mathbb{R}^n$ we obtain $g_1(x)=g_2(x)$, $\forall\ x\in\mathbb{R}^n$ a contradiction. This completes the proof. □

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